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A NOTE ON QUASI-ARMENDARIZ RINGS

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Abstract. A ring R is called a quasi-Armendariz ring if whenever elements $\alpha = a_0+a_1x+a_2x^2+\cdots+a_nx^n$, $\beta = b_0+b_1x+b_2x^2+\cdots+b_mx^m \in R[x]$ satisfy $\alpha R[x]\beta = 0$, then $a_iRb_j = 0$ for each i, j. In this note we consider quasi-Armendariz property of a special subring of the infinite upper triangular matrix ring over a ring R.

All rings considered here are associative with identity. A ring R is called an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. (The converse is always true.) The study of Armendariz rings was initiated by Armendariz [2] and Rege and Chhawchharia [14]. Some properties of Armendariz rings have been studied in Rege and Chhawchharia [14], Anderson and Camillo [1], Kim and Lee [9], Huh, Lee and Smoktunowicz [8], and Lee and Wong [10]. In [7], Hong, Kim and Kwak studied a generalization of Armendariz rings, which they called α-skew Armendariz rings, where α is an endomorphism of R. In [5], Hashemi and Moussavi considered some generalized concepts of Armendariz rings, which we can regard as the Armendariz rings relative to Ore extensions, or skew Laurent polynomial rings or skew Laurent series rings.

The concept of quasi-Armendariz rings is another generalization of Armendariz rings. According to [6], a ring R is called a quasi-Armendariz ring if whenever $f(x) = a_0+a_1x+\cdots+a_mx^m$, $g(x) = b_0+b_1x+\cdots+b_nx^n \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, we have $a_iRb_j = 0$ for each i and j. Clearly every Armendariz ring is quasi-Armendariz.

Let R be a ring. It was shown in [6] that if R is quasi-Armendariz then the $n \times n$-matrix ring $M_n(R)$ and the upper triangular matrix ring $T_n(R)$ are quasi-Armendariz. Here we consider the following ring:

$$S(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots \\ 0 & a & a_{23} & \cdots \\ 0 & 0 & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} |a, a_{ij} \in R \right\}.$$

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In the following, we will show that if R is a left APP-ring then $S(R)$ is quasi-Armendariz. We also give an example which shows that the ring $S(R)$ need not be left APP when R is a left APP-ring.

An ideal I of R is said to be right s-unital if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. It follows from Tominaga ([15], Theorem 1) that I is right s-unital if and only if for any finitely many elements $a_1, a_2, \cdots, a_n \in I$ there exists an element $x \in I$ such that $a_i = a_i x, i = 1, 2, \cdots, n$. According to [13] a ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$. Right APP-rings can be defined analogously. Recall a ring R is a left p.q.-Baer ring if the left annihilator of a principal left ideal of R is generated by an idempotent (see, for example, [3], [4] and [11]). Clearly every left p.q.-Baer ring is a left APP-ring (thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings). A ring R is a right PP-ring if the right annihilator of an element of R is generated by an idempotent. Right PP rings are left APP.

The following result follows from [6] and Example 2.4 of [13].

Proposition 1. *Every left APP-ring is quasi-Armendariz, but not conversely.*

We need a lemma as follows.

Lemma 2. *Let R be a left APP-ring and $a_1, \cdots, a_n, b_1, \cdots, b_m$ belong to R. If $a_i R b_j = 0$ for all i and j, then there exists $e \in R$ such that $a_i = a_i e$ and $eRb_j = 0$ for all i and j.*

Proof. It was shown in [13] that R is a left APP-ring if and only if for every finitely generated left ideal I of R, $l_R(I)$ is right s-unital. Let $I = Rb_1 + \cdots + Rb_m$. Then the conclusion follows from Tominaga ([15], Theorem 1) that an ideal J is right s-unital if and only if for any finitely many elements $a_1, a_2, \cdots, a_n \in J$ there exists an element $e \in J$ such that $a_i = a_i e, i = 1, 2, \cdots, n$. □

