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ON THE FUNDAMENTAL GROUPS OF LOG CONFIGURATION SCHEMES

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Abstract. In the present paper, we study the cuspidalization problem for the fundamental group of a curve by means of the log geometry of the log configuration scheme, which is a natural compactification of the usual configuration space of the curve. The goal of this paper is to show that the fundamental group of the configuration space is generated by the images from morphisms from a group extension of the fundamental groups of the configuration spaces of lower dimension, and that the fundamental group of the configuration space can be partially reconstructed from a collection of data concerning the fundamental groups of the configuration spaces of lower dimension.

1. Introduction

In this paper, we study the cuspidalization problem for the fundamental groups of curves. This problem may be formulated as follows: Let X be a smooth, proper, geometrically connected curve of genus $g \geq 2$ over a field K, and r a natural number. Then the r-th configuration space

$$U_{(r)}$$

of X is, by definition, the complement in the r-th product $\overbrace{X \times_K \cdots \times_K X}^{r}$ of the diagonal divisors $\{(x_1, \cdots, x_r) \in \overbrace{X \times_K \cdots \times_K X}^{r} | x_i = x_j\}$, where $1 \leq i < j \leq r$; moreover, we have projections

$$U_{(r+1)} \longrightarrow U_{(r)}$$

obtained by forgetting the i-th factor, where $1 \leq i \leq r+1$. Roughly speaking, the cuspidalization problem (due to Mochizuki [cf. [11]]) refers the problem of "reconstructing" group-theoretically

the "*output data*" consisting of the fundamental groups

$$\pi_1(U_{(r)})$$

and morphisms

$$\cdots \longrightarrow \pi_1(U_{(r+1)}) \longrightarrow \pi_1(U_{(r)}) \longrightarrow \pi_1(U_{(r-1)}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_1(X) = \pi_1(U_{(1)})$$

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induced by the above projections

from

the "*input data*" consisting of the morphism

$$\pi_1(X) \longrightarrow \pi_1(\mathrm{Spec}\, K) \simeq \mathrm{Gal}(K^{\mathrm{sep}}/K)\,,$$

where K^{sep} is a separable closure of K, induced by the structure morphism of X.

The purpose of this paper is to give an approach to this problem from the viewpoint of log geometry, i.e., we shall make use of the geometry of the log configuration scheme associated to X

$$X^{\log}_{(r)}$$

obtained by equipping a certain canonical compactification $X_{(r)}$ of $U_{(r)}$ with the log structure defined by the divisor with normal crossings $D_{(r)} \stackrel{\text{def}}{=} X_{(r)} \setminus U_{(r)} \subseteq X_{(r)}$. Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of r-pointed stable curves of genus g over K whose r marked points are equipped with an ordering, $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ parametrizing smooth curves, and $\overline{\mathcal{M}}^{\log}_{g,r}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$. Then the log scheme $X^{\log}_{(r)}$ arises as the fiber product of the morphism of stacks $\mathrm{Spec}\, K \to \overline{\mathcal{M}}^{\log}_{g,0}$ determined by the curve X and the morphism of stacks $\overline{\mathcal{M}}^{\log}_{g,r} \to \overline{\mathcal{M}}^{\log}_{g,0}$ obtained by forgetting the marked points. Note that for a set of prime numbers Σ, it follows from the log purity theorem that if $p \notin \Sigma$, where p is the characteristic of K, then the strict open immersion $U_{(r)} \hookrightarrow X^{\log}_{(r)}$ induces an isomorphism of the geometrically pro-Σ fundamental group of $U_{(r)}$ with the geometrically pro-Σ log fundamental group of $X^{\log}_{(r)}$.

The divisor at infinity $D_{(r)} \subseteq X_{(r)}$ admits a decomposition

$$D_{(r)} = \bigcup_{I \subseteq \{1,2,\cdots,r\};\, I^{\sharp} \geq 2} D_{(r)I}$$

by considering the configurations of the r marked points. Our first result is as follows (cf. Theorem 4.1; Remark following Lemma 4.2):

Theorem 1.1. *Let $r \geq 3$ be an integer, and $i, j, k \in \{1, 2, \cdots, r\}$ distinct elements of $\{1, 2, \cdots, r\}$. Then the following hold:*

(i) *The images of the log fundamental groups $\pi_1(D^{\log}_{(r)\{i,j\}})$, $\pi_1(D^{\log}_{(r)\{j,k\}})$, and $\pi_1(D^{\log}_{(r)\{i,j,k\}})$ in $\pi_1(X^{\log}_{(r)})$ topologically generate $\pi_1(X^{\log}_{(r)})$.*

(ii) *If K is of characteristic 0, then there exist exact sequences*

$$1 \longrightarrow \widehat{\mathbb{Z}}(1)(K^{\mathrm{sep}}) \longrightarrow \pi_1(D^{\log}_{(r)\{i,j\}}) \longrightarrow \pi_1(X^{\log}_{(r-1)}) \longrightarrow 1 \ ;$$

$$1 \longrightarrow \widehat{\mathbb{Z}}(1)(K^{\mathrm{sep}}) \longrightarrow \pi_1(D^{\log}_{(r)\{i,j,k\}})$$
$$\longrightarrow \pi_1(X^{\log}_{(r-2)}) \times_{G_K} \pi_1(\mathbb{P}^1_K \setminus \{0,1,\infty\}) \longrightarrow 1 \ .$$

This result can be regarded as a logarithmic analogue of [8], Remark 1.2. Note that even if K is of characteristic $p > 0$, by replacing "log fundamental group" by "geometrically pro-l log fundamental group", where l is a prime number such that $l \neq p$, one can obtain a similar result of (ii) in the statement of the above theorem (cf. Remark following Lemma 4.2).

