COMMUTATIVE GROUP ALGEBRAS
OF ABELIAN GROUPS
WITH UNCOUNTABLE POWERS AND LENGTHS

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ABSTRACT. *Let F be a field of $char(F) = p > 0$ and G an abelian group with p-component G_p of cardinality at most $\aleph_1$ and length at most ω_1.*

The main affirmation on the Direct Factor Problem is that $S(FG)/G_p$ is totally projective whenever F is perfect. This extends results due to May (Contemp. Math., 1989) and Hill-Ullery (Proc. Amer. Math. Soc., 1990).

As applications to the Isomorphism Problem, suppose that for any group H the F-isomorphism $FH \cong FG$ holds. Then if G_p is totally projective, $H_p \cong G_p$. This partially solves a problem posed by May (Proc. Amer. Math. Soc., 1988). In particular, $H \cong G$ provided G is p-mixed of torsion-free rank one so that G_p is totally projective. The same isomorphism $H \cong G$ is fulfilled when G is p-local algebraically compact too. Besides, if F_p is the simple field with p-elements and G_p is a coproduct of torsion complete groups, $F_pH \cong F_pG$ as F_p-algebras implies $H_p \cong G_p$. This expands the central theorem obtained by us in (Rend. Sem. Mat. Univ. Padova, 1999) and partly settles the generalized version of a question raised by May (Proc. Amer. Math. Soc., 1979) as well. As a consequence, when G_p is torsion complete and G is p-mixed of torsion-free rank one, $H \cong G$. Moreover, if G is a coproduct of p-local algebraically compact groups then $H \cong G$. The last attainment enlarges an assertion of Beers-Richman-Walker (Rend. Sem. Mat. Univ. Padova, 1983).

Each of the reported achievements strengthens our statements in this direction (Southeast Asian Bull. Math., 2001-2002) and also continues own studies in this aspect (Hokkaido Math. J., 2000) and (Kyungpook Math. J., 2004).

INTRODUCTION

Everywhere in the text of the present paper, the letter FG designates the group algebra of an abelian group G, written multiplicatively, over a field F of characteristic $p \neq 0$. As usual, $V(FG)$ denotes the group of all normalized invertible elements (often called normed units) in FG, and $S(FG)$ is its Sylow p-subgroup. For an abelian group G, the letter G_p will denote its p-primary component.

For a p-subgroup A of G, we define by $S(RG; A)$ the group $1 + I(RG; A)$ where $I(RG; A)$ is the relative augmentation ideal of RG with respect to A,

and R is a commutative unitary ring of prime characteristic p. All calculated heights are assumed to be p-heights for some arbitrary but a fixed prime number p. All other notations and the terminology not explicitly defined herein are standard and follow essentially the excellent monographs of Ch. Curtis - I. Rainer [2], L. Fuchs [13, 14, 15], Ph. Griffiths [16], I. Kaplansky [22], G. Karpilovsky [23, 24] and D. Passman [29].

In the theory of commutative group algebras there exist two global problems called the *Direct Factor Problem* and the *Isomorphism Problem*, respectively. The first conjectures that $S(FG)/G_p$ is always totally projective whenever F is perfect, and the second says that the p-mixed abelian group G may be retrieved from the F-group algebra FG. The main purpose of this research exploration is to investigate these two conjectures under some minimal restrictions on F and on G. Although F and G are not arbitrary objects, our further established claims are major and they supersede a large number of important classical theorems in this branch (see, for example, the cited bibliography).

DIRECT FACTOR THEOREM

W. May proved in [28] that if G is a p-group of cardinality not exceeding $\aleph_1$ and length not exceeding ω_1 and if F is either perfect or countable, then $S(FG)/G$ is totally projective. P. Hill jointly with W. Ullery extended in [19] this result by removing the restriction on length(G). They, actually, examined successfully even arbitrary coproducts of groups of power $\aleph_1$ (see [21] too).

In [8] we have argued that $S(FG)/G_p$ is totally projective provided F is perfect, G_p is separable and both F and G have cardinalities at most $\aleph_1$. Thus, we give a particular answer of our query formulated in [7].

Now, we shall illustrate that those power conditions on F and G may be dropped as well as the length restriction on G_p can be decreased.

Theorem. *Suppose G is an abelian group whose G_p is with cardinality not exceeding $\aleph_1$ and length not exceeding ω_1, and F is a perfect field of characteristic $p > 0$. Then $S(FG)/G_p$ is a totally projective group and so G_p is a direct factor of $S(FG)$ with totally projective complementary factor.*

In particular, under these circumstances, $V(FG)/G$ is a totally projective p-group and thus G is a direct factor of $V(FG)$ with totally projective complement, provided G is p-mixed.

As immediate valuable consequences, we establish new criteria for total projectivity and summability of $S(FG)$ (e. g. [6, 7, 8, 10, 11] and [12]).

Corollary. *Suppose G is an abelian group whose G_p is with power $\aleph_1$ and F is a perfect field of characteristic $p \neq 0$. Then $S(FG)$ is a direct sum of countable groups if and only if G_p is.*

Remark. This assertion improves the corresponding result in [7, 11].

Corollary. *Suppose G is an abelian group for which G_p is with power $\aleph_1$ and F is a perfect field of characteristic $p \neq 0$. Then $S(FG)$ is summable if and only if G_p is.*

Remark. The last affirmation generalizes a similar fact in [12] proved when either G_p is of countable length or G is of cardinality no more than $\aleph_1$.

