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SERIALLY COALESCENT CLASSES OF LIE ALGEBRAS

Masanobu HONDA and Takanori SAKAMOTO

Abstract. We introduce the concept of serially coalescent classes of Lie algebras corresponding to those of coalescent classes and ascendantly coalescent classes. We show that the class of finite-dimensional and nilpotent, the class of finite-dimensional and the class of finite-dimensional and soluble Lie algebras, are serially coalescent classes for locally finite Lie algebras over any field of characteristic zero. We also introduce the concept of locally serially coalescent classes of Lie algebras and find some locally serially coalescent classes for locally finite Lie algebras.

1. Introduction

Let $\mathfrak{X}$ be a class of Lie algebras. It is said that $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent) if in any Lie algebra L the join of two $\mathfrak{X}$-subideals (resp. ascendant $\mathfrak{X}$-subalgebras) of L is always an $\mathfrak{X}$-subideal (resp. ascendant $\mathfrak{X}$-subalgebra) of L. Several authors proved coalescence and ascendant coalescence for many well-known classes ([1, Chapters 3 and 4] etc.). It is also said that $\mathfrak{X}$ is locally coalescent (resp. locally ascendantly coalescent) if for any two $\mathfrak{X}$-subideals (resp. ascendant $\mathfrak{X}$-subalgebras) H, K of a Lie algebra L and for any finitely generated subalgebra Y of $J = \langle H, K\rangle$ there exists an $\mathfrak{X}$-subideal (resp. an ascendant $\mathfrak{X}$-subalgebra) X of L such that $Y \leq X \leq J$. Some authors proved local coalescence and local ascendant coalescence for many classes ([1], [8] and [9] etc.).

In this paper we introduce the concept of serially coalescent (resp. locally serially coalescent) classes of Lie algebras corresponding to that of ascendantly coalescent (resp. locally ascendantly coalescent) classes and prove that some important classes are serially coalescent (resp. locally serially coalescent) for locally finite Lie algebras.

In Section 3 we shall prove that the classes $\mathfrak{F}\cap\mathfrak{N}$, $\mathfrak{F}$, $\mathfrak{F}\cap\mathrm{E}\mathfrak{A}$ and the classes Min, Min-si, Min-asc, Min-ser, Min-$\lhd$ $\cap$ Max-$\lhd$ etc. are serially coalescent for locally finite Lie algebras over any field of characteristic zero (Theorems 4, 5, 6 and 8). In Section 4 we shall show that $\mathrm{L}\mathfrak{F}\cap\mathrm{L}(\mathrm{ser})\mathfrak{X} = \mathrm{L}\mathfrak{F}\cap\mathrm{J}(\mathrm{ser})\mathfrak{X}$ for any locally serially coalescent class $\mathfrak{X}$ for locally finite Lie algebras (Theorem 11) and that $\mathfrak{X}$ is locally serially coalescent for locally finite Lie algebras if and only if $\mathfrak{Y}$ is locally serially coalescent for locally finite Lie algebras, for any classes $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\mathrm{L}\mathfrak{F}\cap\mathfrak{X} \leq \mathrm{L}\mathfrak{F}\cap\mathfrak{Y} \leq \mathrm{L}\mathfrak{F}\cap\mathrm{L}(\mathrm{ser})\mathfrak{X}$ (Theorem 13). In Section 5 we shall show that if L is locally finite over any field of

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characteristic zero, then $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L) \in \mathrm{L(ser)}\mathfrak{X}$ for $\mathfrak{X} = \mathfrak{F} \cap \mathfrak{N}, \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$ or $\mathfrak{F}$ (Proposition 14), especially, deduce that if L is locally finite over any field of characteristic zero, then $\mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L)$ is the set of all elements x of L such that $\langle x \rangle$ ser L (Corollary 15).

2. Notation and terminology

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [1] for the use of notation and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For a totally ordered set Σ, a series (resp. a weak series) from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma \mid \sigma \in \Sigma\}$ of subalgebras (resp. subspaces) of L such that

(1) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,
(2) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,
(3) $L \backslash H = \cup_{\sigma\in\Sigma}(\Lambda_\sigma \backslash V_\sigma)$,
(4) $V_\sigma \lhd \Lambda_\sigma$ (resp. $[\Lambda_\sigma, H] \subseteq V_\sigma$) for all $\sigma \in \Sigma$.

H is a serial (resp. a weakly serial) subalgebra of L, which we denote by $H \mathrm{ser} L$ (resp. $H \mathrm{wser} L$), if there exists a series (resp. a weak series) from H to L.

For an ordinal σ, H is a σ-step ascendant subalgebra of L, denoted by $H \lhd^\sigma L$, if there exists an ascending chain $(H_\alpha)_{\alpha\leq\sigma}$ of subalgebras of L such that

(1) $H_0 = H$ and $H_\sigma = L$,
(2) $H_\alpha \lhd H_{\alpha+1}$ for any ordinal $\alpha < \sigma$,
(3) $H_\lambda = \cup_{\alpha<\lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is an ascendant subalgebra of L, denoted by $H \mathrm{asc} L$ if $H \lhd^\sigma L$ for some ordinal σ. When σ is finite, H is a subideal of L and denoted by $H \mathrm{si} L$. For an ordinal α, we denote by L^α the α-th term of the transfinite lower central series of L and by $L^{(\alpha)}$ the α-th term of the transfinite derived series of L.