Our second result (cf. Theorem 5.2) asserts that one can "reconstruct" partially the log fundamental groups of the log configuration schemes of higher dimension from a collection of data concerning the log fundamental groups of the log configuration schemes of lower dimension by means of Theorem 1.1; more precisely, one can construct group-theoretically a profinite group "$\pi_1(X^{\log}_{(r+1)})^{\mathcal{G}}$" and projections

$$q_i : \pi_1(X^{\log}_{(r+1)})^{\mathcal{G}} \longrightarrow \pi_1(X^{\log}_{(r)}) \, ,$$

where $1 \leq i \leq r+1$, in such a way that

> *the morphism q_i factors through the projection $\pi_1(X^{\log}_{(r+1)}) \to \pi_1(X^{\log}_{(r)})$ induced by the projection $X^{\log}_{(r+1)} \to X^{\log}_{(r)}$ obtained by forgetting the i-th factor, and, moreover, the first arrow of the factorization*
>
> $$\pi_1(X^{\log}_{(r+1)})^{\mathcal{G}} \longrightarrow \pi_1(X^{\log}_{(r+1)}) \longrightarrow \pi_1(X^{\log}_{(r)})$$
>
> *of q_i is surjective*

from a collection of data concerning the log fundamental groups of the log configuration schemes $X^{\log}_{(k)}$, where $0 \leq k \leq r$. Here, we use the terminology "reconstruct" as a sort of "abbreviation" for the somewhat lengthy but mathematically precise formulation given in the statement of Theorem 5.2.

By this second result, if one can also reconstruct group-theoretically *the kernel of the surjection* $\pi_1(X^{\log}_{(r+1)})^{\mathcal{G}} \to \pi_1(X^{\log}_{(r+1)})$ which appears as the first arrow in the above factorization of q_i, then by taking the quotient by this kernel, one can reconstruct the desired profinite group $\pi_1(X^{\log}_{(r+1)})$ (cf. Proposition 5.1). However, unfortunately, no reconstruction of this kernel is performed in this paper. Moreover, it seems to the author that if such a reconstruction should prove to be possible, it is likely that the method of reconstruction of this kernel should depend on the *arithmetic* of K in an

essential way. On the other hand, in a subsequent paper [4], we obtain a solution of a *pro-l version* of the cuspidalization problem over a finite field by means of the results obtained in this paper.

This paper is organized as follows:

In Section 2, we define the log configuration schemes of curves and consider the scheme-theoretic and log scheme-theoretic properties of log configuration schemes. In Section 3, we study the geometry of the divisors at infinity of log configuration schemes. In Section 4, we study properties of the log fundamental groups of log configuration schemes and their divisors at infinity by means of the results obtained in Sections 2 and 3. In Section 5, we consider a partial reconstruction of the log fundamental groups of higher dimensional log configuration schemes as discussed above.

Notations and Terminologies

Groups:

Let G be a profinite group, Σ a non-empty set of prime numbers, and n an integer. We shall say that n is a Σ-*integer* if the prime divisors of n are in Σ. We shall refer to the quotient

$$\varprojlim G/H$$

of G, where the projective limit is over all open normal subgroups $H \subseteq G$ such that the index $[G : H]$ of H is a Σ-integer, as the *maximal pro-Σ quotient of* G. We shall denote by $G^{(\Sigma)}$ the maximal pro-Σ quotient of G.

Log schemes:

For a log scheme $X^{\log}$, we shall denote by X (respectively, $\mathcal{M}_X$) the underlying scheme (respectively, the sheaf of monoids that defines the log structure) of $X^{\log}$. For a morphism $f^{\log}$ of log schemes, we shall denote by f the underlying morphism of schemes.

Let $\mathcal{P}$ be a property of schemes [for example, "quasi-compact", "connected", "normal", "regular"] (respectively, morphisms of schemes [for example, "proper", "finite", "étale", "smooth"]). Then we shall say that a log scheme (respectively, a morphism of log schemes) satisfies $\mathcal{P}$ if the underlying scheme (respectively, the underlying morphism of schemes) satisfies $\mathcal{P}$.