Before proving the formulated statements, we need some preliminaries starting with

Lemma. *Assume that $1 \in L \leq R$ and that $B \leq A \leq G_p$, $C \leq G$. Then*
$$[AS(RG;B)] \cap S(LC) = (A \cap C)S(LC; B \cap C).$$

Proof. Take an arbitrary element x from the left hand-side. Hence, we may write $x = \sum_{c \in C} \alpha_c c = a \sum_{g \in G} r_g g$ such that $\alpha_c \in L$ with $\sum_{c \in C} \alpha_c = 1$; $a \in A$ and $\sum_{g \in g'B} r_g = 0, g' \notin B$ or $\sum_{g \in g'B} r_g = 1, g' \in B$, for any $g' \in G$. The canonical forms of the two sums imply $\alpha_c = r_g$ and $c = ag$. But $\sum_{g \in G} r_g g$ contains a member that belongs to $B \subseteq A$, say $b \in B \subseteq A$. Thus, $ab \in A \cap C$ and we derive $x = ab \sum_{g \in G} r_g gb^{-1}$. On the other hand, we observe that $gb^{-1} \in C$ and hence $gb^{-1} \in (g'B) \cap C = g'(B \cap C)$ whenever $g' \in C$. Furthermore, $\sum_{gb^{-1} \in c'(B \cap C)} r_g = 0, c' \notin B \cap C$ or $\sum_{gb^{-1} \in c'(B \cap C)} r_g = 1, c' \in B \cap C$, for every $c' \in C$, i.e., equivalently, x lies in the set $(A \cap C)S(LC; B \cap C)$, thus showing the wanted inclusion.

The converse part that the left hand-side contains the right hand-side is trivial. So, the intersection dependence is verified.

We continue with an interesting and, in some special cases, well-known for the specialist formula. Nevertheless, for the sake of completeness and for the convenience of the reader, we shall give a detailed proof because there is no such a proof in the literature yet.

Claim. *Given $A \leq G_p$. Then, for each ordinal α, the following formula is fulfilled*
$$S^{p^\alpha}(RG; A) = S(R^{p^\alpha}G^{p^\alpha}; A^{p^\alpha}).$$

Proof. It is not difficult to observe that it is enough to consider only the case for limit ordinals α since the remaining one is easy. And so, it is elementarily to see that the left hand-side contains the right hand-side. Conversely, take an arbitrary element from the left hand-side. Hence, $x \in \cap_{\beta < \alpha} S^{p^\beta}(RG; A) = \cap_{\beta < \alpha} S(R^{p^\beta} G^{p^\beta}; A^{p^\beta})$ by using the induction hypothesis. Therefore, $x = \sum_{g \in G^{p^\beta}} r_g g = \sum_{a \in G^{p^\gamma}} f_a a = \ldots$, where $r_g \in R$ so that $\sum_{g \in g' A^{p^\beta}} r_g = 0, g' \notin A^{p^\beta}$ or $\sum_{g \in g' A^{p^\beta}} r_g = 1, g' \in A^{p^\beta}$, for each $g' \in G^{p^\beta}$, and $f_a \in R$ so that $\sum_{a \in a' A^{p^\gamma}} r_g = 0, a' \notin A^{p^\gamma}$ or $\sum_{a \in a' A^{p^\gamma}} r_g = 1, a' \in A^{p^\gamma}$,

for each $a' \in G^{p^\gamma}$; $\beta < \gamma \le \alpha$ are arbitrary ordinal numbers. The canonical forms yield that $r_g = f_a$ and $g = a$. On the other hand,

$$\sum_{g \in g'A^{p^\alpha}} r_g = 0, g' \notin A^{p^\alpha} = \cap_{\beta<\alpha}A^{p^\beta}$$

or

$$\sum_{g \in g'A^{p^\alpha}} r_g = 1, g' \in A^{p^\alpha} = \cap_{\beta<\alpha}A^{p^\beta}$$

for each $g' \in G^{p^\alpha} = \cap_{\beta<\alpha}G^{p^\beta}$ that immediately ensures the desired equality. In fact, we foremost consider all elements g from the support of $\sum_{g \in G^{p^\beta}} r_g g$ that belongs to $g'A^{p^\beta}$ for any $g' \in G^{p^\beta}$. Clearly $g' \in G^{p^\gamma}$. Moreover, $g'g_\beta = a_\gamma$ for some $g_\beta \in A^{p^\beta}$; $a_\gamma \in G^{p^\gamma}, g''g'_\beta = a'_\gamma \in a_\gamma A^{p^\gamma}$ for some $g'' \in G^{p^\beta}$, and $g'_\beta \in A^{p^\beta}$. Hence $g'g''^{-1}g_\beta g'_\beta{}^{-1} \in A^{p^\gamma}$. But $f_{a_\gamma} + f_{a'_\gamma} = 0$, i.e., $r_{g'g_\beta} + r_{g''g'_\beta} = 0$, and so $r_{g'g_\beta}g'g_\beta + r_{g''g'_\beta}g''g'_\beta = r_{g'g_\beta}g'g_\beta(1 - g'^{-1}g''g_\beta^{-1}g'_\beta) \in I(R^{p^\gamma}G^{p^\gamma}; A^{p^\gamma})$. By the same token, we obtain similar relations for the other members of this type, as well. Consequently, by what we have shown above, we conclude that the element $\sum_{g \in G^{p^\beta}} r_g g$ belongs to $S(R^{p^\gamma}G^{p^\gamma}; A^{p^\gamma})$ because its group members lie in G^{p^γ} or in $g'A^{p^\gamma}$ for some $g' \in G^{p^\gamma}$.

Further, because we have finite sums of elements and an infinite intersection since $\alpha \ge \omega$, we may without loss of generality presume that such equalities will exist for infinitely many ordinals β. Thus, $g \in \cap_{\beta<\alpha}G^{p^\beta} = G^{p^\alpha}$ and more especially there are elements of the kind gd_α where $d_\alpha \in A^{p^\alpha} = \cap_{\beta<\alpha}A^{p^\beta}$ and sums of coefficients $r_g + r_{gd_\alpha} = 0$. These ratios substantiate our identity, and we are done. The proof is finished.

Remark. It is worthwhile noticing that the same claim is also valid for the ideal $I(RG; A)$ even when A is not p-primary.