Let $\mathfrak{X}, \mathfrak{Y}$ be classes of Lie algebras and let Δ be any of the relations $\leq, \lhd$, si, asc, ser, wser. $\mathfrak{X}\mathfrak{Y}$ is the class of Lie algebras L having an ideal $I \in \mathfrak{X}$ such that $L/I \in \mathfrak{Y}$. A Lie algebra L is said to lie in $\mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra H of L such that $X \subseteq H \;\Delta\; L$ and to lie in $\mathrm{J}(\Delta)\mathfrak{X}$ if L is generated by Δ-subalgebras belonging to $\mathfrak{X}$. In particular we write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\leq)\mathfrak{X}$. When $L \in \mathrm{L}\mathfrak{X}$ (resp. $\mathrm{L(ser)}\mathfrak{X}$), L is called a locally (resp. a serially) $\mathfrak{X}$-algebra. We write Max-Δ (resp. Min-Δ) for the classes of Lie algebras satisfying the maximal (resp. minimal) condition for Δ-subalgebras. $\mathfrak{F}, \mathfrak{G}, \mathfrak{A}, \mathfrak{N}$ and $\mathrm{E}\mathfrak{A}$ are the classes of Lie algebras which are finite-dimensional, finitely generated, abelian, nilpotent and soluble respectively. The $\mathfrak{X}$-residual $\lambda_{\mathfrak{X}}(L)$ of L is the intersection of the ideals I of

L such that $L/I \in \mathfrak{X}$. The Hirsch-Plotkin radical $\rho(L)$ of L is the unique maximal locally nilpotent ideal of L. For a locally finite Lie algebra L the locally soluble radical $\sigma(L)$ of L is the unique maximal locally soluble ideal of L.

3. Serially coalescent classes

We first begin with the $\mathrm{L}\mathfrak{N}$-residual $\lambda_{\mathrm{L}\mathfrak{N}}(L)$ and the $\mathrm{LE}\mathfrak{A}$-residual $\lambda_{\mathrm{LE}\mathfrak{A}}(L)$ of a Lie algebra L. The following two lemmas are key lemmas in this paper.

Lemma 1 ([2, Proposition 2.11]). *If H is a weakly serial subalgebra of a locally finite Lie algebra L, then $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$ and $\lambda_{\mathrm{LE}\mathfrak{A}}(H) \lhd L$.*

Lemma 2. *Let L be a Lie algebra.*
(1) If $L \in \mathfrak{G}$, then $\lambda_{\mathrm{L}\mathfrak{N}}(L) = L^{\omega}$ and $\lambda_{\mathrm{LE}\mathfrak{A}}(L) = L^{(\omega)}$.
(2) If $L \in$ Min-si, then $\lambda_{\mathrm{L}\mathfrak{N}}(L) = L^{\omega}$ and $\lambda_{\mathrm{LE}\mathfrak{A}}(L) = L^{(\omega)}$.
(3) If $L \in$ Max-$\lhd$, then $\lambda_{\mathrm{L}\mathfrak{N}}(L) = L^{\omega}$.
(4) If $L \in$ Min-$\lhd \cap$ Max-$\lhd$, then $\lambda_{\mathrm{LE}\mathfrak{A}}(L) = L^{(\omega)}$.

Proof. In general we have $\lambda_{\mathrm{L}\mathfrak{N}}(L) \leq L^{\omega}$ and $\lambda_{\mathrm{LE}\mathfrak{A}}(L) \leq L^{(\omega)}$.

(1) Let $L \in \mathfrak{G}$. For any ideal I of L such that $L/I \in \mathrm{L}\mathfrak{N}$ (resp. $\mathrm{LE}\mathfrak{A}$), we obtain that

$$L/I \in \mathrm{L}\mathfrak{N} \cap \mathfrak{G} \leq \mathfrak{N} \quad (\text{resp. } \mathrm{LE}\mathfrak{A} \cap \mathfrak{G} \leq \mathrm{E}\mathfrak{A}).$$

Therefore $L^{\omega} \leq I$ (resp. $L^{(\omega)} \leq I$). It follows that

$$L^{\omega} \leq \lambda_{\mathrm{L}\mathfrak{N}}(L) \quad (\text{resp. } L^{(\omega)} \leq \lambda_{\mathrm{LE}\mathfrak{A}}(L)).$$

(2) Let $L \in$ Min-si. For any ideal I of L such that $L/I \in \mathrm{L}\mathfrak{N}$ (resp. $\mathrm{LE}\mathfrak{A}$), we use [1, Proposition 8.5.1] (resp. [1, Corollary 8.5.5]) to see that

$$L/I \in \mathrm{L}\mathfrak{N} \cap \text{Min-si} \leq \mathrm{L}\mathfrak{N} \cap \text{Min-}\lhd^2 = \mathfrak{F} \cap \mathfrak{N} \text{ (resp. } \mathrm{LE}\mathfrak{A} \cap \text{Min-si} = \mathfrak{F} \cap \mathrm{E}\mathfrak{A}).$$

Therefore $L^{\omega} \leq I$ (resp. $L^{(\omega)} \leq I$). It follows that

$$L^{\omega} \leq \lambda_{\mathrm{L}\mathfrak{N}}(L) \quad (\text{resp. } L^{(\omega)} \leq \lambda_{\mathrm{LE}\mathfrak{A}}(L)).$$

(3) Let $L \in$ Max-$\lhd$. For any ideal I of L such that $L/I \in \mathrm{L}\mathfrak{N}$, we use [1, Theorem 8.6.5] to see that

$$L/I \in \mathrm{L}\mathfrak{N} \cap \text{Max-}\lhd \leq \mathfrak{G} \cap \mathfrak{N}.$$

Therefore $L^{\omega} \leq I$. It follows that $L^{\omega} \leq \lambda_{\mathrm{L}\mathfrak{N}}(L)$.

