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A NOTE ON CERTAIN METRICS ON R^4_+

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This work is a continuation of the papers [1], [2], [3] and [4], in which we studied the metrics on $R^4_+ = R^3 \times R_+$ with the canonical coordinates $\{x_1, x_2, x_3, x_4\}$ as

$$(1.1) \quad ds^2 = \frac{1}{x_4x_4}\sum_{b,c=1}^{3}\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right)dx_bdx_c - \frac{1}{x_4x_4(1+ax_4x_4)}dx_4dx_4$$

and

$$(1.2) \quad \begin{aligned} ds^2 =& \frac{1}{x_4x_4}\sum_{b,c=1}^{3}\left(\frac{8}{(x_3+3r)^2}(r^2\delta_{bc} - x_bx_c) + \frac{x_bx_c}{r^2(1+ar^2)}\right)dx_bdx_c \\ & - \frac{1}{x_4x_4(1+ax_4x_4)}dx_4dx_4, \end{aligned}$$

where $r^2 = \sum_{b=1}^{3} x_bx_b$ and a is a constant.

They are derived as special ones from the metric on R^4_+:

$$ds^2 = \frac{1}{u_4u_4}\sum_{i,j=1}^{4} F_{ij}du_idu_j,\ F_{ij} = F_{ji},$$

where $u_1 = r, u_2 = \theta, u_3 = \phi, u_4 = x_4$ and

$$x_1 = r\sin\theta\cos\phi,\ \ x_2 = r\sin\theta\sin\phi,\ \ x_3 = r\cos\theta,$$

and (r, θ, ϕ) are the polar coordinates of R^3, which satisfies the Einstein condition and

$$F_{ij} = F_{ij}(u_1, u_2) \quad \text{except for} \quad F_{44} = F_{44}(u_1, u_2, u_4)$$

and

$$F_{12} = F_{\alpha\lambda} = 0 \quad (\alpha = 1, 2; \lambda = 3, 4).$$

The metric (1.1) is the one such that

$$\frac{\partial F_{11}}{\partial u_2} = \frac{\partial F_{22}}{\partial u_2} = 0 \ \text{ and } \ F_{33} = \psi(u_1)\sin 2u_2,$$

and, as was proved in [4], any geodesic of this metric is a plane curve in R^3. And the metric (1.2) is the one essentially depending on the longitude ϕ and any geodesic of this metric is not plane in R^3 in general. For a geodesic $(x_i(t))$ of the metric (1.2), we have by (1.19) and (1.20) in [4]

$$(1.3) \qquad \frac{d^2x_b}{dt^2} + B\frac{dx_b}{dt} + Ax_b + C\delta_{3b} = 0$$

and

$$\text{(1.4)} \qquad \frac{d^2x_4}{dt^2} - \frac{1+ax_4x_4}{x_4}\left\{\frac{1}{1+ar^2}\left(\frac{dr}{dt}\right)^2 + \frac{8r^2}{(x_3+3r)^2}\left(\sum_b \frac{dx_b}{dt}\frac{dx_b}{dt} - \left(\frac{dr}{dt}\right)^2\right)\right\} + \left(-\frac{2}{x_4} + \frac{1}{x_4(1+ax_4x_4)}\right)\left(\frac{dx_4}{dt}\right)^2 = 0,$$

where

$$B := -2\frac{d}{dt}\log\frac{(x_3+3r)x_4}{r}, \quad C := \frac{1}{x_3+3r}\left(\sum_b \frac{dx_b}{dt}\frac{dx_b}{dt} - \frac{dr}{dt}\frac{dr}{dt}\right)$$