Moreover, by virtue of similar technique, the more general formulae are true, namely:

If $\{R_i\}_{i \in I}, \{G_i\}_{i \in I}$ and $\{A_i\}_{i \in I} \le \{G_i\}_{i \in I}$ are infinite decreasing sequences of unitary commutative rings, groups and subgroups respectively, then $\cap_{i \in I}I(R_iG_i; A_i) = I((\cap_{\in I}R_i)(\cap_{i \in I}G_i); \cap_{i \in I}A_i)$.

For arbitrary chains this intersection equality is wrong, even for the finite case, as the following example shows: $I(RG; A) \cap I(RG; B) \supset I(RG; A \cap B)$ since $(1-a)(1-b) \in [I(RG; A) \cap I(RG; B)] \backslash I(RG; A \cap B)$ whenever $a \in A \backslash B$ and $b \in B \backslash A$.

As a direct efficacious corollary we have the following.

Corollary. *The p-torsion group A is, ever, isotype in $S(RG; A)$.*

Proof. Exploiting the last formula, we elementarily deduce that $A \cap S^{p^\alpha}(RG; A) = A \cap S(R^{p^\alpha}G^{p^\alpha}; A^{p^\alpha}) = A^{p^\alpha}$, as required. The proof is complete.

The following is crucial.

Proposition. *Suppose that $B \leq A \leq G_p$ and that F is perfect. Then*

(a) *$AS(FG; B)$ is isotype in $S(FG; A)$ if B is isotype in G_p. In particular, A is an isotype subgroup of $S(FG; A)$;*

(b) *$AS(FG; B)$ is nice in $S(FG; A)$ if A is isotype in G_p. In particular, A is a nice subgroup of $S(FG; A)$ when A is isotype in G_p;*

(c) *$AS(FG; B)$ is balanced in $S(FG; A)$ if B and A are both isotype in G_p. In particular, A is a balanced subgroup of $S(FG; A)$ when A is isotype in G_p.*

Proof. (a) Let $z \in [AS(FG; B)] \cap [S(FG; A)]^{p^\alpha}$ for any ordinal α and write $z = au$, where $a \in A$ and $u \in S(FG; B)$. Then the Claim applies to show that $au \in [S(FG; A)]^{p^\alpha} = S(FG^{p^\alpha}; A^{p^\alpha}) \subseteq S(FG; A^{p^\alpha}B)$. Therefore, $a \in S(FG; A^{p^\alpha}B)$ and so, $a \in A \cap S(FG; A^{p^\alpha}B) = A^{p^\alpha}B$. Hence, owing to the Lemma, the Claim plus the modular law, we compute that $z \in [A^{p^\alpha}S(FG; B)] \cap S(FG^{p^\alpha}; A^{p^\alpha}) = A^{p^\alpha}[S(FG; B) \cap S(FG^{p^\alpha}; A^{p^\alpha})] \subseteq A^{p^\alpha}[S(FG; B) \cap S(FG^{p^\alpha})] = A^{p^\alpha}S(FG^{p^\alpha}; B \cap G^{p^\alpha}) = A^{p^\alpha}S(FG^{p^\alpha}; B^{p^\alpha}) = A^{p^\alpha}[S(FG; B)]^{p^\alpha} \subseteq [AS(FG; B)]^{p^\alpha}$, which verifies our assertion.

(b) Because A is isotype in G_p, by conforming with the Intersection Lemma in [7] or the Lemma alluded to above, we obtain that

$$S(FG; A) \cap S^{p^\alpha}(FG) = S(FG; A) \cap S(F^{p^\alpha}G^{p^\alpha}) = S(F^{p^\alpha}G^{p^\alpha}; A \cap G^{p^\alpha})$$
$$= S(F^{p^\alpha}G^{p^\alpha}; A^{p^\alpha}) \subseteq S^{p^\alpha}(FG; A),$$

i.e., in other words, $S(FG; A)$ is isotype in $S(FG)$. Thus all heights will be calculated in $S(FG)$ and G_p, respectively. After this, referring to ([15], p. 91, Lemma 79.2) together with the Claim, it suffices to establish for the niceness only that $\cap_{\tau < \alpha}[AS(FG; B)S(FG^{p^\tau}; A^{p^\tau})] = AS(FG; B)S(FG^{p^\alpha}; A^{p^\alpha})$ for each limit ordinal α. To this goal, we choose an arbitrary element x from the left hand-side. Hence $x = a(1 + \sum_{i,j} r_{i,j}g_{i,j}(1 - b_i)).(1 + \sum_{i,j} f_{i,j}g_{i,j}^{(\tau)}(1 - a_i^{(\tau)})) = a'(1 + \sum_{i,j} r_{i,j}'g_{i,j}'(1 - b_i')).(1 + \sum_{i,j} f_{i,j}'g_{i,j}^{(\beta)}(1 - a_i^{(\beta)}) = \ldots$, where $a, a' \in A; r_{i,j}, f_{i,j}, r_{i,j}', f_{i,j}' \in F; g_{i,j}, g_{i,j}' \in G; b_i, b_i' \in B; g_{i,j}^{(\tau)} \in G^{p^\tau}, a_i^{(\tau)} \in A^{p^\tau}; g_{i,j}^{(\beta)} \in G^{p^\beta}, a_i^{(\beta)} \in A^{p^\beta}$ for some arbitrary β with $\tau < \beta < \alpha$. So, we can write $1 + \sum_{i,j} f_{i,j}g_{i,j}^{(\tau)}(1 - a_i^{(\tau)}) = a''(1 + \sum_{i,j} r_{i,j}''g_{i,j}''(1 - b_i'')).(1 + \sum_{i,j} f_{i,j}'g_{i,j}^{(\beta)}(1 - a_i^{(\beta)}))$, where $a'' \in A, r_{i,j}'' \in F, g_{i,j}'' \in G, b_i'' \in B$. Furthermore, this equality assures that $g_{i,j}^{(\tau)} = a''g_{i,j}''.g_{i,j}^{(\beta)}b_i''a_i^{(\beta)}$ and $g_{i,j}^{(\tau)}a_i^{(\tau)} = a''h_{i,j}''.h_{i,j}^{(\beta)}c_i''c_i^{(\beta)}$, where $h_{i,j}'' \in G, c_i'' \in B, h_{i,j}^{(\beta)} \in G^{p^\beta}$ and $c_i^{(\beta)} \in A^{p^\beta}$, etc. for

all indexes i and j. Thus $a_i^{(\tau)} \in gBG^{p^\beta}$ for some $g \in G$. Moreover, we can have eventually some other additional possibilities, namely:

- if $u_{ij}^{(\tau)} = a'' g_{ij}'' b_i'' g_{ij}^{(\beta)}$, then $g_{ij}^{(\tau)} u_{ij}^{(\tau)^{-1}} = a_i^{(\beta)} \in A^{p^\beta}$;
- if $v_{ij}^{(\tau)} = a'' g_{ij}'' g_{ij}^{(\beta)} a_i^{(\beta)}$, then $g_{ij}^{(\tau)} v_{ij}^{(\tau)^{-1}} = b_i'' \in B$;
- if $w_{ij}^{(\tau)} = a'' g_{ij}'' g_{ij}^{(\beta)}$, then $g_{ij}^{(\tau)} w_{ij}^{(\tau)^{-1}} \in BA^{p^\beta}$;
- if $g_{ij}'' \in h_{ij}'' BG^{p^\beta}$, then $a_i^{(\tau)} \in (BG^{p^\beta}) \cap A = BA^{p^\beta}$. In particular, when $g_{ij}'' b_i'' \in h_{ij}'' c_i'' G^{p^\beta}, a_i^{(\tau)} \in A^{p^\beta}$;
- if $g_{ij}^{-(\tau)} = a'' g_i^{-\prime\prime} b_i^{-\prime\prime} g_{ij}^{-(\beta)} a_{ij}^{-(\beta)}$ so that $g_{ij}'' b_i'' \in g_{ij}^{-\prime\prime} b_i^{-\prime\prime} G^{p^\beta}$, then $g_{ij}^{(\tau)} g_{ij}^{-(\tau)^{-1}} \in G^{p^\beta}$;
- if $e_{ij}^{(\tau)} = a'' b_i'' a_i^{(\beta)}$ and $d_{ij}^{(\tau)} = a'' b_i'' g_{ij}^{(\beta)} a_i^{(\beta)}$, then $a'' e_{ij}^{(\tau)^{-1}} \in BA^{p^\beta}$ and $d_{ij}^{(\tau)} e_{ij}^{(\tau)^{-1}} \in G^{p^\beta}$.

And so, as it is not difficult to be seen, $1 + \sum_{i,j} f_{i,j} g_{i,j}^{(\tau)} (1 - a_i^{(\tau)})$ may be written as a finite sum of elements of the types $g_\beta(1 - a_\beta b)$ and $g(1 - a_\beta)$ with coefficients from F, where $g_\beta \in G^{p^\beta}, a_\beta \in A^{p^\beta}, b \in B$ and $g \in G$. But $1 - a_\beta b = 1 - a_\beta + (1 - b)a_\beta$, and $g(1 - a_\beta) = 1 - a_\beta + (g - 1)(1 - a_\beta)$.

We will combine only those members of the sum which belong to $1 + I(FG^{p^\beta}; A^{p^\beta})$. Furthermore, $1 + \sum_{i,j} f_{i,j} g_{i,j}^{(\tau)} (1 - a_i^{(\tau)}) = ayz$, where $a \in A, y \in S(FG; B)$ and $z \in S(FG^{p^\beta}; A^{p\beta}) = S^{p^\beta}(FG; A)$ such that y and z are functions of $g_{i,j}^{(\tau)}$ and $a_i^{(\tau)}$, that is they may exclusively be written by these group elements.

Since we have finite sums of elements, the relationships between the group members are a finite number, while the equalities are infinite because the intersection is infinite taking into account that $\alpha \geq \omega$. Therefore, we can assume that the above presented dependencies between the elements from the group basis are valid infinitely many times, i.e. for almost all ordinals β. Thus, repeating the same procedure for each of these ordinal numbers β, we have $z \in \cap_{\beta < \alpha} S^{p^\beta}(FG; A) = S^{p^\alpha}(FG; A)$ and we are done.

Finally, we infer that $1 + \sum_{i,j} f_{i,j} g_{i,j}^{(\tau)} (1 - a_i^{(\tau)}) \in S(FG^{p^\beta}; A^{p^\beta})$ for all but a finite number of ordinals β, or that $1 + \sum_{i,j} f_{i,j} g_{i,j}^{(\tau)} (1 - a_i^{(\tau)}) \in AS(FG; B)$. That is why, by what we have just argued,

$$x \in [AS(FG; B)][\cap_{\tau < \alpha} S(FG^{p^\tau}; A^{p^\tau})]$$
$$= AS(FG; B)S(FG^{p^\alpha}; A^{p^\alpha}) = AS(FG; B)S^{p^\alpha}(FG; A),$$

as desired. The claim is proved.

(c) Follows immediately combining points (a) and (b).

The proof of the Proposition is completed.

Remark. This Proposition is stronger than Lemmas 4 and 5 used by May in [28].

Question. Whether the assertion in (b) is true even when A is not isotype in G_p?

Without any reference we shall freely use in the sequel the simple facts that height$(g) =$ height(g^{-1}) for every $g \in G$, and that if $1 \neq v \in V(FG;C) \cap V(FG) = [1 + I(FG;C)] \cap V(FG)$ for any subgroup C of G then there exists $1 \neq c \in C$ so that height$_{V(FG)}(v) \leq$ height$_G(c)$.

We are now ready to proceed by proving the following auxiliary technicality.