(4) Let $L \in$ Min-$\lhd \cap$ Max-$\lhd$. For any ideal I of L such that $L/I \in \mathrm{LE}\mathfrak{A}$, we have

$$L/I \in \mathrm{LE}\mathfrak{A} \cap \text{Min-}\lhd \cap \text{Max-}\lhd.$$

As $L/I \in$ Min-$\lhd$ there is a positive integer d such that $(L/I)^{(\omega)} = (L/I)^{(d)}$.

Set $M/I = (L/I)^{(d)}$. Then we have $M/I \lhd L/I$ and $(M/I)^2 = M/I$. Since $L/I \in \mathrm{L}\mathrm{E}\mathfrak{A} \cap \text{Max-}\lhd$ it follows from [1, Lemma 8.6.2] that $M/I = 0$. Therefore $L^{(\omega)} \leq L^{(d)} \leq I$. It follows that $L^{(\omega)} \leq \lambda_{\mathrm{LE}\mathfrak{A}}(L)$. $\square$

We also state the following lemma which is used below.

Lemma 3 ([1, Proposition 13.2.4])**.** *Let L be a locally finite Lie algebra and I an ideal of L. If H is a serial subalgebra of L, then $(H+I)/I$ is a serial subalgebra of L/I.*

Let $\mathfrak{X}$ be a class of Lie algebras. We say that $\mathfrak{X}$ is *serially coalescent* if in any Lie algebra L the join of two serial $\mathfrak{X}$-subalgebras of L is always a serial $\mathfrak{X}$-subalgebra of L. In order to study serially coalescent classes of Lie algebras we need to restrict ourselves to locally finite Lie algebras. We moreover say that $\mathfrak{X}$ is *serially coalescent for locally finite Lie algebras* if in any locally finite Lie algebra L the join of two serial $\mathfrak{X}$-subalgebras of L is always a serial $\mathfrak{X}$-subalgebra of L.

It is well known that the classes $\mathfrak{F} \cap \mathfrak{N}$, $\mathfrak{F}$ and $\mathfrak{F} \cap \mathrm{E}\mathfrak{A}$ are coalescent and ascendanly coalescent over any field of characteristic zero ([1, Theorems 3.2.4 and 3.2.5]). We shall now prove that the three classes above are serially coalescent for locally finite Lie algebras in the following three theorems.

Theorem 4. *Over any field of characteristic zero the class $\mathfrak{F} \cap \mathfrak{N}$ is serially coalescent for locally finite Lie algebras.*

Proof. Let $L \in \mathrm{L}\mathfrak{F}$ and suppose that H, K ser L and $H, K \in \mathfrak{F} \cap \mathfrak{N}$. Since $H, K \in \mathfrak{N}$ we have $\lambda_{\mathrm{L}\mathfrak{N}}(H) = \lambda_{\mathrm{L}\mathfrak{N}}(K) = 0$. It follows from [5, Corollary 6] that $H, K \leq \rho(L)$, so $\langle H, K \rangle \leq \rho(L) \in \mathrm{L}\mathfrak{N}$. Therefore we have $\langle H, K \rangle$ ser $\rho(L) \lhd L$ by [5, Lemma 3]. Hence $\langle H, K \rangle$ ser L. We also obtain $\langle H, K \rangle \in \mathrm{L}\mathfrak{N} \cap \mathfrak{G} \leq \mathfrak{F} \cap \mathfrak{N}$ owing to $H, K \in \mathfrak{F}$. $\square$

Theorem 5. *Over any field of characteristic zero the class $\mathfrak{F}$ is serially coalescent for locally finite Lie algebras.*

Proof. Let $L \in \mathrm{L}\mathfrak{F}$ and suppose that H, K ser L and $H, K \in \mathfrak{F}$. By making use of Lemma 1 and Lemma 2(1), we get

$$H^{\omega} = \lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L \quad \text{and} \quad K^{\omega} = \lambda_{\mathrm{L}\mathfrak{N}}(K) \lhd L.$$

Since $H, K \in \mathfrak{F}$, there are positive integers c, d such that $H^{\omega} = H^{c}, K^{\omega} = K^{d}$. Put $I = H^{c} + K^{d}$. Then $I \lhd L$. It follows from Lemma 3 that

$$(H+I)/I \text{ ser } L/I \quad \text{and} \quad (K+I)/I \text{ ser } L/I.$$

Here we have $(H+I)/I \cong H/H \cap I \cong (H/H^{c})/(H \cap I/H^{c}) \in \mathfrak{F} \cap \mathfrak{N}$. Similarly we have $(K+I)/I \in \mathfrak{F} \cap \mathfrak{N}$. Hence Theorem 4 leads to that

$$\langle (H+I)/I, (K+I)/I \rangle \in \mathfrak{F} \cap \mathfrak{N} \quad \text{and} \quad \langle (H+I)/I, (K+I)/I \rangle \text{ ser } L/I,$$

that is to say,

$$\langle H, K\rangle/I \in \mathfrak{F} \cap \mathfrak{N} \ \text{ and } \ \langle H, K\rangle/I \text{ ser } L/I.$$