For fs log schemes $X^{\log}$, $Y^{\log}$, and $Z^{\log}$, we shall denote by $X^{\log} \times_{Y^{\log}} Z^{\log}$ the fiber product of $X^{\log}$ and $Z^{\log}$ over $Y^{\log}$ in the category of *fs log schemes*. In general, the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is *not naturally isomorphic* to $X \times_Y Z$. However, since strictness (note that a morphism $f^{\log} : X^{\log} \to Y^{\log}$ is called *strict* if the induced morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ on X is an isomorphism) is stable under base-change in the category of

arbitrary log schemes, if $X^{\log} \to Y^{\log}$ is strict, then the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is naturally isomorphic to $X \times_Y Z$. Note that since the natural morphism from the saturation of a fine log scheme to the original fine log scheme is finite, properness and finiteness are stable under fs base-change.

If there exist both schemes and log schemes in a commutative diagram, then we regard each scheme in the diagram as the log scheme obtained by equipping the scheme with the trivial log structure.

We shall refer to the largest open subset (possibly empty) of the underlying scheme of an fs log scheme on which the log structure is trivial as the *interior* of the fs log scheme.

Let $X^{\log}$ and $Y^{\log}$ be log schemes, and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes. Then we shall refer to the quotient of $\mathcal{M}_X$ by the image of the morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ induced by $f^{\log}$ as the *relative characteristic sheaf* of $f^{\log}$. Moreover, we shall refer to the relative characteristic sheaf of the morphism $X^{\log} \to X$ induced by the natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$ as the *characteristic sheaf* of $X^{\log}$.

Fundamental groups:

For a connected scheme X (respectively, log scheme $X^{\log}$) equipped with a geometric point $\overline{x} \to X$ (respectively, log geometric point $\widetilde{x}^{\log} \to X^{\log}$), we shall denote by $\pi_1(X, \overline{x})$ (respectively, $\pi_1(X^{\log}, \widetilde{x}^{\log})$) the fundamental group of X (respectively, log fundamental group of $X^{\log}$). Since one knows that the fundamental group is determined up to inner automorphisms independently of the choice of base-point, we shall often omit the base-point, i.e., we shall often denote by $\pi_1(X)$ (respectively, $\pi_1(X^{\log})$) the fundamental group of X (respectively, log fundamental group of $X^{\log}$).

For a set Σ of prime numbers and a scheme X (respectively, log scheme $X^{\log}$) which is of finite type and geometrically connected over a field K, we shall refer to the quotient of $\pi_1(X)$ (respectively, $\pi_1(X^{\log})$) by the closed normal subgroup obtained as the kernel of the natural projection from $\pi_1(X \otimes_K K^{\mathrm{sep}})$ (respectively, $\pi_1(X^{\log} \otimes_K K^{\mathrm{sep}})$), where K^{sep} is a separable closure of K, to its maximal pro-Σ quotient $\pi_1(X \otimes_K K^{\mathrm{sep}})^{(\Sigma)}$ (respectively, $\pi_1(X^{\log} \otimes_K K^{\mathrm{sep}})^{(\Sigma)}$) as the *geometrically pro-Σ fundamental group* of X (respectively, *geometrically pro-Σ log fundamental group* of $X^{\log}$). Thus, the geometrically pro-Σ fundamental group $\pi_1(X)^{(\Sigma)}$ of X (respectively, geometrically pro-Σ log fundamental group $\pi_1(X^{\log})^{(\Sigma)}$ of $X^{\log}$) fits into the following exact sequence:

$$1 \longrightarrow \pi_1(X \otimes_K K^{\mathrm{sep}})^{(\Sigma)} \longrightarrow \pi_1(X)^{(\Sigma)} \longrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow 1$$

(respectively,

$$1 \longrightarrow \pi_1(X^{\log} \otimes_K K^{\mathrm{sep}})^{(\Sigma)} \longrightarrow \pi_1(X^{\log})^{(\underline{\Sigma})} \longrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow 1).$$

2. Log configuration schemes

In this Section, we define the *log configuration scheme* of a curve over a field and consider the geometry of such log configuration schemes.

Throughout this Section, we shall denote by X a smooth, proper, geometrically connected curve of genus $g \geq 2$ over a field K whose (not necessarily positive) characteristic we denote by p.

Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of r-pointed stable curves of genus g over K whose r marked points are equipped with an ordering, and $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ parametrizing smooth curves (cf. [7]). Let us write $\overline{\mathcal{M}}_g \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g,0}$ and $\mathcal{M}_g \stackrel{\text{def}}{=} \mathcal{M}_{g,0}$. For $1 \leq i \leq r+1$, we shall denote by $p^{\mathcal{M}}_{(r)i} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ the morphism of stacks obtained by forgetting the i-th marked point. Then by considering the morphism of stacks $p^{\mathcal{M}}_{(r)r+1} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$, we obtain a natural isomorphism of $\overline{\mathcal{M}}_{g,r+1}$ with the universal r-pointed stable curve over $\overline{\mathcal{M}}_{g,r}$ (cf. [7], Corollary 2.6). Now the complement $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ is a divisor with normal crossings in $\overline{\mathcal{M}}_{g,r}$ (cf. [7], Theorem 2.7). Let us denote by $\overline{\mathcal{M}}^{\log}_{g,r}$ the fs log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$. Then since a natural action of the symmetric group on r letters $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ given by permuting the marked points preserves the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$, the action of $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ extends to an action on $\overline{\mathcal{M}}^{\log}_{g,r}$.

