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BELYI FUNCTION ON $X_0(49)$ OF DEGREE 7

Appendix to: "The Belyi functions and dessin d'enfants corresponding to the non-normal inclusions of triangle groups" by K.Hoshino

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In [H] §4, we identified two types of the non-normal inclusions of triangle groups (type A and type C from Singerman's list [S]) as those corresponding to subcovers of the Klein quartic. N. D. Elkies [E] closely studied those subcovers in view of modular curves, i.e., as subcovers under the elliptic modular curve $X_7 = X(7)$ identified with the Klein quartic defined by $X^3Y + Y^3Z + Z^3Y = 0$. Let $\mathbb{C}(X_7)$ be the function field of X_7 generated by $y := Y/X$ and $z := Z/X$ with a relation $y + y^3z + z^3 = 0$, and consider two automorphisms of $\mathbb{C}(X_7)$ defined by

$$\text{(A1)} \qquad \sigma : \begin{cases} y \mapsto \zeta^3 y, \\ z \mapsto \zeta z, \end{cases} \qquad \tau : \begin{cases} y \mapsto 1/z, \\ z \mapsto y/z, \end{cases}$$

where $\zeta := e^{2\pi i/7}$. Then, σ and τ are automorphisms respectively of order 7 and 3, and they form an automorphism group H of $\mathbb{C}(X_7)$ of order 21, a subgroup of the full automorphism group G of order 168 of the Klein quartic. In this respect, the genus zero covers of type A and type C of [S] are respectively $X_1(7) \to X(1)$ and $X_0(7) \to X(1)$ arising from the inclusion relations of $\langle\sigma\rangle \subset H \subset G$. Remarkably, Elkies [E] studied deep arithmetic properties of a genus one subcover E fixed by $\langle\tau\rangle$ with showing that E is $\mathbb{Q}$-isomorphic to $X_0(49)$. Especially, he explicitly presented its function field $\mathbb{C}(E)$ as $\mathbb{C}(E) = \mathbb{C}(u, v)$ with $v^2 = 4u^3 + 21u^2 + 28u$, where

$$\text{(A2)} \qquad u = -\frac{(y+z+yz)^2}{(1+y+z)yz}, \quad v = -\frac{(2-y-z+2y^2-yz+2z^2)(y+z+yz)}{yz(1+y+z)}$$

Recalling also from [E] that standard coordinates of $X_1(7) \cong \mathbf{P}^1_t$ and $X_0(7) \cong \mathbf{P}^1_s$ may be given as

$$t := -y^2 z, \quad s := t + \frac{1}{1-t} + \frac{t-1}{t},$$

we would like to interpret the degree 7 cover $E \to X_0(7)$ arising from $\langle\tau\rangle \subset H$ by expressing s by u, v explicitly. In this note, we show

Proposition A. *Notations being as above, the covering of $\mathbf{P}^1_s$ by the elliptic curve $E : v^2 = 4u^3+21u^2+28u$ is ramified only above $s = 3\rho, 3\rho^{-1}, \infty$ (where $\rho = e^{2\pi i/6}$), and the equation is given by*

$$s = \frac{1}{2}\big\{(u^2 + 7u + 7)v + (7u^3 + 35u^2 + 49u + 16)\big\}.$$

Thus $\beta = \frac{s-3\rho}{3\rho^{-1}-3\rho}$ *gives a Belyi function (i.e., unramified outside* $\beta = 0, 1, \infty$*) of degree 7 on* E *with valency list* $[331, 331, 7]$.

In effect, one can induce an isomorphism of covers

$$\begin{array}{ccc} X_0(49) & \xrightarrow{\sim} & E \\ \downarrow & & \downarrow \\ X_0(7) & \xrightarrow{\sim} & X_0(7) \end{array}$$

from the conjugacy by the 'Fricke involution' $\pm\frac{1}{\sqrt{7}}\left(\begin{smallmatrix}0 & -1\\ 7 & 0\end{smallmatrix}\right)$ between the modular group $\Gamma_0(49)$ and $\{\pm\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z}) : b \equiv c \equiv 0 \bmod 7\}$ in $\mathrm{PSL}_2(\mathbb{R})$ (cf. [E] p.90). Therefore, the computation of $E \to X_0(7)$ may be reduced to combining classically well known equations that relate $X_0(7)$, $X_0(49)$ with the J-line $X(1)$ found in, e.g., [F] pp.395–403. Here, however, we shall employ an alternative enjoyable discussion following the Elkies scheme:

$$\begin{array}{ccc} X_1(7) = \mathbf{P}^1_t & \xleftarrow[\langle\sigma\rangle]{f} & X_7 \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle q}\,\langle\tau\rangle \\ X_0(7) = \mathbf{P}^1_s & \xleftarrow{p} & E. \end{array}$$