Thus $\langle H, K\rangle$ ser L. Since $I = H^c + K^d \in \mathfrak{F}$ and $\langle H, K\rangle/I \in \mathfrak{F}$, we obtain that $\langle H, K\rangle \in \mathfrak{F}$. □

Theorem 6. *Over any field of characteristic zero the class* $\mathfrak{F}\cap \mathrm{E}\mathfrak{A}$ *is serially coalescent for locally finite Lie algebras.*

Proof. Let $L \in \mathrm{L}\mathfrak{F}$ and suppose that H, K ser L and $H, K \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$. Owing to Theorem 5 we get $\langle H, K\rangle$ ser L and $\langle H, K\rangle \in \mathfrak{F}$. By using Lemma 1 and Lemma 2(1), we get

$$H^\omega = \lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L \ \text{ and } \ K^\omega = \lambda_{\mathrm{L}\mathfrak{N}}(K) \lhd L.$$

Since $H, K \in \mathfrak{F}$, there are positive integers c, d such that $H^\omega = H^c, K^\omega = K^d$. Put $I = H^c + K^d$. Then $I \lhd L$. As in the proof of Theorem 5 we have

$$\langle H, K\rangle/I \in \mathfrak{F} \cap \mathfrak{N} \le \mathrm{E}\mathfrak{A}.$$

On the other hand, since $H^c, K^d \in \mathrm{E}\mathfrak{A}$ and $H^c, K^d \lhd L$, we have $I = H^c + K^d \in \mathrm{E}\mathfrak{A}$. Therefore $\langle H, K\rangle \in \mathrm{EE}\mathfrak{A} = \mathrm{E}\mathfrak{A}$. □

Remark. Hartley[1, Lemma 3.1.1] constructed a finite-dimensional Lie algebra L over a field of characteristic $p > 0$, in which there exist $x, y \in L$ such that $\langle x\rangle, \langle y\rangle$ si L, but $\langle x, y\rangle$ is not a subideal of L. This example indicates that the classes $\mathfrak{F} \cap \mathfrak{N}$, $\mathfrak{F}$, $\mathfrak{F} \cap \mathrm{E}\mathfrak{A}$ are not serially coalescent over any field of characteristic $p > 0$.

The following classes are also coalescent and ascendantly coalescent over any field of characteristic zero ([1, Theorem 3.2.5]):

$$\text{Min}, \ \text{Min-}\lhd^\sigma \ (\sigma \ge 2), \ \text{Min-si}, \ \text{Min-asc}, \ \text{Min-}\lhd \cap \text{Max-}\lhd\ .$$

Before we show that the classes above and Min-ser are serially coalescent for locally finite Lie algebras we need the following lemma, which is a direct consequence from [1, Theorem 8.1.4 and Corollary 10.2.2], [3, Corollary 1.6 and Theorem 1.7] and [4, Theorem].

Lemma 7. *Over any field of characteristic zero, we have*

$$\begin{aligned} &\mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd \ > \ \mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd^2 \ = \ \mathrm{L}\mathfrak{F} \cap \text{Min-si} \\ = \ &\mathrm{L}\mathfrak{F} \cap \text{Min-asc} \ = \ \mathrm{L}\mathfrak{F} \cap \text{Min-ser} \ > \ \mathrm{L}\mathfrak{F} \cap \text{Min} \ = \ \mathfrak{F}. \end{aligned}$$

Theorem 8. *Over any field of characteristic zero, we have*

(1) Min *is serially coalescent for locally finite Lie algebras,*

(2) Min-$\lhd^\sigma$ $(\sigma \ge 2)$*,* Min-si*,* Min-asc *and* Min-ser *are serially coalescent for locally finite Lie algebras,*

(3) Min-$\lhd \cap$ Max-$\lhd$ *is serially coalescent for locally finite Lie algebras.*

Proof. (1) Since $\mathrm{L}\mathfrak{F}\cap\mathrm{Min} = \mathfrak{F}$ by Lemma 7, Theorem 5 leads to the assertion.

(2) Let $\mathfrak{X}$ be one of the classes Min-$\lhd^{\sigma}$ ($\sigma \geq 2$), Min-si, Min-asc and Min-ser. Since $\mathrm{L}\mathfrak{F} \cap \mathfrak{X} = \mathrm{L}\mathfrak{F} \cap$ Min-si by Lemma 7, it is enough to show the assertion in the case of $\mathfrak{X} =$Min-si. Let $L \in \mathrm{L}\mathfrak{F}$ and suppose that H, K ser L and $H, K \in$Min-si. By Lemma 1 and Lemma 2(2), we get

$$H^{(\omega)} = \lambda_{\mathrm{LE}\mathfrak{A}}(H) \lhd L \text{ and } K^{(\omega)} = \lambda_{\mathrm{LE}\mathfrak{A}}(K) \lhd L.$$

