Math. J. Okayama Univ. **51** (2009), 149–157

THE SPACE $\mathcal{L}_q$ OF DOUBLE SEQUENCES

Feyzi BAŞAR and Yurdal SEVER

Abstract. The spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences have recently been studied by Altay and Başar [J. Math. Anal. Appl. **309**(1)(2005), 70–90]. In this work, following Altay and Başar [1], we introduce the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examine some properties of the space $\mathcal{L}_q$. Furthermore, we determine the $\beta(\upsilon)$-dual of the space and establish that the α- and γ-duals of the space $\mathcal{L}_q$ coincide with the $\beta(\upsilon)$-dual; where $1 \leq q < \infty$ and $\upsilon \in \{p,\ bp,\ r\}$.

1. Introduction

By w and Ω, we denote the set of all real valued single and double sequences which are the vector spaces with coordinatewise addition and scalar multiplication. Any vector subspaces of w and Ω are called as the *single* and *double sequence spaces*, respectively. The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u := \left\{ x = (x_{mn}) \in \Omega \, : \, \|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty \right\},$$

which is a Banach space with the norm $\|\cdot\|_\infty$; where $\mathbb{N}$ denotes the set of all positive integers. Consider a sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{R}$ such that

$$|x_{mn} - l| < \varepsilon$$

for all $m,\ n > n_0$ then we call that the double sequence x is *convergent* in the *Pringsheim's sense* to the limit l and write $p-\lim x_{mn} = l$; where $\mathbb{R}$ denotes the real field. By $\mathcal{C}_p$ we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. So, we can consider the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and the set of all such sequences denoted by $\mathcal{C}_r$. Also by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$, we denote the spaces of all double sequences converging to 0 contained

Mathematics Subject Classification. Primary: 46A45; Secondary: 40C05.

Key words and phrases. Double sequence space, paranormed sequence space, α-, β-, γ-duals of a double sequence space.

in the sequence spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively. Móricz [7] proved that $\mathcal{C}_{bp}$, $\mathcal{C}_{bp0}$, $\mathcal{C}_r$ and $\mathcal{C}_{r0}$ are Banach spaces with the norm $\|\cdot\|_\infty$.

Let us consider the isomorphism T which plays an essential role for the present study, defined by Zeltser [11, p. 36] as

$$\begin{array}{cccccl} T & : & \Omega & \longrightarrow & w \\ & & x & \longmapsto & z=(z_i):=(x_{\psi^{-1}(i)}), \end{array} \tag{1.1}$$

where $\psi:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is a bijection defined by

$$\begin{array}{rcl} \psi[(1,1)] & = & 1, \\ \psi[(1,2)] & = & 2,\ \psi[(2,2)]=3,\ \psi[(2,1)]=4, \\ & \vdots & \\ \psi[(1,n)] & = & (n-1)^2+1,\ \psi[(2,n)]=(n-1)^2+2,\ \ldots, \\ \psi[(n,n)] & = & (n-1)^2+n,\ \psi[(n,n-1)]=n^2-n+2,\ \ldots,\ \psi[(n,1)]=n^2, \\ & \vdots & \end{array}$$

Let us consider a double sequence $x=(x_{mn})$ and define the sequence $s=(s_{mn})$ via x by

$$s_{mn}:=\sum_{i,j=1}^{m,n} x_{ij}\ ;\ (m,\ n\in\mathbb{N}), \tag{1.2}$$

which will be used throughout. For the sake of brevity, here and in what follows, we abbreviate the summations $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}$, $\sum_{i=1}^{m}\sum_{j=1}^{n}$ and $\sum_{i=1}^{n}\sum_{j=1}^{n}$ by $\sum_{i,j}$, $\sum_{i,j=1}^{m,n}$ and $\sum_{i,j=1}^{n}$, respectively. Then the pair (x,s) and the sequence $s=(s_{mn})$ are called as a *double series* and the *sequence of partial sums* of the double series, respectively. Let λ be the space of double sequences, converging with respect to some linear convergence rule $\upsilon-\lim:\lambda\to\mathbb{R}$. The sum of a double series $\sum_{i,j}x_{ij}$ with respect to this rule is defined by $\upsilon-\sum_{i,j}x_{ij}:=\upsilon-\lim s_{mn}$.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t):=\left\{(x_{mn})\in\Omega:\sup_{m,n\in\mathbb{N}}|x_{mn}|^{t_{mn}}<\infty\right\},$$