Proposition Structure. *Suppose A is a countable isotype p-subgroup of the abelian group G and C is a subgroup of A. Then S(FG; A)/AS(FG; C) is totally projective, provided F is a perfect field of characteristic p.*

Proof. We write down $A = \cup_{n<\omega} A_n$, where $A_n \subseteq A_{n+1}$ and all A_n are finite groups. By utilizing (b) and its proof, $S(FG;A)$ is an isotype subgroup of $S(FG)$, whence all heights will be computed in the whole group $S(FG)$. Moreover, the subgroup $AS(FG;C)$ is nice in $S(FG;A)$, hence $S(FG;A)/AS(FG;C)$ is a group of countable length since length$(A) < \omega_1$ and so $S(FG;A)$ is with such length by employing the Claim.

After this, we construct the subgroups $S_n = \langle x^{(n)} = f_1^{(n)} g_1^{(n)} + f_2^{(n)} g_2^{(n)} + \ldots + f_{s_n}^{(n)} g_{s_n}^{(n)} \in S(FG; A_n) | f_1^{(n)}, f_2^{(n)}, \ldots, f_{s_n}^{(n)} \in F$ with $f_1^{(n)} + f_2^{(n)} + \ldots + f_{s_n}^{(n)} = 1; g_1^{(n)}, g_2^{(n)}, \ldots, g_{s_n}^{(n)} \in G \rangle$ such that for all possible finite products between the degrees of group members: height$_G(g_i^{(n)\pm\varepsilon_i} g_j^{(n)\pm\xi_j} \cdot \cdots \cdot g_k^{(n)\pm\eta_k}) \in M = \{0, 1, \ldots, n\} \cup \{$the height spectrum of $A_n\} \cup \{\geq$ length$(A)\}$ whenever $0 \leq \varepsilon_i, \xi_j, \ldots, \eta_k < \min($order$(x^{(n)})$, order$(g_i^{(n)})$, order$(g_j^{(n)})$, $\ldots$, order $(g_k^{(n)}))$; $1 \leq i \leq s_n \in \mathbf{N}$. Clearly, S_n are correctly defined generating subgroups because the height conditions are really satisfied, $A_n \subseteq S_n \subseteq S_{n+1}$ and $S(FG;A) = \cup_{n<\omega} S_n = \cup_{n<\omega} S(FG; A_n)$.

We will show now that all S_n are height-finite in $S(FG)$. Indeed, every element from S_n is of the form $x_1^{(n)\varepsilon_1} \cdot \cdots \cdot x_t^{(n)\varepsilon_t}$, where $x_1^{(n)}, \ldots, x_t^{(n)}$ are generating elements and $0 \leq \varepsilon_i \leq$ order $(x_i^{(n)})$; $1 \leq i \leq t \in \mathbf{N}$. By construction, we see that $x_i^{(n)\varepsilon_i}$ has height as computed in $S(FG)$ that belongs to M for all indices i. It is also worthwhile noticing that if $g_1^{(n)} \in G^{p^\tau}, \ldots, g_{s_n}^{(n)} \in G^{p^\tau}$, then $x^{(n)} \in S(FG^{p^\tau}) \cap S(FG;A) = S^{p^\tau}(FG) \cap S(FG;A) = S^{p^\tau}(FG;A) = 1$, whenever $\tau =$ length(A). Further, if $1 \neq x_1^{(n)\varepsilon_1} \cdot \cdots \cdot x_t^{(n)\varepsilon_t}$, then this product as an element of $S(FG;A_n)$ possesses height less than or equal to the height of some $a_n \in A_n$. That is why, we presume that in the canonical form of

$x_1^{(n)^{\varepsilon 1}} \cdots . x_t^{(n)^{\varepsilon t}}$ all elements of A_n have infinite heights; otherwise we are done.

Besides, we can assume that $t = 2$ and that $\varepsilon_1 = \varepsilon_2 = 1$, since the general case follows by making use of the same arguments presented below.

Next, write $x_1^{(n)} = f_1^{(1n)} g_1^{(1n)} + f_2^{(1n)} g_2^{(1n)} + \ldots + f_s^{(1n)} g_s^{(1n)}$ and $x_2^{(n)} = r_1^{(2n)} a_1^{(2n)} + r_2^{(2n)} a_2^{(2n)} + \ldots + r_s^{(2n)} a_s^{(2n)}$, hence $x_1^{(n)} x_2^{(n)} = f_1^{(1n)} r_1^{(2n)} g_1^{(1n)} a_1^{(2n)} + \ldots + f_1^{(1n)} r_s^{(2n)} g_1^{(1n)} a_s^{(2n)} + \ldots + f_s^{(1n)} r_1^{(2n)} g_s^{(1n)} a_1^{(2n)} + \ldots + f_s^{(1n)} r_s^{(2n)} g_s^{(1n)} a_s^{(2n)}$.

We note that there are no zero divisors in the coefficient field F, i.e. $f_i^{(1n)} r_j^{(2n)} \neq 0$ for $1 \leq i, j \leq s \in \mathbf{N}$.