Since $H, K \in$Min-si, there are positive integers c, d such that

$$H^{(\omega)} = H^{(c)},\ K^{(\omega)} = K^{(d)}.$$

Put $I = H^{(c)} + K^{(d)}$. Then $I \lhd L$ and $H^{(c)}, K^{(d)} \in$Min-si. Hence

$$I/H^{(c)} = (H^{(c)} + K^{(d)})/H^{(c)} \cong K^{(d)}/H^{(c)} \cap K^{(d)} \in \text{Min-si}.$$

Therefore $I \in \mathrm{E}$Min-si $=$ Min-si. By Lemma 3 $(H+I)/I, (K+I)/I$ ser L/I. As $(H+I)/I \cong H/H\cap I \in$ Min-si, it follows from [1, Corollary 8.5.5] that

$$(H+I)/I \in \mathrm{E}\mathfrak{A} \cap \text{Min-si} = \mathfrak{F} \cap \mathrm{E}\mathfrak{A}.$$

Similarly $(K+I)/I \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$. Therefore by using Theorem 6 we have

$$\langle (H+I)/I, (K+I)/I\rangle \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A} \text{ and } \langle (H+I)/I, (K+I)/I\rangle \text{ ser } L/I,$$

that is to say,

$$\langle H, K\rangle/I \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A} \text{ and } \langle H, K\rangle/I \text{ ser } L/I.$$

Thus $\langle H, K\rangle$ ser L and $\langle H, K\rangle \in \mathrm{E}$Min-si $=$ Min-si.

(3) Let $L \in \mathrm{L}\mathfrak{F}$ and suppose that H, K ser L and $H, K \in$ Min-$\lhd \cap$ Max-$\lhd$. By Lemma 1 and Lemma 2 (4), we get

$$H^{(\omega)} = \lambda_{\mathrm{LE}\mathfrak{A}}(H) \lhd L \text{ and } K^{(\omega)} = \lambda_{\mathrm{LE}\mathfrak{A}}(K) \lhd L.$$

Since $H, K \in$ Min-$\lhd$, there are positive integers c, d such that

$$H^{(\omega)} = H^{(c)},\ K^{(\omega)} = K^{(d)}.$$

Put $I = H^{(c)} + K^{(d)}$ and $J = \langle H, K\rangle$. Then

$$I \lhd L,\ I \leq J \text{ and } (H+I)/I, (K+I)/I \text{ ser } L/I.$$

Moreover we have

$$(H+I)/I \cong H/H\cap I \in \mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd \cap \mathrm{E}\mathfrak{A} \leq \mathrm{L}\mathfrak{F} \cap \mathrm{E}\mathfrak{A} \cap \mathfrak{G} \leq \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$$

by using [1, Lemma 8.6.1]. Similarly $(K+I)/I \in \mathfrak{F}\cap\mathrm{E}\mathfrak{A}$. Then by Theorem 6 we have

$$\langle (H+I)/I, (K+I)/I\rangle \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A} \text{ and } \langle (H+I)/I, (K+I)/I\rangle \text{ ser } L/I,$$

that is to say,

$$J/I \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A} \text{ and } J/I \text{ ser } L/I.$$

Therefore we get J ser L. On the other hand, since $H, K \in \text{Min-}\lhd \cap \text{Max-}\lhd$,

$$\{S : S \lhd J,\ S \leq H^{(c)}\} \text{ and } \{S : S \lhd J,\ S \leq K^{(d)}\}$$

satisfy both the minimal condition and the maximal condition. Since

$$\{T : T \lhd J,\ T \leq I\}$$

satisfies both the minimal condition and the maximal condition by virtue of [7, Lemma 2.2.7]. Thus we have that $\{T \cap I : T \lhd J\}$ and $\{(T+I)/I : T \lhd J\}$ satisfy both the minimal condition and the maximal condition by noting $J/I \in \mathfrak{F}$. Therefore we conclude that $J \in \text{Min-}\lhd \cap \text{Max-}\lhd$ by using [1, Theorem 1.7.3]. □

4. Locally serially coalescent classes

Let $\mathfrak{X}$ be a class of Lie algebras. We say that $\mathfrak{X}$ is *locally serially coalescent for locally finite Lie algebras* if for any two serial $\mathfrak{X}$-subalgebras H, K of a locally finite Lie algebra L and for any finitely generated subalgebra Y of $J = \langle H, K\rangle$ there exists a serial $\mathfrak{X}$-subalgebra X of L such that $Y \leq X \leq J$.

First we obtain the following lemma as [8, Theorem 3.1] and [9, Theorem 4.4].

Lemma 9. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes of Lie algebras such that*

$$\mathrm{L}\mathfrak{F} \cap \mathfrak{X} \leq \mathrm{L}\mathfrak{F} \cap \mathfrak{Y} \leq \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X}.$$

Then the following statements are equivalent.