$$\mathcal{C}_p(t):=\left\{(x_{mn})\in\Omega:p-\lim_{m,n\to\infty}|x_{mn}-l|^{t_{mn}}=0\ \text{ for some } l\in\mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t):=\left\{(x_{mn})\in\Omega:p-\lim_{m,n\to\infty}|x_{mn}|^{t_{mn}}=0\right\},$$

$$\mathcal{L}_u(t):=\left\{(x_{mn})\in\Omega:\sum_{m,n}|x_{mn}|^{t_{mn}}<\infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \quad \text{and } \mathcal{C}_{0bp}(t) := \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,\ n \in \mathbb{N}$. In the case $t_{mn} = 1$ for all $m,\ n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [4, 5] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α-, β-, γ-duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [11] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [8] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [9] and Mursaleen and Edely [10] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series.

In the present paper, we introduce the space $\mathcal{L}_q$

$$\mathcal{L}_q := \left\{ (x_{ij}) \in \Omega : \sum_{i,j} |x_{ij}|^q < \infty \right\}, \ (1 \leq q < \infty)$$

of double sequences corresponding to the space ℓ_q of single sequences and examine some properties of the space.

2. The Double Sequence Space $\mathcal{L}_q$

In this section, we give the theorem which states that $\mathcal{L}_q$ is a sequence space and is a Banach space with the norm $\|\cdot\|_q$, firstly. Subsequent to giving some inclusion relations concerning the space $\mathcal{L}_q$, we establish that the α- and γ-duals of a space of double sequences are identical whenever it is solid, and $\mathcal{L}_q$ is soid if $q > 1$ and determine the $\beta(\upsilon)$-dual of the space $\mathcal{L}_q$ for $\upsilon \in \{p,\ bp,\ r\}$ which coincides with the α- and γ-duals of the space $\mathcal{L}_q$.

Theorem 2.1. *The set $\mathcal{L}_q$ becomes a linear space with the coordinatewise addition and scalar multiplication and $\mathcal{L}_q$ is a Banach space with the norm*

$$\|x\|_q = \left(\sum_{i,j} |x_{ij}|^q\right)^{1/q}, \tag{2.1}$$

where $1 \leq q < \infty$.

Proof. The proof of the first part of the theorem is a routine verification and so we omit the detail.

Furthermore, the statement "*a sequence space ν is a Banach space with the norm $\|\cdot\|_\nu$ if and only if the sequence space $T^{-1}(\nu) = \lambda$ is a Banach space with the norm $\|\cdot\|_\lambda$*" holds by Boos [3, Corollary 6.3.41]. Therefore, the restriction of the transformation defined by (1.1) to the space $\mathcal{L}_q$ which is norm preserving isomorphism yields the fact that $\mathcal{L}_q = T^{-1}(\ell_q)$ is also a Banach space with the norm $\|\cdot\|_q$ defined by (2.1) because of ℓ_q is a Banach space.