and

$$\begin{aligned} A :&= \left\{-\frac{3}{r^2} + \frac{1}{r^2(1+ar^2)} + \frac{8(1+ar^2)}{(x_3+3r)^2} + \frac{3}{r(x_3+3r)}\right\}\frac{dr}{dt}\frac{dr}{dt} \\ &\quad + \left\{\frac{3}{r(x_3+3r)} - \frac{2(1+ar^2)}{(x_3+3r)^2}\right\}\sum_b \frac{dx_b}{dt}\frac{dx_b}{dt} + \frac{2}{r(x_3+3r)}\frac{dr}{dt}\frac{dx_3}{dt} \\ &= \left\{-\frac{3}{r^2} + \frac{1}{r^2(1+ar^2)} + \frac{6(1+ar^2)}{(x_3+3r)^2} + \frac{6}{r(x_3+3r)}\right\}\frac{dr}{dt}\frac{dr}{dt} \\ &\quad + \left\{\frac{3}{r} - \frac{2(1+ar^2)}{x_3+3r}\right\}C + \frac{2}{r(x_3+3r)}\frac{dr}{dt}\frac{dx_3}{dt}, \end{aligned}$$

from which we obtain

$$\frac{A}{C} = \frac{3}{r} - \frac{2(1+ar^2)}{x_3+3r} + \frac{\Phi\frac{dr}{dt}\frac{dr}{dt} + \frac{2}{r}\frac{dr}{dt}\frac{dx_3}{dt}}{\sum_b \frac{dx_b}{dt}\frac{dx_b}{dt} - \frac{dr}{dt}\frac{dr}{dt}},$$

where

$$\Phi = -\frac{2+3ar^2}{r^2(1+ar^2)}(x_3+3r) + \frac{6(1+ar^2)}{x_3+3r} + \frac{6}{r}.$$

The curve $(x_b(t))$ in R^3 for the metric (1.2) is plane if and only if

$$\frac{d}{dt}\left(\frac{A}{C}\right)\left(x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}\right) = 0$$

by Proposition 2 in [4].

In this work we shall try to express these metrics in more concrete forms. When $a > 0$, we put $a = 1/\alpha^2$ $(\alpha > 0)$. Then we have

$$\frac{dx_4dx_4}{x_4x_4(1+ax_4x_4)} = \frac{\alpha^2 dx_4dx_4}{x_4x_4(\alpha^2+x_4x_4)} \qquad (x_4 > 0)$$

and

$$\frac{\alpha dx_4}{x_4\sqrt{\alpha^2+x_4x_4}} = -\frac{1}{2}d\log\frac{\sqrt{\alpha^2+x_4x_4}+\alpha}{\sqrt{\alpha^2+x_4x_4}-\alpha}.$$

Putting as

$$\log \frac{\sqrt{\alpha^2 + x_4x_4} + \alpha}{x_4} = \zeta, \ \text{ i.e., } \ \frac{\sqrt{\alpha^2 + x_4x_4} + \alpha}{x_4} = e^{\zeta},$$

we have

$$e^{-\zeta} = \frac{x_4}{\sqrt{\alpha^2 + x_4x_4} + \alpha} = \frac{\sqrt{\alpha^2 + x_4x_4} - \alpha}{x_4},$$

from which we obtain

$$\frac{e^{\zeta} - e^{-\zeta}}{2} = \frac{\alpha}{x_4}, \qquad \frac{e^{\zeta} + e^{-\zeta}}{2} = \frac{\sqrt{\alpha^2 + x_4x_4}}{x_4},$$

hence

$$x_4 = \frac{\alpha}{\sinh\zeta}, \qquad \frac{x_4}{\sqrt{\alpha^2 + x_4x_4}} = \frac{1}{\cosh\zeta}.$$