On the other hand, it is a routine matter to check that the following can be realized: In the canonical form of $x_1^{(n)}$ and $x_2^{(n)}$ must exist elements from A_n with nonzero coefficients, say for instance $g_1^{(1n)} \in A_n$ and $a_1^{(2n)} \in A_n$. The same is also true for the canonical form of the product $x_1^{(n)} x_2^{(n)}$. Henceforth, with no loss of generality, we may assume in this situation that $g_1^{(1n)} a_1^{(2n)} \in A_n$. Consider now the member $g_1^{(1n)} a_s^{(2n)}$; we shall further estimate its height. In fact, if $\text{height}(g_1^{(1n)} a_s^{(2n)}) \geq \text{height}(g_1^{(1n)} a_1^{(2n)})$, we are done. In the remaining case when $\text{height}(g_1^{(1n)} a_s^{(2n)}) < \text{height}(g_1^{(1n)} a_1^{(2n)})$, we derive that $\text{height}(g_1^{(1n)} a_s^{(2n)}) = \text{height}(g_1^{(1n)} a_s^{(2n)} . g_1^{(1n)^{-1}} a_1^{(2n)^{-1}}) = \text{height}(a_s^{(2n)} . a_1^{(2n)^{-1}})$. By the same token, we obtain that $\text{height}(g_s^{(1n)} a_1^{(2n)}) \geq \text{height}(g_1^{(1n)} a_1^{(2n)})$ or $\text{height}(g_s^{(1n)} a_1^{(2n)}) = \text{height}(g_s^{(1n)} . g_1^{(1n)^{-1}})$, and $\text{height}(g_s^{(1n)} a_s^{(2n)}) \geq \text{height}(g_1^{(1n)} a_s^{(2n)})$ or $\text{height}(g_s^{(1n)} a_s^{(2n)}) = \text{height}(g_s^{(1n)} . g_1^{(1n)^{-1}})$. The difficulty is when $\text{height}(g_1^{(1n)} a_s^{(2n)}) < \text{height}(g_s^{(1n)} a_s^{(2n)}) < \text{height}(g_1^{(1n)} a_1^{(2n)})$, where $g_1^{(1n)} a_s^{(2n)}$ and $g_s^{(1n)} a_1^{(2n)}$ have zero coefficients and $g_s^{(1n)} a_s^{(2n)}$ has nonzero coefficient in the support of $x_1^{(n)} x_2^{(n)}$. Without harm of generality, let $g_1^{(1n)} a_s^{(2n)} = g_s^{(1n)} a_1^{(2n)}$ with $f_1^{(1n)} r_s^{(2n)} + f_s^{(1n)} r_1^{(2n)} = 0$, since we can replace the indices. Therefore, $\text{height}(g_s^{(1n)} a_s^{(2n)}) = \text{height } (g_s^{(1n)} g_1^{(1n)^{-1}} a_s^{(2n)} a_1^{(2n)^{-1}}) = \text{height}((g_s^{(1n)} . g_1^{(1n)^{-1}})^2)$, and etc. for all other analogous variants. Indeed, if $g_1^{(1n)} a_s^{(2n)} \neq g_s^{(1n)} a_1^{(2n)}$, then $g_1^{(1n)} a_s^{(2n)}$ and $g_s^{(1n)} a_1^{(2n)}$ are equal to some products of group elements from the support of $x_1^{(n)} x_2^{(n)}$. Thus, it is not hard to be seen that (we omit the details because of technical difficulties and leave them to the reader) we can obtain the explicit kind for $g_s^{(1n)} a_s^{(2n)}$ such that either $\text{height}(g_s^{(1n)} a_s^{(2n)}) \geq \min[\text{height}(g_t^{(1n)} a_1^{(2n)}), \text{height}(g_1^{(1n)} a_t^{(2n)})]$ for some $1 \leq t \leq s$ for which at least one of $g_t^{(1n)} a_1^{(2n)}$ and $g_1^{(1n)} a_t^{(2n)}$ lies in the support, or $\text{height}(g_s^{(1n)} a_s^{(2n)}) = \text{height}(d)$ for some element $d \in G$ that is a multiplication of $g_i^{(1n)}$'s or of $a_i^{(2n)}$'s for $1 \leq i \leq s$, or $\text{height}(g_s^{(1n)} a_s^{(2n)}) \geq$

height(c_n) for some $c_n \in A_n$. Hence, height$(g_s^{(1n)}a_s^{(2n)}) \geq \delta \in M$. We may obtain similar inequalities for the other members from the canonical sum of $x_1^{(n)}x_2^{(n)}$ as well. Finally, taking into account that height$(x_1^{(n)}x_2^{(n)}) =$ min$\{$height$(g_1^{(1n)}a_1^{(2n)}), \ldots,$ height$(g_1^{(1n)}a_s^{(2n)}), \ldots,$ height$(g_s^{(1n)}a_1^{(2n)}), \ldots,$ height$(g_s^{(1n)}a_s^{(2n)})\}$, where, of course, these elements eventually have nonzero coefficients in the canonical form of $x_1^{(n)}x_2^{(n)}$, we argue the claim that we pursue. So, employing the Hill-Ullery necessary and sufficient condition for total projectivity in [20], we detect that $S(FG; A)$ is therefore totally projective.

After this, an appeal to the Proposition and more especially to the half on niceness leads us to the fact that height$(v_n AS(FG; C)) =$ height$(v_n aw_n)$ for every $v_n \in S_n$ and some $a \in A$ and $w_n \in S(FG; C)$. Apparently height$(v_n) \leq$ height$(v_n AS(FG; C))$ and hence height$(aw_n) \geq$ height(v_n); otherwise height$(v_n) \leq$ height$(aw_n) <$ height(v_n) which is a contradiction. Furthermore, let height$(aw_n) =$ height(v_n) since in the remaining case height$(v_n AS(FG; C)) =$ height(v_n) and, by what we have already shown, there is nothing to prove. It is self-evident that height$(v_n aw_n) \geq$ height(v_n). Write $w_n = r_1 c_1 + \ldots + r_k c_k$ whence $aw_n = r_1 ac_1 + \ldots + r_k ac_k$. Besides, there exists a member of the canonical sum of aw_n, say ac_1, such that height$(ac_1) \in M$. Next, we shall study the product $v_n aw_n$. First, we write $v_n = \alpha_1 d_1 + \ldots + \alpha_k d_k$. Thus $v_n aw_n = (\alpha_1 d_1 + \ldots + \alpha_k d_k).(r_1 ac_1 + \ldots + r_k ac_k) = \sum_i \sum_j \alpha_i r_j d_i ac_i$. What we need to have is the canonical form of this element and to evaluate its height. First of all, we observe that $\alpha_i r_j \neq 0$ because the field possesses no zero divisors. Next, since $v_n \in S_n$, we shall presume that d_1 lies in $A_n \subseteq A \cap S_n$. Thereby, according to the form of $v_n AS(FG; C)$, we may assume also that $d_1 = 1$. Consequently, $v_n aw_n = \alpha_1 r_1 ac_1 + \ldots + \alpha_1 r_k ac_k + \ldots + \alpha_k r_1 d_k ac_1 + \ldots + \alpha_k r_k d_k ac_k$. In the case when ac_1 has a nonzero coefficient in the canonical form of $v_n aw_n$, we are done. But it is of a real possibility some relation between the group members of the last sum to exist such that ac_1 to be with zero coefficient in the final canonical record. In this situation, bearing in mind that some of $d_2, \ldots, d_k$ has height in M, specifically this with minimal height, and that $a \in A_r$ for some natural number r, and adapting the same technology as to the above presented, we can compute that the heights height$(d_k ac_1)$ etc. lie in M, whenever the elements $d_k ac_1, \ldots$ have nonzero support. By the same token, we may successfully estimate the other heights of elements with nonzero coefficients, i.e. those which belong to the support of $v_n aw_n$. We consequently can deduce that height$(v_n aw_n) \in M$, hence $S_n AS(FG; C)/AS(FG; C)$ are subgroups of $S(FG; A)/AS(FG; C)$ with finite height spectrum, as expected. Finally, since $S(FG; A)/AS(FG; C) = \cup_{n<\omega}[S_n AS(FG; C)/AS(FG; C)]$, we can apply again the aforementioned