This step concludes the proof. □

Theorem 2.2. *Let $1 \leq q < s < \infty$. Then, the inclusions $\mathcal{L}_q \subset \mathcal{L}_s \subset \mathcal{C}_{r0} \subset \mathcal{M}_u$ hold.*

Proof. Let us take any $x = (x_{ij}) \in \mathcal{L}_q$. Then, $\sum_{\max\{i,j\}>n_0} |x_{ij}|^q < \varepsilon < 1$ for sufficiently large $n_0 \in \mathbb{N}$. Since $q < s$, it is obvious that $|x_{ij}|^q \geq |x_{ij}|^s$ for all $i, j \in \mathbb{N}$ such that $\max\{i,j\} > n_0$. Thus,

$$\begin{aligned}
\sum_{i,j} |x_{ij}|^s &= \sum_{i,j=1}^{n_0} |x_{ij}|^s + \sum_{\max\{i,j\}>n_0} |x_{ij}|^s \\
&\leq A + \sum_{\max\{i,j\}>n_0} |x_{ij}|^q \\
&\leq A + \varepsilon,
\end{aligned}$$

which leads us to the fact that $x \in \mathcal{L}_s$, where $A = \sum_{i,j=1}^{n_0} |x_{ij}|^s$. Hence, $\mathcal{L}_q \subset \mathcal{L}_s$.

Besides one can easily deduce, by means of the suitable restrictions of the isomorphism T defined by (1.1) and taking into account the fact that the space $\mathcal{C}_{r0}$ consists of all sequences $x = (x_{mn})$ such that $\lim_{\max\{m,n\}\to\infty} x_{mn} = 0$, from the known inclusions $\ell_s \subset c_0 \subset \ell_\infty$ for $1 \leq s < \infty$ that

$$T^{-1}(\ell_s) = \mathcal{L}_s \subset \mathcal{C}_{r0} = T^{-1}(c_0) \subset T^{-1}(\ell_\infty) = \mathcal{M}_u.$$

This step completes the proof. □

The *α-dual* λ^α, *β(υ)-dual* $\lambda^{\beta(\upsilon)}$ with respect to the υ-convergence for $\upsilon \in \{p, bp, r\}$ and the *γ-dual* λ^γ of a double sequence space λ are respectively defined by

$$\lambda^\alpha := \left\{ (a_{ij}) \in \Omega : \sum_{i,j} |a_{ij} x_{ij}| < \infty \text{ for all } (x_{ij}) \in \lambda \right\},$$

$$\lambda^{\beta(\upsilon)} := \left\{ (a_{ij}) \in \Omega : \upsilon - \sum_{i,j} a_{ij} x_{ij} \text{ exists for all } (x_{ij}) \in \lambda \right\}$$

and

$$\lambda^\gamma := \left\{ (a_{ij}) \in \Omega : \sup_{k,l \in \mathbb{N}} \left| \sum_{i,j=1}^{k,l} a_{ij} x_{ij} \right| < \infty \text{ for all } (x_{ij}) \in \lambda \right\}.$$

It is easy to see for any two spaces λ, μ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$ and $\lambda^\alpha \subset \lambda^\gamma$. Additionally, it is known that the inclusion $\lambda^\alpha \subset \lambda^{\beta(\upsilon)}$ holds while the inclusion $\lambda^{\beta(\upsilon)} \subset \lambda^\gamma$ does not hold, since the υ-convergence of the sequence of partial sums of a double series does not imply its boundedness.

The space λ of double sequences is said to be *solid* if and only if

$$\tilde{\lambda} = \{(u_{kl}) \in \Omega : \exists (x_{kl}) \in \lambda \text{ such that } |u_{kl}| \leq |x_{kl}| \text{ for all } k, l \in \mathbb{N}\} \subset \lambda.$$

The space λ of double sequences is also said to be *monotone* if and only if $m_0\lambda \subset \lambda$, where m_0 is the span of the set of all sequences of zeros and ones and $m_0\lambda = \{ax = (a_{ij}x_{ij}) : a \in m_0,\ x \in \lambda\}$. If λ is monotone, then $\lambda^\alpha = \lambda^{\beta(\upsilon)}$ (cf. Zeltser [11, p. 36]) and λ is monotone whenever λ is solid.

Prior to giving the theorem which asserts that the α- and γ-duals of a solid space of double sequences are identical, we quote two lemmas which are needed in proving the theorem.