First, we define the *log configuration scheme* $X^{\log}_{(r)}$ as follows:

Definition 1. Let r be a natural number. Then we shall define $X_{(r)}$ as the fiber product of the classifying morphism of stacks $\mathrm{Spec}\, K \to \overline{\mathcal{M}}_g$ determined by the curve $X \to \mathrm{Spec}\, K$ and the morphism of stacks $\overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_g$ obtained by forgetting the marked points. Since $\overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_g$ is representable, $X_{(r)}$ is a *scheme*. We shall denote by $X^{\log}_{(r)}$ the fs log scheme obtained by equipping $X_{(r)}$ with the log structure induced by the log structure of $\overline{\mathcal{M}}^{\log}_{g,r}$. We shall denote by $U_{X_{(r)}}$ the interior of $X^{\log}_{(r)}$, and by $D_{X_{(r)}}$ the reduced closed subscheme of $X_{(r)}$ obtained as the complement of $U_{X_{(r)}}$ in $X_{(r)}$. Note that by definition, the scheme $U_{X_{(r)}}$ is naturally isomorphic to the usual r-th configuration space of X, and the action of $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ (respectively, $\overline{\mathcal{M}}^{\log}_{g,r}$) determines an action on $X_{(r)}$ (respectively, $X^{\log}_{(r)}$). For simplicity, we shall write $U_{(r)}$ (respectively, $D_{(r)}$) instead of $U_{X_{(r)}}$ (respectively, $D_{X_{(r)}}$) when there is no danger of confusion.

As is well-known, the pull-back of the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ via $p^{\mathcal{M}}_{(r)r+1} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ is a subdivisor of the divisor $\overline{\mathcal{M}}_{g,r+1} \setminus \mathcal{M}_{g,r+1}$ (cf. [7], the proof of Theorem 2.7). Thus, there exists a unique morphism of log stacks $p^{\mathcal{M}\log}_{(r)r+1} : \overline{\mathcal{M}}^{\log}_{g,r+1} \to \overline{\mathcal{M}}^{\log}_{g,r}$ whose underlying morphism of stacks is $p^{\mathcal{M}}_{(r)r+1}$. Moreover, for an integer $1 \leq i \leq r$, since $p^{\mathcal{M}}_{(r)i}$ is obtained as the composite of the automorphism of $\overline{\mathcal{M}}_{g,r+1}$ determined by the action of an element of $\mathcal{S}_r$ and $p^{\mathcal{M}}_{(r)r+1}$, the morphism of stacks $p^{\mathcal{M}}_{(r)i}$ also extends to a morphism of log stacks $\overline{\mathcal{M}}^{\log}_{g,r+1} \to \overline{\mathcal{M}}^{\log}_{g,r}$. We shall denote this morphism of log stacks by $p^{\mathcal{M}\log}_{(r)i}$.

Definition 2. Let r be a natural number.

(i) Let $1 \leq i \leq r+1$ be an integer. Then $p^{\mathcal{M}}_{(r)i} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ (respectively, $p^{\mathcal{M}\log}_{(r)i} : \overline{\mathcal{M}}^{\log}_{g,r+1} \to \overline{\mathcal{M}}^{\log}_{g,r}$) determines a morphism $X_{(r+1)} \to X_{(r)}$ (respectively, $X^{\log}_{(r+1)} \to X^{\log}_{(r)}$). We shall denote this morphism by $p_{X_{(r)}i}$ (respectively, $p^{\log}_{X_{(r)}i}$). Note that by the definition of stable curves, $p_{X_{(r)}i}$ is proper, flat, geometrically connected, and geometrically reduced. For simplicity, we shall write $p_{(r)i}$ (respectively, $p^{\log}_{(r)i}$) instead of $p_{X_{(r)}i}$ (respectively, $p^{\log}_{X_{(r)}i}$) when there is no danger of confusion.

(ii) Let $I = \{i_1, i_2, \cdots, i_{I^\sharp}\}$, where $i_1 < i_2 < \cdots < i_{I^\sharp}$, be a non-empty subset of $\{1, 2, \cdots, r\}$. Then we shall denote by

$$\mathrm{pr}^{\log}_{X_{(r)}I} : X^{\log}_{(r)} \longrightarrow X^{\log}_{(r-I^\sharp)}$$

the morphism obtained as the compactification of the projection $U_{X_{(r)}} \to U_{X_{(r-I^\sharp)}}$ given by mapping $(x_1, \cdots, x_r)$ to $(x_{i_1}, \cdots, x_{i_{I^\sharp}})$, i.e., if the morphism $U_{X_{(r)}} \to U_{X_{(r-I^\sharp)}}$ which induces a morphism