Hill-Ullery's criterion for total projectivity in [20] to finish the proof in general after all.

Remark. The last proposition supersedes [28, Lemma 6].

Now, we have at our disposal all the information necessary for proving the central Theorem. We shall develop the idea of May materialized in [28].

Proof of the Theorem. Owing to the hypothesis from the formulation, we write down $G_p = \cup_{\alpha<\omega_1} G_\alpha$, where $G_0 = 1, G_\alpha \subseteq G_{\alpha+1}, |G_\alpha| \leq \aleph_0$ and for every limit ordinal $\beta < \omega_1$ it is true that $G_\beta = \cup_{\alpha<\beta} G_\alpha$, i.e., in other words, the union is smooth. Moreover, in accordance with ([15], p. 98, Exercise 5), all G_α may be chosen to be isotype in G_p. Invoking to the Structure Proposition, $S(FG; G_{\alpha+1})/G_{\alpha+1}S(FG; G_\alpha)$ is a totally projective factor-group. On the other hand, point (c) gives that $G_{\alpha+1}S(FG; G_\alpha)$ is balanced in $S(FG; G_{\alpha+1})$. Therefore, as is well-known (see [15]), we may write $S(FG; G_{\alpha+1}) = (G_{\alpha+1}S(FG; G_\alpha)) \times T_\alpha$, for some totally projective p-group T_α. Next, we shall prove via ordinary transfinite induction that $S(FG; G_\alpha) = G_\alpha \times (\coprod_{\beta<\alpha} T_\beta)$ for each ordinal α. In fact, in [11] we have proved that $S(FG; G_1) = G_1 \times T_0$ for some p-torsion totally projective p-group T_0. Consider now the group $S(FG; G_{\alpha+1})$. Using the foregoing interpolation formula and the induction hypothesis, we yield that $S(FG; G_{\alpha+1}) = [G_{\alpha+1}(G_\alpha \times (\coprod_{\beta<\alpha} T_\beta))] \times T_\alpha = [G_{\alpha+1} \times (\coprod_{\beta<\alpha} T_\beta)] \times T_\alpha = G_{\alpha+1} \times (\coprod_{\beta\leq\alpha} T_\beta) = G_{\alpha+1} \times (\coprod_{\beta<\alpha+1} T_\beta)$ since $G_{\alpha+1} \cap (\coprod_{\beta<\alpha} T_\beta) = G_{\alpha+1} \cap S(FG; G_\alpha) \cap (\coprod_{\beta<\alpha} T_\beta) = G_\alpha \cap (\coprod_{\beta<\alpha} T_\beta) = 1$. This completes the induction after all. Further, our approach exploits routine set-theoretical arguments, namely both the obtained decomposition and the Main Lemma in [3] insure that

$$
\begin{aligned}
S(FG) = S(FG; G_p) &= S(FG; \cup_{\alpha<\omega_1} G_\alpha) \\
&= \cup_{\alpha<\omega_1} S(FG; G_\alpha) = \cup_{\alpha<\omega_1} [G_\alpha \times (\coprod_{\beta<\alpha} T_\beta)] \\
&= [\cup_{\alpha<\omega_1} G_\alpha] \times [\cup_{\alpha<\omega_1} (\coprod_{\beta<\alpha} T_\beta)] = G_p \times (\coprod_{\beta<\omega_1} T_\beta),
\end{aligned}
$$

where the complementary group is totally projective, as well. The proof is finished.

We continue with checking of the validity of the corollaries.

Proof of the first Corollary. Because $\text{length}(S(FG)) = \text{length}(G_p) \leq \omega_1$ (for instance, cf. [15]), we will employ the Direct Factor Theorem to extract that $S(FG) \cong G_p \times S(FG)/G_p$, where $S(FG)/G_p$ is a direct sum of countable groups. Consequently, the statement holds invoking to [15]. The proof is completed.

Proof of the second Corollary. Since $\text{length}(S(FG)) = \text{length}(G_p) \leq \omega_1$ (see, for example, [15]), we apply the Direct Factor Theorem to obtain $S(FG) \cong G_p \times S(FG)/G_p$, where $S(FG)/G_p$ is a direct sum of countable groups. Therefore, the assertion follows directly in virtue of [15]. The proof is finished.

We shall describe now the more general configuration by formulating of the so-called *isotype tower* of abelian groups. So, we can give

Definition. We shall say that $\{G\}_{\alpha < \lambda}$ is an isotype tower (or, in other words, an isotype family) of abelian groups which form the abelian group G if $G = \cup_{\alpha < \lambda} G_\alpha, G_\alpha \subseteq G_{\alpha+1}, G_\alpha = \cup_{\beta < \alpha} G_\beta$, whenever α is limit, and all groups G_α are isotype in G such that $|G_\alpha| < |G|$. It is worth noting that every uncountable abelian group has a pure tower (see [17]).