Theorem 3. *Let R be a left APP-ring. Then $S(R)$ is quasi-Armendariz.*

Proof. Suppose that R is left APP and $\sum_{i=1}^k A_i x^i$, $\sum_{j=1}^l B_j x^j \in S(R)[x]$ such that $(\sum_{i=1}^k A_i x^i) S(R)[x] (\sum_{j=1}^l B_j x^j) = 0$. We will show that $A_i S(R) B_j = 0$ for all i and j. Suppose that

$$A_i = \begin{pmatrix} a^i & a_{12}^i & a_{13}^i & \cdots \\ 0 & a^i & a_{23}^i & \cdots \\ 0 & 0 & a^i & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B_j = \begin{pmatrix} b^j & b_{12}^j & b_{13}^j & \cdots \\ 0 & b^j & b_{23}^j & \cdots \\ 0 & 0 & b^j & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Set $f = \sum_{i=1}^{k} a^i x^i$, $f_{pq} = \sum_{i=1}^{k} a^i_{pq} x^i$, $g = \sum_{j=1}^{l} b^j x^j$, and $g_{pq} = \sum_{j=1}^{l} b^j_{pq} x^j$ for any p, q with $1 \le p < q$. Then from $(\sum_{i=1}^{k} A_i x^i) S(R)[x] (\sum_{j=1}^{l} B_j x^j) = 0$ it follows that for any λ and $\lambda_{pq} \in R[x]$ with $1 \le p < q$

$$\begin{pmatrix} f & f_{12} & f_{13} & \cdots \\ 0 & f & f_{23} & \cdots \\ 0 & 0 & f & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \lambda & \lambda_{12} & \lambda_{13} & \cdots \\ 0 & \lambda & \lambda_{23} & \cdots \\ 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g & g_{12} & g_{13} & \cdots \\ 0 & g & g_{23} & \cdots \\ 0 & 0 & g & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

Thus, for any $\lambda \in R[x]$

$$f \lambda g = 0$$

and for any s, t with $1 \le s < t$, and any $\lambda, \lambda_{kl} \in R[x]$ with $s < k < l < t$,

$$\begin{aligned} f\lambda g_{st} + f\lambda_{st} g + f_{st}\lambda g \quad &+ \sum_{l=s+1}^{t-1} f\lambda_{sl} g_{lt} + \sum_{l=s+1}^{t-1} f_{sl}\lambda g_{lt} + \sum_{l=s+1}^{t-1} f_{sl}\lambda_{lt} g \\ &+ \sum_{l=s+1}^{t-1} \sum_{k=s+1}^{l-1} f_{sk}\lambda_{kl} g_{lt} = 0. \end{aligned}$$

Fixed s and t with $s < t$. Let $\lambda = \lambda_{st} = \lambda_{sl} = \lambda_{lt} = 0$ for any l with $s + 1 \le l \le t - 1$. Then

$$\sum_{l=s+1}^{t-1} \sum_{k=s+1}^{l-1} f_{sk}\lambda_{kl} g_{lt} = 0.$$

That is

$$\sum_{s<k<l<t} f_{sk}\lambda_{kl} g_{lt} = 0.$$

For any $k_0 < l_0$, if we take $\lambda_{kl} = 0$ when $k \ne k_0$ or $l \ne l_0$, then it follows that $f_{sk_0}\lambda_{k_0 l_0} g_{l_0 t} = 0$. This means that for any $\lambda \in R[x]$,

$$f_{sk}\lambda g_{lt} = 0, \quad \forall s < k < l < t. \tag{1}$$

Since R is quasi-Armendariz, we have

$$a^i_{sk} R b^j_{lt} = 0, \quad \forall s < k < l < t,\ \forall i,\ \forall j.$$

Take $\lambda_{st} = 0$ for any s and t. Then it follows that for any $1 \le s < t$

$$f\lambda g_{st} + f_{st}\lambda g + \sum_{l=s+1}^{t-1} f_{sl}\lambda g_{lt} = 0. \tag{2}$$