First of all, since $\frac{ds}{dt} = \frac{(t^2-t+1)^2}{(t-1)^2t^2}$, the 3-cyclic cover g is ramified only over $s = 3\rho, 3\rho^{-1}$ at $t = \rho, \rho^{-1}$, and the fiber over $s = \infty$ is formed by the three points $t = 0, 1, \infty$. We next chase the fibers of f over these points and their images in E by q. Noticing that $\mathbb{C}(X_7) = \mathbb{C}(y, z) = \mathbb{C}(y, t)$ with $y^7 = \frac{t^3}{1-t}$, we see that f is totally ramified over $t = 0, 1, \infty$, and their images by q coincide at the infinity point on $E : v^2 = 4u^3 + 21u^2 + 28u$. From this follows that $p : E \to \mathbf{P}^1_s$ is totally ramified at the infinity point of E over $s = \infty$, hence s is of the form $s = F(u) + G(u)v$ with $F, G \in \mathbb{C}[u]$, $\deg(F) = 3$, $\deg(G) = 2$.

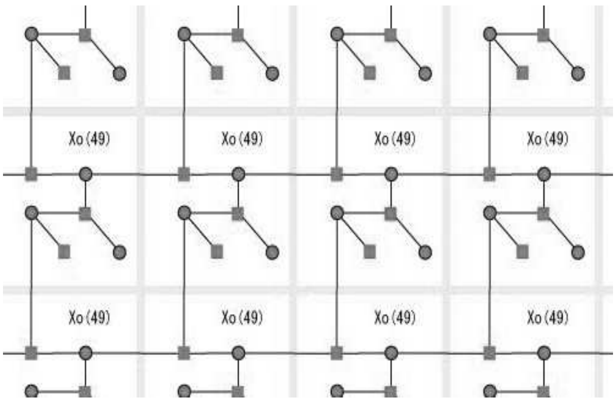
The fiber of f over $t = \rho$ forms one orbit under the action of σ ($y \mapsto \zeta y$) whose points are represented by the set of their y-coordinates $S_\rho := \{\xi^2, \xi^5, \xi^8, \xi^{11}, \xi^{14}, \xi^{17}, \xi^{20}\}$, where $\xi := e^{2\pi i/21}$. The action of τ preserves $t = \rho$ and transforms those y-coordinates (over $t = \rho$) as $y \mapsto y^2\xi^{-14}$, hence decomposes S_ρ into the three τ-orbits $S^1_\rho := \{\xi^2, \xi^{11}, \xi^8\}$, $S^2_\rho := \{\xi^5, \xi^{17}, \xi^{20}\}$, and $S^3_\rho := \{\xi^{14} = \rho^4\}$ (This also explains the branch type of p over $s = 3\rho$ is '331'). Then we compute the u-coordinates u_1, u_2, u_3 of the images of these orbits by q after the formula (A2); it turns out that $u_1 = -\xi^2 - \xi^{11} - \xi^8 - 1$, $u_2 = -\xi^7 + \xi^2 + \xi^{11} + \xi^8 - 2$ and $u_3 = 3\rho - 1$. Eliminating v from $F(u) +$

$G(u)v = 3\rho$, we should then have an equation of the form

$$(4u^3 + 21u^2 + 28u)G(u)^2 - (F(u) - 3\rho)^2 = (u - u_1)^3(u - u_2)^3(u - u_3).$$

Using a symbolic computation software such as MAPLE to compare the coefficients of the both sides above (plus a slight more consideration of signs), we conclude $F(u) = \frac{1}{2}(7u^3 + 35u^2 + 49u + 16)$, $G(u) = \frac{1}{2}(u^2 + 7u + 7)$ as stated in Proposition A. $\square$

Remark. The monodromy representation associated with the above Belyi function β is given by $x = (142)(356)(7)$, $y = (175)(346)(2)$, $z = (1234567)$. The following picture illustrates the uniformization of this Grothendieck dessin.



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