And so, adopting the proof of our central Theorem for the direct factor, we may pose the following scheme of proof of the Generalized Direct Factor Problem for mixed abelian groups with uncountable cardinality (for countable groups everything was made in [30], [20], [7]), namely:

If for any isotype subgroup $A \leq (G_\alpha)_p$ with $|G_\alpha| < |G|$ is fulfilled that $S(FG_\alpha)/(G_\alpha)_p S(FG_\alpha; A)$ is totally projective, then the same is true for $S(FG)/G_p$. Thus the Direct Factor Problem holds for all abelian p-mixed groups which have an isotype tower.

Notice that a reformulation of the Direct Factor Problem in terms of σ-summable groups was made in [18] (see [3] and [10] as well).

As an application of our group attainment, listed above, we state

ISOMORPHISM THEOREMS

We begin with proofs of the applied results.

Theorem 1. *Let G be a p-mixed abelian group for which G_p has cardinality at most $\aleph_1$ and length at most ω_1, and let F be a field with characteristic p. Then $FH \cong FG$ as F-algebras for arbitrary group H implies that there is a totally projective p-group T with the property $H \times T \cong G \times T$.*

Proof. Referring to [24], we deduce $|H_p| \leq \aleph_1$. Therefore, our Direct Factor Theorem does imply that $G \times V(FG)/G \cong V(FG) \cong V(FH) \cong H \times V(FH)/H$, where $V(FG)/G$ and $V(FH)/H$ are both totally projective p-groups. Choosing a totally projective p-group T with cardinal Ulm-Kaplansky functions more than the maximal of the Ulm-Kaplansky invariants of $V(FG)/G$ and $V(FH)/H$, we find that $T \times V(FG)/G \cong T \cong T \times V(FH)/H$. Finally, it is easy to see that $G \times T \cong H \times T$, and so we are done.

The following theorem partly resolves a problem of May from [27] (see [3], [4], [7] and [11] too).

Theorem 2. *Assume that G is an abelian group so that G_p is with cardinality not exceeding $\aleph_1$ a coproduct of countable groups. Then the F-isomorphism $FH \cong FG$ for some group H yields $H_p \cong G_p$. In particular, $H \cong G$, provided G is p-mixed of torsion-free rank one.*

Proof. By making use of [24], we derive $|H_p| \leq \aleph_1$. So, our Direct Factor Theorem guarantees that $G_p \times S(FG)/G_p \cong S(FG) \cong S(FH) \cong H_p \times S(FH)/H_p$, where $S(FG)/G_p$ is totally projective. Thus $S(FG)$ is totally projective and H_p as its direct factor is one also (see [15] too). But, according to [25], the Ulm-Kaplansky invariants of G_p and H_p are equal. Hence, $G_p \cong H_p$. After this, we can apply our algorithm developed in [6] together with a theorem of Megibben-Wallace (see, for example, [15] or [31]) to conclude that G and H must be isomorphic. The proof is completed.

The following theorem refines the main result in [5] and answers a question due to W. May [26].

Theorem 3. *Suppose that G is an abelian group such that G_p is with cardinality $\aleph_1$ a coproduct of torsion complete groups and F_p is the simple field of char$(F_p) = p$. Then for any group H the F_p–isomorphism $F_pH \cong F_pG$ does imply that $H_p \cong G_p$. In particular, $H \cong G$, when G is p-mixed of torsion-free rank one and G_p is torsion complete of power $\aleph_1$.*

Proof. In the sense of an affirmation due to Beers-Richman-Walker [1], it is sufficient to show only that H_p is a coproduct of torsion complete groups. In this direction, as we have observed above, $G_p \times S(F_pG)/G_p \cong S(F_pG) \cong S(F_pH) \cong H_p \times S(F_pH)/H_p$, where $S(F_pG)/G_p$ is totally projective. Besides, G_p being separable ensures that $S(F_pG)$ is separable whence the same holds for $S(F_pG)/G_p$ as "a subgroup". That is why, $S(F_pG)/G_p$ is a coproduct of cyclic groups. Furthermore, H_p as being a direct factor of a coproduct of torsion complete groups is with the same property (see cf. [15]), as required. After this, we may apply the method developed by us in [6] along with a result of Megibben (e.g., cf. [15]) to infer that G and H are isomorphic. The proof is finished.

Theorem 4. *Let G be a coproduct of p-local algebraically compact groups whose G_p has cardinality at most $\aleph_1$ and F_p be the finite field with char$(F_p) = p$. Then $F_pH \cong F_pG$ as F_p–algebras for some group H if and only if $H \cong G$.*

Proof. As we have just shown above, $G \times V(F_pG)/G \cong V(F_pG) \cong V(F_pH) \cong H \times V(F_pH)/H$, where $V(F_pG)/G$ is a totally projective p-group. Because of the separability of G_p, we yield that $V(F_pG)/G \cong S(F_pG)/G_p$ is also separable and so it is a coproduct of cyclic groups. Consequently, H as a direct factor of a coproduct of p-local algebraically compact groups is

one also (for instance, see [15]). Therefore, invoking to a result of Beers-Richman-Walker [1], we derive that G and H are isomorphic groups. The proof is completed.

The following isomorphism affirmation improves the corresponding one in [9].

Theorem 5. *Suppose G is a p-local algebraically compact group such that G_p is of power not exceeding $\aleph_1$. Then $FH \cong FG$ as F-algebras for arbitrary group H if and only if $H \cong G$.*

Proof. First, we exploit [24] to conclude that $|H_p| \leq \aleph_1$. After this, because $\text{length}(G_p) = \text{length}(H_p) \leq \omega$, the main Direct Factor Theorem implies that $G \times V(FG)/G \cong V(FG) \cong V(FH) \cong H \times V(FH)/H$, where $V(FG)/G$ is a coproduct of cyclic p-groups. Adapting the technique described in [15], we are in a position to write $H = A \times C$, where A is p-local algebraically compact and C is a coproduct of p-primary cyclic groups. Consequently, we can employ the scheme for a proof developed by us in [9] to conclude that G and H are isomorphic. The proof is finished.

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