$$\begin{array}{ccc} U_{X_{(r)}}(S) & \longrightarrow & U_{X_{(r-I^\sharp)}}(S) \\ (x_1, \cdots, x_r) & \mapsto & (x_{i_1}, \cdots, x_{i_{I^\sharp}}) \end{array}$$

for any scheme S over K, where x_i is an S-valued point of X, is given as the compsite

$$p_{U_{X_{(r-I^\sharp)}}} j_{r-I^\sharp} \circ p_{U_{X_{(r-I^\sharp+1)}}} j_{r-I^\sharp+1} \circ \cdots \circ p_{U_{X_{(r-2)}}} j_{r-2} \circ p_{U_{X_{(r-1)}}} j_{r-1} ,$$

where $p_{U_{X_{(i)}}} j$ is the projection $U_{X_{(i+1)}} \to U_{X_{(i)}}$ obtained by forgetting the j-th factor, then we shall write

$$\mathrm{pr}^{\log}_{X_{(r)}I} \stackrel{\mathrm{def}}{=} p^{\log}_{X_{(r-I^\sharp)}} j_{r-I^\sharp} \circ p^{\log}_{X_{(r-I^\sharp+1)}} j_{r-I^\sharp+1} \circ \cdots \circ p^{\log}_{X_{(r-2)}} j_{r-2} \circ p^{\log}_{X_{(r-1)}} j_{r-1} .$$

Note that it is immediate that this composite only depends on I, i.e., this composite is independent of the choice of the sequence $(j_{r-I^\sharp}, \cdots, j_{r-1})$. For simplicity, we shall write $\mathrm{pr}^{\log}_{(r)I}$ instead of $\mathrm{pr}^{\log}_{X_{(r)}I}$ when there is no danger of confusion.

Next, let us consider the scheme-theoretic and log scheme-theoretic properties of $X^{\log}_{(r)}$ in more detail.

Proposition 2.1. *Let r be a natural number, $1 \leq i \leq r+1$ an integer, and I a non-empty subset of $\{1, 2, \cdots, r\}$.*

(i) *The scheme $X_{(r)}$ is connected.*
(ii) *The morphism $p^{\log}_{(r)i}$ is log smooth. In particular, $\mathrm{pr}^{\log}_{(r)I}$ is also log smooth.*
(iii) *The log scheme $X^{\log}_{(r)}$ is log regular.*
(iv) *The scheme $X_{(r)}$ is regular, and the log structure of $X^{\log}_{(r)}$ is given by a divisor with normal crossings.*

Proof. Assertion (i) follows from the fact that $X_{(0)} = \mathrm{Spec}\, K$ is connected, together with the fact that the $p_{(r)i}$'s are proper and geometrically connected.

Next, we prove assertion (ii). The assertion for $p^{\log}_{(r)r+1}$ follows from the fact that $p^{\mathcal{M}\log}_{(r)r+1} : \overline{\mathcal{M}}^{\log}_{g,r+1} \to \overline{\mathcal{M}}^{\log}_{g,r}$ is log smooth (cf. [5], Section 4). Since $p^{\log}_{(r)i}$ is a composite of an automorphism of $X^{\log}_{(r)}$ obtained by permuting of the marked points and $p^{\log}_{(r)r+1}$, the morphism $p^{\log}_{(r)i}$ is also log smooth.

Assertion (iii) follows from assertion (ii), together with the log regularity of the log scheme $X^{\log}_{(0)} = \mathrm{Spec}\, K$.

Finally, we prove assertion (iv). Since the natural morphism $X^{\log}_{(r)} \to \overline{\mathcal{M}}^{\log}_{g,r}$ is strict, for any geometric point $\overline{x} \to X_{(r)}$, the stalk $(\mathcal{M}_{X_{(r)}}/\mathcal{O}^*_{X_{(r)}})_{\overline{x}}$ of the characteristic sheaf of $X^{\log}_{(r)}$ at $\overline{x} \to X_{(r)}$ is isomorphic to $\mathbb{N}^{\oplus n}$ for some natural number n. Thus, assertion (iv) follows from assertion (iii). $\square$

Proposition 2.2. *Let r be a natural number, $1 \leq i \leq r+1$ an integer, and $\overline{x}^{\log} \to X^{\log}_{(r)}$ a strict geometric point, i.e., a strict morphism whose underlying morphism of schemes is a geometric point (cf.* [3]*, Definition 1.1,* (i)*). Then the following sequence is exact:*

$$\varprojlim \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}^{\log}_{\lambda}) \overset{\text{via } \mathrm{pr}_1}{\longrightarrow} \pi_1(X^{\log}_{(r+1)}) \overset{\text{via } p^{\log}_{(r)i}}{\longrightarrow} \pi_1(X^{\log}_{(r)}) \longrightarrow 1\,.$$

Here, the projective limit is over all reduced covering points $\overline{x}_\lambda^{\log} \to \overline{x}^{\log}$, *i.e., a morphism obtained as the composite of a connected Kummer finite log étale covering* $(\overline{x}'_\lambda)^{\log} \to \overline{x}^{\log}$ *and the strict morphism* $\overline{x}_\lambda^{\log} \to (\overline{x}'_\lambda)^{\log}$ *whose underlying morphism of schemes is the closed immersion* $\overline{x}_\lambda \stackrel{\text{def}}{=} (\overline{x}'_\lambda)_{\text{red}} \hookrightarrow \overline{x}'_\lambda$ *defined by the ideal generated by the nilpotent elements of* $\mathcal{O}_{\overline{x}'_\lambda}$ (*cf.* [3], *Definition 1.1,* (ii)).