(1) *$\mathfrak{Y}$ is locally serially coalescent for locally finite Lie algebras.*

(2) *For any finite number of serial $\mathfrak{X}$-subalgebras $H_1, \ldots, H_n$ of a locally finite Lie algebra L and for any finitely generated subalgebra Y of $J = \langle H_1, \ldots, H_n\rangle$ there exists a serial $\mathfrak{Y}$-subalgebra X of L such that $Y \leq X \leq J$.*

Proof. (1) ⇒ (2) : We show the implication by using induction on n. When $n = 1$ the assertion is trivial. Let $n > 1$ and suppose that the assertion is true for $n - 1$. Since $Y \in \mathfrak{G}$ there exist finitely generated subalgebras Y_i of H_i $(1 \leq i \leq n)$ such that $Y \leq \langle Y_1, \ldots, Y_n\rangle$. Here $\langle Y_1, \ldots, Y_{n-1}\rangle$ is a finitely generated subalgebra of $\langle H_1, \ldots, H_{n-1}\rangle$. By inductive hypothesis there exists a serial $\mathfrak{Y}$-subalgebra X_{n-1} of L such that

$$\langle Y_1, \ldots, Y_{n-1}\rangle \leq X_{n-1} \leq \langle H_1, \ldots, H_{n-1}\rangle.$$

Hence $\langle Y_1, \ldots, Y_{n-1}, Y_n\rangle$ is a finitely generated subalgebra of $\langle X_{n-1}, H_n\rangle$. As X_{n-1} and H_n are serial $\mathfrak{Y}$-subalgebras of L, it follows from (1) that there exists a serial $\mathfrak{Y}$-subalgebra X_n of L such that

$$\langle Y_1, \ldots, Y_n\rangle \leq X_n \leq \langle X_{n-1}, H_n\rangle.$$

That is to say, there exists a serial $\mathfrak{Y}$-subalgebra X_n of L such that $Y \leq X_n \leq J$. Hence the assertion is true for n.

(2) $\Rightarrow$ (1) : Let $L \in \mathrm{L}\mathfrak{F}$ and let H, K be serial $\mathfrak{Y}$-subalgebras of L. Suppose that Y is any finitely generated subalgebra of $\langle H, K\rangle$. Since $H, K \in \mathrm{L}\mathfrak{F} \cap \mathfrak{Y} \leq \mathrm{J}(\mathrm{ser})\mathfrak{X}$, we have

$$H = \langle H_\alpha : \alpha \in A\rangle,\ K = \langle K_\beta : \beta \in B\rangle,$$

where H_α (resp. K_β) is a serial $\mathfrak{X}$-subalgebra of H (resp. K) for any $\alpha \in A$ (resp. $\beta \in B$). As Y is a finitely generated subalgebra of

$$\langle H, K\rangle = \langle H_\alpha, K_\beta : \alpha \in A, \beta \in B\rangle,$$

there exist $\alpha_i \in A$ $(1 \leq i \leq m)$ and $\beta_j \in B$ $(1 \leq j \leq n)$ such that

$$Y \leq \langle H_{\alpha_1}, \dots, H_{\alpha_m}, K_{\beta_1}, \dots, K_{\beta_n}\rangle.$$

Since H_{α_i} $(1 \leq i \leq m)$ and K_{β_j} $(1 \leq j \leq n)$ are serial $\mathfrak{X}$-subalgebras of L, it follows from (2) that there exists a serial $\mathfrak{Y}$-subalgebra X of L such that

$$Y \leq X \leq \langle H_{\alpha_1}, \dots, H_{\alpha_m}, K_{\beta_1}, \dots, K_{\beta_n}\rangle \leq \langle H, K\rangle.$$

Therefore we get the assertion (1). □

If we put $\mathfrak{X} = \mathfrak{Y}$ in Lemma 9, then we obtain the following

Corollary 10. *The following statements are equivalent.*

(1) *$\mathfrak{X}$ is locally serially coalescent for locally finite Lie algebras.*

(2) *For any finite number of serial $\mathfrak{X}$-subalgebras $H_1, \dots, H_n$ of a locally finite Lie algebra L and for any finitely generated subalgebra Y of $J = \langle H_1, \dots, H_n\rangle$ there exists a serial $\mathfrak{X}$-subalgebra X of L such that $Y \leq X \leq J$.*

The following theorem corresponds to [9, Proposition 4.1].

Theorem 11. *Let $\mathfrak{X}$ be a class of Lie algebras. If $\mathfrak{X}$ is locally serially coalescent for locally finite Lie algebras, then $\mathrm{L}\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{X} = \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X}$.*

Proof. It is clear that $\mathrm{L}\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{X} \leq \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X}$. Let $L \in \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X}$. Then we have

$$L = \langle H_\alpha : \alpha \in A\rangle$$

where H_α is a serial $\mathfrak{X}$-subalgebra of L for any $\alpha \in A$. Now, for any finitely generated subalgebra Y of L, there are $\alpha_i \in A$ $(1 \leq i \leq n)$ such that $Y \leq \langle H_{\alpha_1}, \dots, H_{\alpha_n}\rangle$. Therefore Corollary 10 indicates that there exists a serial $\mathfrak{X}$-subalgebra X of L such that $Y \leq X \leq \langle H_{\alpha_1}, \dots, H_{\alpha_n}\rangle$. Thus we get $L \in \mathrm{L}(\mathrm{ser})\mathfrak{X}$ and conclude that $\mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X} \leq \mathrm{L}\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{X}$. □