Using these expressions, (1.1) and (1.2) are expressed as

$$(1.1') \qquad ds^2 = \frac{1}{\alpha^2}\sinh^2\zeta \sum_{b,c=1}^{3}\left(\delta_{bc} - \frac{x_bx_c}{\alpha^2 + r^2}\right)dx_bdx_c - d\zeta d\zeta \quad (\zeta > 0)$$

and

$$(1.2') \qquad ds^2 = \frac{1}{\alpha^2}\sinh^2\zeta \sum_{b,c=1}^{3}\Bigg(\frac{8}{(x_3 + 3r)^2}(r^2\delta_{bc} - x_bx_c) + \frac{\alpha^2 x_bx_c}{r^2(\alpha^2 + r^2)}\Bigg)dx_bdx_c - d\zeta d\zeta.$$

And (1.4) is expressed as

$$(1.4') \qquad \frac{d^2\zeta}{dt^2} + \frac{\sinh 2\zeta}{2(\alpha^2 + r^2)}\left(\frac{dr}{dt}\right)^2 + \frac{4r^2\sinh 2\zeta}{\alpha^2(x_3 + 3r)}C = 0.$$

Next, when $a < 0$ we put $a = -1/\alpha^2 (\alpha > 0)$. Then we have

$$\frac{dx_4dx_4}{x_4x_4(1 + ax_4x_4)} = \frac{\alpha^2 dx_4dx_4}{x_4x_4(\alpha^2 - x_4x_4)}.$$

For $0 < x_4 < \alpha$, we have

$$\frac{\alpha^2 dx_4dx_4}{x_4x_4(\alpha^2 - x_4x_4)} = \left(d\log\frac{\sqrt{\alpha^2 - x_4x_4} + \alpha}{x_4}\right)^2.$$

Putting

$$\log \frac{\sqrt{\alpha^2 - x_4x_4} + \alpha}{x_4} = \zeta, \ \text{ i.e., } \ \frac{\sqrt{\alpha^2 - x_4x_4} + \alpha}{x_4} = e^{\zeta},$$

we have

$$e^{-\zeta} = \frac{\alpha - \sqrt{\alpha^2 - x_4x_4}}{x_4},$$

from which we obtain

$$\frac{e^\zeta + e^{-\zeta}}{2} = \frac{\alpha}{x_4}, \quad \frac{e^\zeta - e^{-\zeta}}{2} = \frac{\sqrt{\alpha^2 - x_4x_4}}{x_4},$$

hence

$$x_4 = \frac{\alpha}{\cosh\zeta}, \quad \frac{x_4}{\sqrt{\alpha^2 - x_4x_4}} = \frac{1}{\sinh\zeta}.$$

Therefore (1.1) and (1.2) can be written in this case as

$$(1.1'') \qquad ds^2 = \frac{\cosh^2\zeta}{\alpha^2}\sum_{b,c=1}^{3}(\delta_{bc} + \frac{x_bx_c}{\alpha^2 - r^2})dx_bdx_c - d\zeta d\zeta,$$

and

$$(1.2'') \quad ds^2 = \frac{\cosh^2\zeta}{\alpha^2}\sum_{b,c=1}^{3}\left(\frac{8}{(x_3+3r)^2}(r^2\delta_{bc} - x_bx_c) + \frac{\alpha^2x_bx_c}{r^2(\alpha^2-r^2)}\right)dx_bdx_c - d\zeta d\zeta$$

where $a = -1/\alpha^2$ and $\zeta > 0$. For a geodesic $(x_i(t))$ of the metric (1.2″), the equation (1.4) becomes as follows:

$$\frac{d^2x_4}{dt^2} - \frac{\alpha^2 - x_4x_4}{\alpha^2x_4}\left\{\frac{\alpha^2}{\alpha^2 - r^2}\left(\frac{dr}{dt}\right)^2 + \frac{8r^2}{x_3+3r}C\right\} - \frac{\alpha^2 - 2x_4x_4}{x_4(\alpha^2 - x_4x_4)}\left(\frac{dx_4}{dt}\right)^2 = 0.$$