Now we by induction on $t - s$ show that for every $\alpha \in R[x]$

$$f\alpha g_{st} = 0,\ f_{st}\alpha g = 0,\ f_{sl}\alpha g_{lt} = 0, \forall s < l < t. \tag{3}$$

Let $t-s=1$. Then
$$f\alpha g_{st}+f_{st}\alpha g=0.$$
Since $f\alpha g=0$ for every $\alpha\in R[x]$, we have $a^iRb^j=0$ for all i and j. Thus, by Lemma 2, there exists $e\in R$ such that $a^i=a^ie$ and $eRb^j=0$ for all i and j and, so $f=fe$ and $eR[x]g=0$. Substitute $e\beta$ for α in $f\alpha g_{st}+f_{st}\alpha g=0$ to yield
$$f\beta g_{st}=fe\beta g_{st}=fe\beta g_{st}+f_{st}e\beta g=0.$$
Hence $f_{st}\alpha g=0$.

Now suppose that $t-s>1$ and (3) holds for $t-s<m$. We will show that (3) holds for $t-s=m$. For $l=s+1,s+2,\cdots,t-1$, since $t-l<m$, we have $f\alpha g_{lt}=0$ by the induction hypothesis. Thus $fR[x]g_{lt}=0$. Since R is quasi-Armendariz, we have $a^iRb^j=0$ and $a^iRb^j_{lt}=0$ for all i and j. Thus, by Lemma 2, there exists $e\in R$ such that $a^i=a^ie$ and $eRb^j=0$, $eRb^j_{lt}=0$ for all i, j and l with $s+1\le l\le t-1$. Hence $f=fe$ and $eR[x]g=0$, $eR[x]g_{lt}=0$. Now substitute $e\beta$ for λ in (2) to obtain
$$0=fe\beta g_{st}+f_{st}e\beta g+\sum_{s<l<t}f_{sl}e\beta g_{lt}=fe\beta g_{st}=f\beta g_{st}.$$
Thus
$$f_{st}\beta g+\sum_{s<l<t}f_{sl}\beta g_{lt}=0. \tag{4}$$
For $k=s+1,s+2,\cdots,t-1$, since $k-s<m$, by the induction hypothesis, we have $f_{sk}\alpha g=0$. Thus $f_{sk}R[x]g=0$ and, so $a^i_{sk}Rb^j=0$ for all i and j since R is quasi-Armendariz. By Lemma 2 again, there exists $c\in R$ such that $a^i_{sk}=a^i_{sk}c$ and $cRb^j=0$ for all i, j and all k with $s<k<t$. Hence $f_{sk}=f_{sk}c$ and $cR[x]g=0$. Now substitute $c\gamma$ for β in (4) to yield
$$0=f_{st}c\gamma g+\sum_{s<l<t}f_{sl}c\gamma g_{lt}=\sum_{s<l<t}f_{sl}\gamma g_{lt}. \tag{5}$$
By (1) it follows that for every $\gamma\in R[x]$,
$$f_{s,s+1}\gamma g_{s+2,t}=0,\quad \cdots,f_{s,s+1}\gamma g_{t-1,t}=0.$$
This means that $f_{s,s+1}R[x]g_{s+2,t}=0$, $\cdots$, $f_{s,s+1}R[x]g_{t-1,t}=0$. Thus
$$a^i_{s,s+1}Rb^j_{s+2,t}=0,\cdots,a^i_{s,s+1}Rb^j_{t-1,t}=0$$
for all i and j. Hence there exists $d\in R$ such that $a^i_{s,s+1}=a^i_{s,s+1}d$ and $dRb^j_{s+2,t}=0$, $\cdots$, $dRb^j_{t-1,t}=0$ for all i and j, which implies that $f_{s,s+1}=f_{s,s+1}d$ and $dR[x]g_{s+2,t}=0$, $\cdots$, $dR[x]g_{t-1,t}=0$. Substitute $d\delta$ for γ in (5) to yield
$$\begin{aligned}f_{s,s+1}\delta g_{s+1,t}&=f_{s,s+1}d\delta g_{s+1,t}\\&=f_{s,s+1}d\delta g_{s+1,t}+f_{s,s+2}d\delta g_{s+2,t}+\cdots+f_{s,t-1}d\delta g_{t-1,t}\end{aligned}$$