For a class $\mathfrak{X}$ of Lie algebras it is clear that if $\mathfrak{X}$ is serially coalescent for locally finite Lie algebras, then $\mathfrak{X}$ is locally serially coalescent for locally finite Lie algebras. Therefore Theorems 4, 5, 6 and 8 assert that over any field of characteristic zero the following classes are all locally serially coalescent for locally finite Lie algebras:

$$\mathfrak{F}\cap\mathfrak{N},\quad \mathfrak{F},\quad \mathfrak{F}\cap \mathrm{E}\mathfrak{A},\quad \mathrm{Min},\quad \mathrm{Min}\text{-}\lhd^{\sigma}\ (\sigma\geq 2),$$
$$\mathrm{Min}\text{-si},\quad \mathrm{Min}\text{-asc},\quad \mathrm{Min}\text{-ser},\quad \mathrm{Min}\text{-}\lhd\cap\mathrm{Max}\text{-}\lhd.$$

Thus Theorem 11 leads to the following corollary which contains some results in [6, Theorem 2].

Corollary 12. *Over any field of characteristic zero we have*

$$\mathrm{L}(\mathrm{ser})(\mathfrak{F}\cap\mathfrak{N})=\mathrm{L}\mathfrak{F}\cap\mathrm{J}(\mathrm{ser})(\mathfrak{F}\cap\mathfrak{N}),\quad \mathrm{L}(\mathrm{ser})\mathfrak{F}=\mathrm{L}\mathfrak{F}\cap\mathrm{J}(\mathrm{ser})\mathfrak{F},$$
$$\mathrm{L}(\mathrm{ser})(\mathfrak{F}\cap\mathrm{E}\mathfrak{A})=\mathrm{L}\mathfrak{F}\cap\mathrm{J}(\mathrm{ser})(\mathfrak{F}\cap\mathrm{E}\mathfrak{A}).$$

Furthermore if $\mathfrak{X}$ *is any of the classes* Min, Min-$\lhd^{\sigma}$ ($\sigma\geq 2$), Min-si, Min-asc, Min-ser *and* Min-$\lhd\cap$ Max-$\lhd$*, then*

$$\mathrm{L}\mathfrak{F}\cap\mathrm{L}(\mathrm{ser})\mathfrak{X}=\mathrm{L}\mathfrak{F}\cap\mathrm{J}(\mathrm{ser})\mathfrak{X}.$$

The following theorem corresponds to [8, Theorem 3.2] and [9, Theorem 4.2].

Theorem 13. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes of Lie algebras such that*

$$\mathrm{L}\mathfrak{F}\cap\mathfrak{X}\leq\mathrm{L}\mathfrak{F}\cap\mathfrak{Y}\leq\mathrm{L}\mathfrak{F}\cap\mathrm{L}(\mathrm{ser})\mathfrak{X}.$$

Then the following statements are equivalent.

(1) $\mathfrak{X}$ *is locally serially coalescent for locally finite Lie algebras.*
(2) $\mathfrak{Y}$ *is locally serially coalescent for locally finite Lie algebras.*

Proof. (1) $\Rightarrow$ (2) : Suppose that $\mathfrak{X}$ is locally serially coalescent for locally finite Lie algebras. Let $L\in\mathrm{L}\mathfrak{F}$ and let H,K be serial $\mathfrak{Y}$-subalgebras of L. Suppose that Y is any finitely generated subalgebra of $J=\langle H,K\rangle$. Then there exist finitely generated subalgebras A of H and B of K such that $Y\leq\langle A,B\rangle$. Since

$$A\leq H\in\mathrm{L}\mathfrak{F}\cap\mathfrak{Y}\leq\mathrm{L}(\mathrm{ser})\mathfrak{X},$$

there is a serial $\mathfrak{X}$-subalgebra M of H such that $A\leq M$. Then M is a serial $\mathfrak{X}$-subalgebra of L. Similarly there is a serial $\mathfrak{X}$-subalgebra N of L such that $B\leq N\leq K$. Since $Y\leq\langle A,B\rangle\leq\langle M,N\rangle$ it follows from the hypothesis that there exists a serial $\mathfrak{X}$-subalgebra X of L such that $Y\leq X\leq\langle M,N\rangle$. Here $X\in\mathrm{L}\mathfrak{F}\cap\mathfrak{X}\leq\mathfrak{Y}$ and $Y\leq X\leq J$. Thus $\mathfrak{Y}$ is locally serially coalescent for locally finite Lie algebras.

(2) $\Rightarrow$ (1) : Suppose that $\mathfrak{Y}$ is locally serially coalescent for locally finite Lie algebras. Let $L\in\mathrm{L}\mathfrak{F}$ and let H,K be serial $\mathfrak{X}$-subalgebras of L. Suppose that Y is any finitely generated subalgebra of $J=\langle H,K\rangle$. Since $H,K\in\mathrm{L}\mathfrak{F}\cap\mathfrak{X}\leq\mathfrak{Y}$ it follows from the hypothesis that there exists a serial $\mathfrak{Y}$-subalgebra X of L such that $Y\leq X\leq J$. Here $X\in\mathrm{L}\mathfrak{F}\cap\mathfrak{Y}\leq\mathrm{L}(\mathrm{ser})\mathfrak{X}$, so there is a serial $\mathfrak{X}$-subalgebra X_1 of X such that $Y\leq X_1\leq X$. Therefore

X_1 is a serial $\mathfrak{X}$-subalgebra of L and $Y \leq X_1 \leq J$. Thus $\mathfrak{X}$ is locally serially coalescent for locally finite Lie algebras. □

5. Radicals

For a class $\mathfrak{X}$ of Lie algebras and a Lie algebra L we denote by $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L)$ the subalgebra generated by all the serial $\mathfrak{X}$-subalgebras of L, and call it the $\mathfrak{X}$-ser radical of L. If $\mathfrak{X}$ is one of the classes $\mathfrak{F} \cap \mathfrak{N}, \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$ and $\mathfrak{F}$, then we have the following proposition about the $\mathfrak{X}$-ser radical of a locally finite Lie algebra L.