Using the relation $x_4 = \alpha/\cosh\zeta$, we obtain

$$\frac{dx_4}{dt} = -\frac{\alpha\sinh\zeta}{\cosh^2\zeta}\frac{d\zeta}{dt},$$

$$\frac{d^2x_4}{dt^2} = -\frac{\alpha\sinh\zeta}{\cosh^2\zeta}\frac{d^2\zeta}{dt^2} - \frac{1-\sinh^2\zeta}{\cosh^3\zeta}\frac{d\zeta}{dt}\frac{d\zeta}{dt},$$

and (1.4) is reduced to the equation

$$(1.4'') \qquad \frac{d^2\zeta}{dt^2} + \frac{\sinh 2\zeta}{2(\alpha^2 - r^2)}\left(\frac{dr}{dt}\right)^2 + \frac{4r^2\sinh 2\zeta}{\alpha^2(x_3+3r)}C = 0.$$

Last for $x_4 > \alpha$, we have

$$\frac{\alpha^2dx_4dx_4}{x_4x_4(\alpha^2 - x_4x_4)} = -\alpha^2\frac{dx_4dx_4}{x_4x_4(x_4x_4 - \alpha^2)} = -\alpha^2\left(\frac{dx_4}{x_4\sqrt{x_4x_4 - \alpha^2}}\right)^2$$

and

$$\frac{dx_4}{x_4\sqrt{x_4x_4 - \alpha^2}} = \frac{1}{\alpha}d\tan^{-1}\frac{\sqrt{x_4x_4 - \alpha^2}}{\alpha}.$$

Putting

$$\zeta = \tan^{-1}\frac{\sqrt{x_4x_4-\alpha^2}}{\alpha}, \text{ i.e., } \tan\zeta = \frac{\sqrt{x_4x_4-\alpha^2}}{\alpha},$$

we have

$$1+\tan^2\zeta = \frac{1}{\cos^2\zeta} = \frac{x_4x_4}{\alpha^2},\ x_4 = \frac{\alpha}{\cos\zeta} = \alpha\sec\zeta \quad \left(0<\zeta<\frac{\pi}{2}\right).$$

Therefore we obtain

$$ds^2 = \frac{\cos^2\zeta}{\alpha^2}\sum_{b,c=1}^{3}\left(\delta_{bc}+\frac{x_bx_c}{\alpha^2-r^2}\right)dx_bdx_c + d\zeta d\zeta \tag{1.1'''}$$

and

$$\begin{aligned} ds^2 =& \frac{\cos^2\zeta}{\alpha^2}\sum_{b,c=1}^{3}\left(\frac{8}{(x_3+3r)^2}(r^2\delta_{bc}-x_bx_c)+\frac{\alpha^2x_bx_c}{r^2(\alpha^2-r^2)}\right)dx_bdx_c \\ &+ d\zeta d\zeta \end{aligned} \tag{1.2'''}$$

where $a=-1/\alpha^2$ and $0<\zeta<\pi/2$. For a geodesic $(x_i(t))$ of the metric (1.2‴), by means of the equalities:

$$x_4 = \frac{\alpha}{\cos\zeta}, \quad \frac{dx_4}{dt} = \frac{\alpha\sin\zeta}{\cos^2\zeta}\frac{d\zeta}{dt},$$

and

$$\frac{d^2x_4}{dt^2} = \frac{\alpha\sin\zeta}{\cos^2\zeta}\frac{d^2\zeta}{dt^2} + \frac{\alpha(1+\sin^2\zeta)}{\cos^3\zeta}\left(\frac{d\zeta}{dt}\right)^2$$

the equation (1.4) is reduced to the expression for this case as follows :

$$\frac{d^2\zeta}{dt^2} + \frac{\sin 2\zeta}{2(\alpha^2-r^2)}\left(\frac{dr}{dt}\right)^2 + \frac{4r^2\sin 2\zeta}{\alpha^2(x_3+3r)}C = 0. \tag{1.4'''}$$

Thus we have a proposition as follows.