$$= \quad 0.$$

Thus

$$f_{s,s+2}\delta g_{s+2,t} + \cdots + f_{s,t-1}\delta g_{t-1,t} = 0.$$

Continuing this procedure yields

$$f_{s,s+2}\delta g_{s+2,t} = 0, \cdots, f_{s,t-1}\delta g_{t-1,t} = 0.$$

Now we have shown that $f\alpha g_{st} = 0, f_{st}\alpha g = 0, f_{sl}\alpha g_{lt} = 0$ for all $s < l < t$, by induction, which implies that $fR[x]g_{st} = 0, f_{st}R[x]g = 0, f_{sl}R[x]g_{lt} = 0$ for all $s < l < t$. Thus $a^iRb^j_{st} = 0, a^i_{st}Rb^j = 0$, $a^i_{sl}Rb^j_{lt} = 0$ for all i, j and $s < l < t$. Now it is easy to see that $A_iS(R)B_j = 0$ for all i and j. This completes the proof. □

Proposition 4. *If $S(R)$ is quasi-Armendariz then so is R.*

Proof. Suppose that $f = \sum a_ix^i$ and $g = \sum b_jx^j$ are in $R[x]$ such that $fR[x]g = 0$. Then for any α, $\alpha_{ij} \in R[x]$,

$$\begin{pmatrix} f & 0 & 0 & \cdots \\ 0 & f & 0 & \cdots \\ 0 & 0 & f & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \alpha & \alpha_{12} & \alpha_{13} & \cdots \\ 0 & \alpha & \alpha_{23} & \cdots \\ 0 & 0 & \alpha & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g & 0 & 0 & \cdots \\ 0 & g & 0 & \cdots \\ 0 & 0 & g & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

Thus

$$\begin{pmatrix} a_i & 0 & 0 & \cdots \\ 0 & a_i & 0 & \cdots \\ 0 & 0 & a_i & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} S(R) \begin{pmatrix} b_j & 0 & 0 & \cdots \\ 0 & b_j & 0 & \cdots \\ 0 & 0 & b_j & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0,$$

for all i and j, which implies that $a_iRb_j = 0$ for all i, j. □

The following example shows that the left APP property of R does not imply the left APP property of $S(R)$.

Example 5. *Let F be a field and consider the ring $S(F)$. Let*

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

belong to $S(F)$. Then

$$S(F)A = \left\{ \begin{pmatrix} 0 & a & a & a & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} | a \in F \right\}.$$

Thus it is easy to see that

$$l_{S(F)}(S(F)A) = \left\{ \begin{pmatrix} 0 & x_{12} & x_{13} & \cdots \\ 0 & 0 & x_{23} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} | x_{ij} \in F \right\}.$$

Now let

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in l_{S(F)}(S(F)A).$$

If $S(F)$ is left APP, then there exists $C \in l_{S(F)}(S(F)A)$ such that $B = BC$. But this contradicts with the fact

$$BC = B \begin{pmatrix} 0 & c_{12} & c_{13} & \cdots \\ 0 & 0 & c_{23} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_{23} & c_{24} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus $S(F)$ is not left APP.

Remark. *We do not know whether or not the ring $S(R)$ is quasi-Armendariz when R is quasi-Armendariz. In* [12] *an example was given to show that for a quasi-Armendariz ring R, the ring*

$$T(R,R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R \right\}$$

need not be quasi-Armendariz.

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