Proof. This follows immediately from Proposition 2.1; [3], Theorem 2.3. □

3. Divisors at infinity of log configuration schemes

We continue with the notation of the preceding Section. In this Section, we consider the scheme-theoretic and log scheme-theoretic properties of the divisors defining the log structures of the log configuration schemes.

Definition 3. Let $r \geq 2$ be an integer, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^{\#} \geq 2$ equipped with the natural ordering. Then we shall denote by

$$(C_{(r)I} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1})$$

the r-pointed stable curve of genus g obtained by applying the clutching morphism of stacks (cf. [7], Definition 3.8)

$$\beta_{g,0,\{1,2,\cdots,r\}\setminus I,I} : \overline{\mathcal{M}}_{g,r-I^{\#}+1} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to \overline{\mathcal{M}}_{g,r}\,,$$

where $\{1, 2, \cdots, r\} \setminus I$ is equipped with the natural ordering, to the $(r - I^{\#} + 1)$-pointed stable curve of genus g

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1}$$

obtained by base-changing $p_{X_{(r-I^{\#}+1)}r-I^{\#}+2} : X_{(r-I^{\#}+2)} \to X_{(r-I^{\#}+1)}$ and the $(I^{\#} + 1)$-pointed stable curve of genus 0

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1}$$

obtained by base-changing the universal curve $\overline{\mathcal{M}}_{0,I^{\#}+2} \to \overline{\mathcal{M}}_{0,I^{\#}+1}$ over $\overline{\mathcal{M}}_{0,I^{\#}+1}$. Note that the clutching locus of

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1}$$

$$(\text{respectively, } X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1})$$

is the $(r - I^{\#} + 1)$-st (respectively, $(I^{\#} + 1)$-st) section (cf. [7], Definition 3.8).

Then it is immediate that the classifying morphism of stacks $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to \overline{\mathcal{M}}_{g,r}$ determined by the r-pointed stable curve of genus g

$$(C_{(r)I} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1})$$

factors through $X_{(r)}$, and the resulting morphism $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to X_{(r)}$ is a closed immersion since it is a proper monomorphism. We shall denote by $\delta_{X_{(r)}I}$ this closed immersion, by $D_{X_{(r)}I}$ the scheme-theoretic image of $\delta_{X_{(r)}I}$, by $D^{\log}_{X_{(r)}I}$ the log scheme obtained by equipping $D_{X_{(r)}I}$ with the log structure induced by the log structure of $X^{\log}_{(r)}$, and by $\delta^{\log}_{X_{(r)}I} : D^{\log}_{X_{(r)}I} \to X^{\log}_{(r)}$ the strict closed immersion whose underlying morphism of schemes is $\delta_{X_{(r)}I}$. For simplicity, we shall write $D_{(r)I}$ (respectively, $D^{\log}_{(r)I}$; respectively, $\delta_{(r)I}$; respectively, $\delta^{\log}_{(r)I}$) instead of $D_{X_{(r)}I}$ (respectively, $D^{\log}_{X_{(r)}I}$; respectively, $\delta_{X_{(r)}I}$; respectively, $\delta^{\log}_{X_{(r)}I}$) when there is no danger of confusion.

Proposition 3.1. *Let $r \geq 2$; $1 \leq i \leq r+1$ be integers, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^{\#} \geq 2$.*

(i) *The scheme $D_{(r)I}$ is irreducible.*

(ii) *The divisor $D_{(r)}$ of $X_{(r)}$ is $\bigcup_J D_{(r)J}$, where J ranges over the subsets of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2.*

(iii) *The closed subscheme of $X_{(r+1)}$ determined by the composite of the natural closed immersions defined in Definition 3*

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \hookrightarrow C_{(r)I} \hookrightarrow X_{(r+1)}$$

$$\text{(respectively, } X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \hookrightarrow C_{(r)I} \hookrightarrow X_{(r+1)}\text{)}$$

is $D_{(r+1)I\cup\{r+1\}}$ (respectively, $D_{(r+1)I}$).

(iv) *Let J be a subset of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2. Then $D_{(r)I} \cap D_{(r)J} \neq \emptyset$ if and only if $I \subseteq J$, $J \subseteq I$, or $I \cap J = \emptyset$.*

(v) *The inverse image of $D_{(r)I} \subseteq X_{(r)}$ via $p_{(r)i}$ is $D_{(r+1)(I\cup\{r+1\})^{\delta_i}} \cup D_{(r+1)I^{\delta_i}}$, where*

$$\delta_i = ((1, 2, \cdots, r+1) \mapsto (1, 2, \cdots, i-1, i+1, \cdots, r, r+1, i)) \in \mathcal{S}_{r+1}\,,$$

and $I^{\delta_i} = \{\delta_i(k) \mid k \in I\}$.