Proposition A. For any geodesic $(x_i(t))$ in R^4_+ of the metric (1.2), the function $\zeta(t)$ defined as above for the expressions (1.2′), (1.2″) and (1.2‴) of (1.2) satisfies very simple equations :

$$\frac{d^2\zeta}{dt^2} + \frac{\sinh 2\zeta}{2(\alpha^2+r^2)}\left(\frac{dr}{dt}\right)^2 + \frac{4r^2\sinh 2\zeta}{\alpha^2(x_3+3r)}C = 0, \tag{1.4'}$$

$$\frac{d^2\zeta}{dt^2} + \frac{\sinh 2\zeta}{2(\alpha^2-r^2)}\left(\frac{dr}{dt}\right)^2 + \frac{4r^2\sinh 2\zeta}{\alpha^2(x_3+3r)}C = 0 \tag{1.4''}$$

and

$$\frac{d^2\zeta}{dt^2} + \frac{\sin 2\zeta}{2(\alpha^2-r^2)}\left(\frac{dr}{dt}\right)^2 + \frac{4r^2\sin 2\zeta}{\alpha^2(x_3+3r)}C = 0 \tag{1.4'''}$$

respectively, where

$$C = \frac{1}{x_3 + 3r}\left(\sum_b \frac{dx_b}{dt}\frac{dx_b}{dt} - \frac{dr}{dt}\frac{dr}{dt}\right).$$

Remark B. By means of specializing the quantities in $(1.4') \sim (1.4''')$, we may construct special theories on the orbit of the geodesics.

Finally, regarding Proposition A we consider the case in which $C = 0$. $C = 0$ means that

$$\sum_b \frac{dx_b}{dt}\frac{dx_b}{dt} = \frac{dr}{dt}\frac{dr}{dt}$$

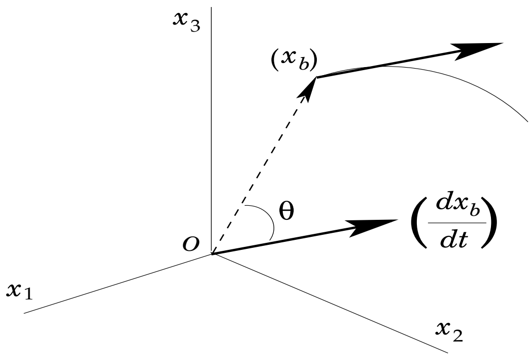
or

$$\left|\left(\frac{dx_b}{dt}\right)\right| = \left|\frac{dr}{dt}\right|.$$

Since $r^2 = \sum_b x_b x_b$, we obtain

$$\frac{dr}{dt} = \sum_b \frac{x_b}{r}\frac{dx_b}{dt} = \left|\left(\frac{dx_b}{dt}\right)\right| \cos\theta$$

where θ denotes the angle between two vectors (x_b) and $\left(\frac{dx_b}{dt}\right)$ in R^3. Hence we have $\cos\theta = 1$ or -1, and so we may put $x_b(t) = a_b(1 \pm t)$ and $r^2 = W^2(1 \pm t)^2$, where $W = \sqrt{\sum_b a_b a_b}$.



Hence, $(1.4')$, $(1.4'')$ and $(1.4''')$ become for this case as follows :

$$(1.5') \qquad \frac{d^2y}{dt^2} + \frac{\sinh 2y}{2(\beta^2 + (1 \pm t)^2)} = 0,$$

$$(1.5'') \qquad \frac{d^2y}{dt^2} + \frac{\sinh 2y}{2(\beta^2 - (1 \pm t)^2)} = 0$$

and

$$(1.5''') \qquad \frac{d^2y}{dt^2} + \frac{\sin 2y}{2(\beta^2 - (1 \pm t)^2)} = 0$$

respectively, where $y=\zeta$ and $\beta=\alpha/W$. We solve these differential equations as follows. First, we set