Lemma 2.3. [6, Theorem 2, p. 279] *A positive term double series converges to its l.u.b. (that is the l.u.b. of its partial sums) if it is bounded above. Otherwise it diverges to* $+\infty$.

Lemma 2.4. [2, p. 382] *A double series is absolutely convergent if and only if the set*

$$\left\{ \sum_{i,j=1}^{m,n} |x_{ij}| : m, n \in \mathbb{N} \right\}$$

is a bounded set of real numbers.

Now, we may give the theorem

Theorem 2.5. *If a given double sequence space λ is solid, then the equality $\lambda^{\alpha} = \lambda^{\gamma}$ holds.*

Proof. To prove the theorem, it is enough to show that the inclusion $\lambda^{\gamma} \subset \lambda^{\alpha}$ holds. Suppose that the sequence space λ is solid and take any $y = (y_{kl}) \in \lambda^{\gamma}$. Then,

$$\sup_{m,n\in\mathbb{N}} \left| \sum_{k,l=1}^{m,n} x_{kl} y_{kl} \right| < \infty$$

for any $x = (x_{kl}) \in \lambda$. Now, define the sequence $z = (z_{kl})$ via the sequence $x = (x_{kl}) \in \lambda$ by $z_{kl} := x_{kl} sgn(x_{kl} y_{kl})$ for all $k, l \in \mathbb{N}$. Then, $z = (z_{kl}) \in \lambda$ since λ is solid and $|z_{kl}| \leq |x_{kl}|$ for all $k, l \in \mathbb{N}$. Therefore,

$$\begin{aligned} \sup_{m,n\in\mathbb{N}} \sum_{k,l=1}^{m,n} |x_{kl} y_{kl}| &= \sup_{m,n\in\mathbb{N}} \sum_{k,l=1}^{m,n} x_{kl} y_{kl} sgn(x_{kl} y_{kl}) \\ &= \sup_{m,n\in\mathbb{N}} \left| \sum_{k,l=1}^{m,n} y_{kl} z_{kl} \right| < \infty. \end{aligned}$$

This shows that the positive term double series $\sum_{k,l} |x_{kl} y_{kl}|$ is bounded which is convergent by Lemma 2.3. Therefore, one can see by Lemma 2.4 that $(x_{kl} y_{kl})_{k,l\in\mathbb{N}} \in \mathcal{L}_1$. Since $x \in \lambda$ is arbitrary, y must be in λ^{α}, i.e., the inclusion $\lambda^{\gamma} \subset \lambda^{\alpha}$ holds.

This step terminates the proof. □

As an easy consequence of Theorem 2.5, we have

Corollary 2.6. *If λ is solid then $\lambda^{\alpha} = \lambda^{\beta(\upsilon)} = \lambda^{\gamma}$.*

One can easily observe that the double sequence space $\mathcal{L}_q$ is solid, if $q > 1$. This yields to us that the double sequence space $\mathcal{L}_q$ is monotone which implies the fact that the α- and the $\beta(\upsilon)$-duals of the space $\mathcal{L}_q$ are identical.

Now, we may give the theorem on the $\beta(\upsilon)$-dual of the space $\mathcal{L}_q$.

Theorem 2.7. *The $\beta(\upsilon)$-dual of the space $\mathcal{L}_q$ is the space $\mathcal{L}_{q'}$, where $q > 1$ and $q^{-1} + q'^{-1} = 1$.*

Proof. Let $q > 1$ and $q^{-1} + q'^{-1} = 1$. Let us take any $x \in \mathcal{L}_{q'}$ and $y \in \mathcal{L}_q$. Consider the inequalities

$$|x_{mn} y_{mn}| \leq \frac{|x_{mn}|^{q'}}{q'} + \frac{|y_{mn}|^q}{q} \leq |x_{mn}|^{q'} + |y_{mn}|^q$$

satisfied for all $m, n \in \mathbb{N}$. Therefore, we derive that

$$\sum_{m,n} |x_{mn} y_{mn}| \leq \sum_{m,n} |x_{mn}|^{q'} + \sum_{m,n} |y_{mn}|^q < \infty,$$

which leads us to the fact that $x \in \mathcal{L}_q^\alpha$, i.e., the inclusions

$$\mathcal{L}_{q'} \subset \mathcal{L}_q^\alpha \subset \mathcal{L}_q^{\beta(\upsilon)} \tag{2.2}$$

hold.