Proposition 14. *Over any field of characteristic zero, for any locally finite Lie algebra L we have*
(1) $\mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L) \in \mathrm{L}(\mathrm{ser})(\mathfrak{F} \cap \mathfrak{N}) = \mathrm{L}\mathfrak{N}$,
(2) $\mathrm{R}_{\mathfrak{F}\cap\mathrm{E}\mathfrak{A}\text{-ser}}(L) \in \mathrm{L}(\mathrm{ser})(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$,
(3) $\mathrm{R}_{\mathfrak{F}\text{-ser}}(L) \in \mathrm{L}(\mathrm{ser})\mathfrak{F}$.

Proof. Let $\mathfrak{X}$ be one of the classes $\mathfrak{F} \cap \mathfrak{N}$, $\mathfrak{F} \cap \mathrm{E}\mathfrak{A}$ and $\mathfrak{F}$. By virtue of Theorems 4, 5 and 6 $\mathfrak{X}$ is serially coalescent for locally finite Lie algebras. Let $L \in \mathrm{L}\mathfrak{F}$. For any finitely generated subalgebra Y of $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L)$ there exist a finite number of serial $\mathfrak{X}$-subalgebras H_i $(1 \leq i \leq n)$ of L such that $Y \leq \langle H_1, H_2, \ldots, H_n \rangle$. Then we can show that $\langle H_1, H_2, \ldots, H_n \rangle$ is a serial $\mathfrak{X}$-subalgebra of L by using induction on n. Therefore $\langle H_1, H_2, \ldots, H_n \rangle$ is a serial $\mathfrak{X}$-subalgebra of $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L)$ containing Y. Thus $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L) \in \mathrm{L}(\mathrm{ser})\mathfrak{X}$. We also have $\mathrm{L}(\mathrm{ser})(\mathfrak{F} \cap \mathfrak{N}) = \mathrm{L}\mathfrak{N}$ by [6, Theorem 4]. □

As a corollary of Proposition 14, we get the following result which corresponds to [1, Theorem 6.2.1].

Corollary 15. *Over any field of characteristic zero, for any locally finite Lie algebra L we have*

$$\begin{aligned}\mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L) &= \{x \in L : \langle x \rangle \text{ ser } L\} = \cup\{H : \mathfrak{F} \cap \mathfrak{N} \ni H \text{ ser } L\} \\ &= \cup\{H : \mathfrak{N} \ni H \text{ ser } L\} = \cup\{H : \mathrm{L}\mathfrak{N} \ni H \text{ ser } L\}.\end{aligned}$$

Proof. It is trivial that

$$\begin{aligned}&\{x \in L : \langle x \rangle \text{ ser } L\} \subset \cup\{H : \mathfrak{F} \cap \mathfrak{N} \ni H \text{ ser } L\} \\ &\subset \cup\{H : \mathfrak{N} \ni H \text{ ser } L\} \subset \cup\{H : \mathrm{L}\mathfrak{N} \ni H \text{ ser } L\}.\end{aligned}$$

Let H be a serial $\mathrm{L}\mathfrak{N}$-subalgebra of L and let x be any element of H. As any subalgebra is serial in a locally nilpotent Lie algebra, $\langle x \rangle$ ser H, therefore $\langle x \rangle$ ser L. It follows that $H \subset \{x \in L : \langle x \rangle \text{ ser } L\}$. Thus we get

$$\begin{aligned}&\{x \in L : \langle x \rangle \text{ ser } L\} = \cup\{H : \mathfrak{F} \cap \mathfrak{N} \ni H \text{ ser } L\} \\ &= \cup\{H : \mathfrak{N} \ni H \text{ ser } L\} = \cup\{H : \mathrm{L}\mathfrak{N} \ni H \text{ ser } L\}.\end{aligned}$$

Also it is clear that $\cup\{H : \mathfrak{F} \cap \mathfrak{N} \ni H \text{ ser } L\} \subset \mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L)$. Let $x \in \mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L)$. As in the proof of Proposition 14 there exists a serial $\mathfrak{F} \cap \mathfrak{N}$-subalgabra K of L such that $x \in K$. Thus we have $\mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L) \subset \cup\{H : \mathfrak{F} \cap \mathfrak{N} \ni H \text{ ser } L\}$. $\square$

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Masanobu HONDA
Faculty of Pharmaceutidal Sciences
Niigata University of Pharmacy and Applied Life Sciences
Niigata 956-8603 Japan
e-mail address: honda@nupals.ac.jp

Takanori SAKAMOTO
Department of Mathematics
Fukuoka University of Education
Munakata, Fukuoka 811-4192 Japan
e-mail address: sakamott@fukuoka-edu.ac.jp

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