Proof. First, we prove assertion (i). The log smoothness of the morphism $p^{\log}_{(s)s+1} : X^{\log}_{(s+1)} \to X^{\log}_{(s)}$ and the morphism $\overline{\mathcal{M}}^{\log}_{0,t+4} \to \overline{\mathcal{M}}^{\log}_{0,t+3}$ obtained by forgetting the $(t+4)$-th marked point, where s, t are natural numbers, imply the log regularity of $X^{\log}_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^{\#}+1}$; therefore, $D_{(r)I}$ is normal (cf. [6], Theorem 4.1). Moreover, by a similar argument to the argument used in the proof of Proposition 2.1, (i), $D_{(r)I}$ is connected. Thus, in light of the normality just observed, $D_{(r)}$ is irreducible.

Assertions (ii), (iii), (iv), and (v) follow from the construction of $D_{(r)I}$. $\square$

Proposition 3.2. *Let $r \geq 2$ be an integer, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^\sharp \geq 2$. Then the morphism $D^{\log}_{(r)I} \to D_{(r)I} = X_{(r-I^\sharp+1)} \times_K \overline{\mathcal{M}}_{0,I^\sharp+1}$ induced by the natural inclusion $\mathcal{O}^*_{D_{(r)I}} \hookrightarrow \mathcal{M}_{D_{(r)I}}$ determines a morphism*

$$\nu^{\log}_{X_{(r)I}} : D^{\log}_{(r)I} \longrightarrow X^{\log}_{(r-I^\sharp+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^\sharp+1}$$

which is of type $\mathbb{N}$, i.e., the underlying morphism of schemes is an isomorphism, and the relative characteristic sheaf is locally constant with stalk isomorphic to $\mathbb{N}$ (cf. [3], Definition 4.1). Moreover, let $I^{[i]}$ be the unique subset of $\{1, 2, \cdots, r-1\}$ such that for $j \in \{1, 2, \cdots, r-1\}$, $j \in I^{[i]}$ if and only if

$$\begin{cases} j \in I & (\text{if } j < i) \\ j+1 \in I & (\text{if } j \geq i)\,. \end{cases}$$

Then the following hold:

(i) *If $i \in I$, then $(I^{[i]})^\sharp = I^\sharp - 1$, and the diagram*

$$\begin{array}{ccccc} X^{\log}_{(r-I^\sharp+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^\sharp+1} & \xleftarrow{\nu^{\log}_{X_{(r)I}}} & D^{\log}_{(r)I} & \xrightarrow{\delta^{\log}_{(r)I}} & X^{\log}_{(r)} \\ \downarrow & & \downarrow{\scriptstyle \text{via } p^{\log}_{(r-1)i}} & & \downarrow{\scriptstyle p^{\log}_{(r-1)i}} \\ X^{\log}_{(r-I^\sharp+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^\sharp} & \xleftarrow[\nu^{\log}_{X_{(r-1)I^{[i]}}}]{} & D^{\log}_{(r-1)I^{[i]}} & \xrightarrow[\delta^{\log}_{(r-1)I^{[i]}}]{} & X^{\log}_{(r-1)} \end{array}$$

commutes, where if $I = \{i_1, i_2, \cdots, i_{I^\sharp}\}$, $i_1 < i_2 < \cdots < i_{I^\sharp}$, and $i = i_j$, then the left-hand vertical arrow is the morphism induced by the morphism $\overline{\mathcal{M}}^{\log}_{0,I^\sharp+1} \to \overline{\mathcal{M}}^{\log}_{0,I^\sharp}$ obtained by forgetting the j-th marked point.

(ii) *If $i \notin I$, then $(I^{[i]})^\sharp = I^\sharp$, and the diagram*

$$\begin{array}{ccccc} X^{\log}_{(r-I^\sharp+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^\sharp+1} & \xleftarrow{\nu^{\log}_{X_{(r)I}}} & D^{\log}_{(r)I} & \xrightarrow{\delta^{\log}_{(r)I}} & X^{\log}_{(r)} \\ \downarrow & & \downarrow{\scriptstyle \text{via } p^{\log}_{(r-1)i}} & & \downarrow{\scriptstyle p^{\log}_{(r-1)i}} \\ X^{\log}_{(r-I^\sharp)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^\sharp+1} & \xleftarrow[\nu^{\log}_{X_{(r-1)I^{[i]}}}]{} & D^{\log}_{(r-1)I^{[i]}} & \xrightarrow[\delta^{\log}_{(r-1)I^{[i]}}]{} & X^{\log}_{(r-1)} \end{array}$$

commutes, where if $\{1, 2, \cdots, r\} \setminus I = \{i_1, i_2, \cdots, i_{r-I^\sharp}\}$, $i_1 < i_2 < \cdots < i_{r-I^\sharp}$, and $i = i_j$, then the left-hand vertical arrow is the morphism induced by $p^{\log}_{(r-I^\sharp)j} : X^{\log}_{(r-I^\sharp+1)} \to X^{\log}_{(r-I^\sharp)}$.