Conversely, take any $y = (y_{mn}) \in \mathcal{L}_q^{\beta(\upsilon)}$. For establishing the inclusion $\mathcal{L}_q^{\beta(\upsilon)} \subset \mathcal{L}_{q'}$, we use the analogous idea employing by Boos [3, p. 344, Theorem 7.1.11.c] for single sequences. Let us consider the linear functional f_n and the double sequence $y^{[n]}$ defined by

$$\begin{array}{cccc} f_n & : & \mathcal{L}_q & \longrightarrow & \mathbb{R} \\ & & x = (x_{kl}) & \longmapsto & f_n(x) := \sum_{k,l=1}^n x_{kl} y_{kl} \end{array}$$

and

$$y^{[n]} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} & 0 & \cdots \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2n} & 0 & \cdots \\ y_{31} & y_{32} & y_{33} & \cdots & y_{3n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ y_{n1} & y_{n2} & y_{n3} & \cdots & y_{nn} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{bmatrix}$$

for every $n \in \mathbb{N}$. Then, since $y^{[n]} \in \mathcal{L}_{q'}$, we obtain by Hölder's inequality that

$$|f_n(x)| \leq \sum_{k,l=1}^n |x_{kl} y_{kl}| = \sum_{k,l} \left| x_{kl} y_{kl}^{[n]} \right| \leq \|x\|_q \left\| y^{[n]} \right\|_{q'}$$

for each $x = (x_{kl}) \in \mathcal{L}_q$ which yields the continuity of the linear functionals f_n. Therefore, we have

$$\|f_n\| \leq \left\| y^{[n]} \right\|_{q'} \quad \text{for each } n \in \mathbb{N}. \tag{2.3}$$

Let us consider the sequence $x^{(n)} = \{x_{kl}^{(n)}\}_{k,l \in \mathbb{N}}$ to prove the reverse inequality, defined by

$$x_{kl}^{(n)} := \begin{cases} \frac{|y_{kl}|^{q'}}{y_{kl}} & , \quad (\text{if } \ y_{kl} \neq 0, \text{ and } k, l \leq n), \\ 0 & , \quad (\text{otherwise}). \end{cases}$$

Then, it is clear that $x^{(n)} \in \mathcal{L}_q$ and one can see that

$$\left\|x^{(n)}\right\|_q = \left(\sum_{k,l=1}^{n} |y_{kl}|^{(q'-1)q}\right)^{1/q} = \left(\sum_{k,l=1}^{n} |y_{kl}|^{q'}\right)^{1/q} = \left(\left\|y^{[n]}\right\|_{q'}\right)^{q'/q}.$$

This leads us to the consequence for all $n \in \mathbb{N}$ that

$$\frac{\left|f_n\left(x^{(n)}\right)\right|}{\left\|x^{(n)}\right\|_q} = \frac{\sum_{k,l=1}^{n} |y_{kl}|^{q'}}{\left\|x^{(n)}\right\|_q} = \left\|y^{[n]}\right\|_{q'}.$$

Hence,

$$\left\|y^{[n]}\right\|_{q'} \leq \|f_n\| \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