$$Q(y)=\int_1^y \frac{1}{\sinh 2y}dy \qquad \text{for} \quad y>0 \tag{1.7}$$

and another auxiliary function as follows:

$$P(y)=\int_0^y \frac{1}{\cosh y}dy \qquad \text{for} \quad y\geq 0. \tag{1.8}$$

Then we have

$$\frac{dQ(y)}{dt}=\frac{dQ}{dy}\frac{dy}{dt}=\frac{1}{\sinh 2y}\frac{dy}{dt}$$

and

$$\begin{aligned}\frac{d^2Q(y)}{dt^2} &= \frac{1}{\sinh 2y}\frac{d^2y}{dt^2}-\frac{2\cosh 2y}{\sinh^2 2y}\frac{dy}{dt}\frac{dy}{dt}\\ &= \frac{1}{\sinh 2y}\frac{d^2y}{dt^2}-2\cosh 2y\left(\frac{d}{dt}Q(y)\right)^2,\end{aligned}$$

hence we obtain

$$\begin{aligned}\frac{1}{\sinh 2y}\frac{d^2y}{dt^2} &= \frac{d^2Q}{dt^2}+2\cosh 2y\left(\frac{dQ}{dt}\right)^2\\ &= \frac{d^2Q}{dt^2}+2\left(\frac{dQ}{dt}\right)^2+4\left(\sinh y\frac{dQ}{dt}\right)^2\\ &= \frac{d^2Q}{dt^2}+2\left(\frac{dQ}{dt}\right)^2+\left(\frac{dP}{dt}\right)^2.\end{aligned}$$

Therefore $(1.5')\sim(1.5''')$ become as follows.

$$\frac{d^2Q}{dt^2}+2\left(\frac{dQ}{dt}\right)^2+\left(\frac{dP}{dt}\right)^2+\frac{1}{2(\beta^2+(1\pm t)^2)}=0, \tag{1.6$'$}$$

$$\frac{d^2Q}{dt^2}+2\left(\frac{dQ}{dt}\right)^2+\left(\frac{dP}{dt}\right)^2+\frac{1}{2(\beta^2-(1\pm t)^2)}=0 \tag{1.6$''$}$$

and

$$\frac{d^2Q}{dt^2}+2\left(\frac{dQ}{dt}\right)^2+\left(\frac{dP}{dt}\right)^2+\frac{1}{2(\beta^2-(1\pm t)^2)}=0 \tag{1.6$'''$}$$

respectively. Now, since

$$\frac{dQ}{dy}=\frac{1}{\sinh 2y}=\frac{1}{2\cosh y\sqrt{\cosh^2 y-1}},$$

we have the relation

$$\left\{\left(\frac{dP}{dy}\right)^2+2\left(\frac{dQ}{dy}\right)^2\right\}^2=4\left(\frac{dQ}{dy}\right)^2\left(1+\left(\frac{dQ}{dy}\right)^2\right).$$

Thus we obtain

$$\left(\frac{dP}{dy}\right)^2=2\frac{dQ}{dy}\left\{\sqrt{1+\left(\frac{dQ}{dy}\right)^2}-\frac{dQ}{dy}\right\}$$

and so

$$\left(\frac{dP}{dt}\right)^2=2\frac{dQ}{dt}\left\{\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dQ}{dt}\right)^2}-\frac{dQ}{dt}\right\}.$$

Therefore $(1.6')$ becomes

$$\frac{d^2Q}{dt^2}+2\frac{dQ}{dt}\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dQ}{dt}\right)^2}+\frac{1}{2(\beta^2+(1\pm t)^2)}=0 \tag{1.7'}$$

and $(1.6'')$ and $(1.6''')$ become analogous expressions. They are essentially equivalent to $(1.5')\sim(1.5''')$.