Therefore, we have by (2.3) and (2.4) that

$$\|f_n\| = \left\|y^{[n]}\right\|_{q'} \text{ for all } n \in \mathbb{N}.$$

By applying the Banach-Steinhauss Theorem, one can observe by our hypothesis that the sequence (f_n) of linear functionals converges pointwise. Since $(\mathcal{L}_q, \|\cdot\|_q)$ and $(\mathbb{C}, |\cdot|)$ are the Banach spaces, the linear functional defined by

$$\begin{array}{rccl} f_y & : \quad \mathcal{L}_q & \longrightarrow & \mathbb{R} \\ & x = (x_{kl}) & \longmapsto & f_y(x) := \lim_{n\to\infty} f_n(x) = \sum_{k,l} x_{kl} y_{kl} \end{array}$$

is continuous, and

$$\|f_y\| \leq \sup_{n\in\mathbb{N}} \|f_n\| = \sup_{n\in\mathbb{N}} \left\|y^{[n]}\right\|_{q'} < \infty$$

holds. Thus, we have $y \in \mathcal{L}_{q'}$, because of

$$\|f_y\| \leq \sup_{n\in\mathbb{N}} \left\|y^{[n]}\right\|_{q'} = \sup_{n\in\mathbb{N}} \left(\sum_{k,l=1}^{n} |y_{kl}|^{q'}\right)^{1/q'} = \left(\sum_{k,l} |y_{kl}|^{q'}\right)^{1/q'} < \infty.$$

That is to say that the inclusion

$$\mathcal{L}_q^{\beta(\upsilon)} \subset \mathcal{L}_{q'} \tag{2.5}$$

holds.

By combining the inclusions (2.2) and (2.5), the desired result immediately follows.

This completes the proof. □

As a direct consequence of Theorem 2.7, we have

Corollary 2.8. *The α-, $\beta(\upsilon)$- and γ-duals of the space $\mathcal{L}_q$ are the space $\mathcal{L}_{q'}$, where $q > 1$ and $q^{-1} + q'^{-1} = 1$.*

Acknowledgements

We have benefited a lot from discussions with Professor Bilâl Altay, Matematik Eğitimi Bölümü, İnönü Üniversitesi, Malatya-44280/Türkiye, about this work. We would like to express our gratitude for his valuable helps. Finally, we thank to the reviewer for his/her careful reading and making a useful comment which improved the presentation and the readability of the paper.

References

[1] B. Altay, F. Başar, *Some new spaces of double sequences*, J. Math. Anal. Appl. **309**(1)(2005), 70–90.

[2] R.G. Bartle, *The Elements of Real Analysis*, John Wiley & Sons Inc., New York, 1964.

[3] J. Boos, *Classical and Modern Methods in Summability*, Oxford University Press Inc., New York, 2000.

[4] A. Gökhan, R. Çolak, *The double sequence spaces* $c_2^P(p)$ *and* $c_2^{PB}(p)$, Appl. Math. Comput. **157**(2)(2004), 491–501.

[5] A. Gökhan, R. Çolak, *Double sequence space* $\ell_2^\infty(p)$, ibid. **160**(1)(2005), 147–153.

[6] V.G. Iyer, *Mathematical Analysis*, Tata McGraw-Hill Publishing Company Ltd., New Delhi, 1985.

[7] F. Móricz, *Extensions of the spaces* c *and* c_0 *from single to double sequences,* Acta Math. Hung. **57**(1-2)(1991), 129–136.

[8] Mursaleen, O.H.H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288**(1)(2003), 223–231.

[9] Mursaleen, *Almost strongly regular matrices and a core theorem for double sequences*, J. Math. Anal. Appl. **293**(2)(2004), 523–531.

[10] Mursaleen, O.H.H. Edely, *Almost convergence and a core theorem for double sequences*, J. Math. Anal. Appl. **293**(2)(2004), 532–540.

[11] M. Zeltser, *Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods*, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.

Feyzi Başar
Fatih Üniversitesi,
Fen- Edebiyat Fakültesi,
Matematik Bölümü,
Büyükçekmece Kampüsü,
34500-İstanbul, Türkiye
e-mail address: fbasar@fatih.edu.tr, feyzibasar@gmail.com

Yurdal Sever
Fen Lisesi Matematik Öğretmeni,
Karakavak, 44110-Malatya, Türkiye
e-mail address: ysever@hotmail.com

(*Received May 22, 2007*)
(*Revised June 19, 2007*)