Proposition. The solution of $(1.5')$ is given as follows. Setting $y(0)=0$, $y(t)=\sum_{n=1}^{\infty}a_nt^n$, and $y^m=\sum_{n=m}^{\infty}b_{mn}t^n$ with $b_{mn}=0$ for $n<m$, we have

$$(\beta^2-1)a_2=0,\ 6(\beta^2-1)a_3\mp 4a_2+a_1=0,$$
$$12(\beta^2-1)a_4\mp 12a_3-a_2=0,\ 40(\beta^2-1)a_5\mp 48a_4-10a_3+\frac{4}{3}{a_1}^3=0,$$
$$30(\beta^2-1)a_6\mp 40a_5-11a_4+2{a_1}^2a_2=0,\ 84(\beta^2-1)a_7\mp 120a_6-38a_5$$
$$+4({a_1}^2a_3+a_1{a_2}^2)+\frac{4}{15}{a_1}^5=0,$$
$$2(\beta^2-1)(n+2)(n+1)a_{n+2}\mp 4(n+1)na_{n+1}$$
$$-2n(n-1)a_n+\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}b_{(2m+1)n}=0,$$

from which we obtain in Case I : $\beta^2\neq 1$,

$$a_2=0,\ a_3=-\frac{1}{6(\beta^2-1)}a_1,\ a_4=\mp\frac{1}{6(\beta^2-1)^2}a_1,$$
$$a_5=-\left(\frac{1}{5(\beta^2-1)^3}+\frac{1}{24(\beta^2-1)^2}\right)a_1-\frac{1}{30(\beta^2-1)}{a_1}^3,$$

$$\begin{aligned}a_6&=\frac{1}{30(\beta^2-1)}(\pm 40a_5+11a_4)\\&=\frac{\mp 1}{30(\beta^2-1)}\left(\left(\frac{7}{2(\beta^2-1)^2}+\frac{8}{(\beta^2-1)^3}\right)a_1+\frac{4}{3(\beta^2-1)}{a_1}^3\right)\ \ldots\ldots,\end{aligned}$$

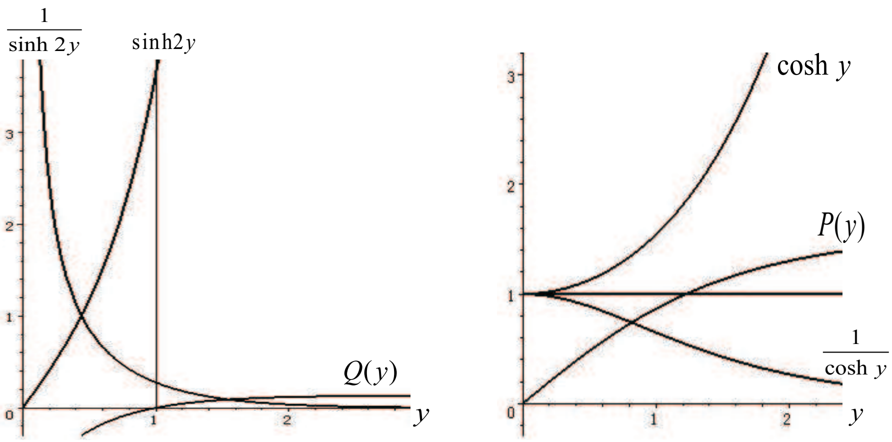
and in Case II : $\beta^2 = 1$,

$$a_2 = \pm\frac{1}{4}a_1,\ a_3 = \mp\frac{1}{12}a_2,\ a_4 = \pm\frac{1}{36}\left({a_1}^3 + \frac{5}{32}a_1\right),$$

$$a_5 = \frac{7}{80\times 18}{a_1}^3 - \frac{11}{8\times 32\times 36}a_1,$$

$$a_6 = \pm\frac{19\text{x}11}{60\times 8\times 36\times 32}a_1 \mp \frac{13}{60\times 80\times 18}{a_1}^3 \pm \frac{1}{15\times 30}{a_1}^5, \ldots\ldots,$$

respectively.



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