Proof. By the definition of $D^{\log}_{(r)I}$, the log scheme obtained by forgetting the portion of the log structure of $D^{\log}_{(r)I}$ defined by the divisor $D_{(r)I}$ of $X_{(r)}$ is isomorphic to $X^{\log}_{(r-I^\sharp+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^\sharp+1}$; moreover, it follows from [3], Lemma 4.12, (i), together with the definition of the log structure of $D^{\log}_{(r)I}$, that the morphism $D^{\log}_{(r)I} \to D_{(r)I}$ induced by the natural inclusion $\mathcal{O}^*_{D_{(r)I}} \hookrightarrow \mathcal{M}_{D_{(r)I}}$ factors through the morphism obtained by forgetting the portion of the log structure of $D^{\log}_{(r)I}$ defined by the divisor $D_{(r)I}$ of $X_{(r)}$. Thus, we obtain a morphism $D^{\log}_{(r)I} \to X^{\log}_{(r-I^\sharp+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^\sharp+1}$. On the other hand, by construction, it is a morphism of type $\mathbb{N}$.

The assertion that the diagrams in the statement of Proposition 3.2 commute follows from Proposition 3.1, (v), together with the definitions of $D^{\log}_{(r)I}$ and $\nu^{\log}_{X_{(r)}I}$.

□

Definition 4. Let $r \geq 2$ be an integer, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^\sharp \geq 2$. Then we shall denote by $\nu_{X_{(r)}I}$ the underlying morphism of schemes of the morphism $\nu^{\log}_{X_{(r)}I}$, and by $U_{X_{(r)}I}$ the open subscheme of $D_{X_{(r)}I}$ determined by the open immersion

$$U_{X_{(r-I^\sharp+1)}} \times_K \mathcal{M}_{0,I^\sharp+1} \hookrightarrow X_{(r-I^\sharp+1)} \times_K \overline{\mathcal{M}}_{0,I^\sharp+1} \overset{\nu^{-1}_{X_{(r)}I}}{\overset{\sim}{\to}} D_{X_{(r)}I}\,.$$

For simplicity, we shall write $\nu^{\log}_{(r)I}$ (respectively, $\nu_{(r)I}$; respectively, $U_{(r)I}$) instead of $\nu^{\log}_{X_{(r)}I}$ (respectively, $\nu_{X_{(r)}I}$; respectively, $U_{X_{(r)}I}$) when there is no danger of confusion.

4. Fundamental groups of the log configuration schemes

We continue with the notation of the preceding Section. In this Section, we study fundamental facts concerning the log fundamental groups of log configuration schemes and their divisors at infinity. Let Σ be a non-empty set of prime numbers. We shall fix a separable closure K^{sep} of K and denote by G_K the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K. Moreover, we shall denote by Λ the maximal pro-Σ quotient of $\widehat{\mathbb{Z}}(1)(K^{\mathrm{sep}})$.

Definition 5. Let r be a natural number. Then we shall denote by $\Pi^{\log}_{X_{(r)}}$ (respectively, $\Pi^{\log}_{\mathcal{M}_{r+3,K}}$) the geometrically pro-$\Sigma$ log fundamental group of $X^{\log}_{(r)}$ (respectively, the log *scheme* $\overline{\mathcal{M}}^{\log}_{0,r+3}$) and write $\Pi_X \overset{\mathrm{def}}{=} \Pi^{\log}_{X_{(1)}}$, and

$\Pi^{\log}_{\mathbb{P}_K} \stackrel{\text{def}}{=} \Pi^{\log}_{\mathcal{M}_{4,K}}$. Moreover, if $r \geq 2$, then for a subset I of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2, we shall denote by $\Pi^{\log}_{X_{(r)I}}$ the geometrically pro-Σ log fundamental group of $D^{\log}_{X_{(r)I}}$. For simplicity, we shall write $\Pi^{\log}_{(r)}$ (respectively, $\Pi^{\log}_{\mathcal{M}_{r+3}}$; respectively, $\Pi^{\log}_{\mathbb{P}}$; respectively, $\Pi^{\log}_{(r)I}$) instead of $\Pi^{\log}_{X_{(r)}}$ (respectively, $\Pi^{\log}_{\mathcal{M}_{r+3,K}}$; respectively, $\Pi^{\log}_{\mathbb{P}_K}$; respectively, $\Pi^{\log}_{X_{(r)}I}$) when there is no danger of confusion.

First, we show the following theorem.

Theorem 4.1. *Let X be a smooth, proper, geometrically connected curve of genus ≥ 2 over a field K, $r \geq 3$ an integer, and $i, j, k \in \{1, 2, \cdots, r\}$ distinct elements of $\{1, 2, \cdots, r\}$. Then, for any non-empty set of prime numbers Σ, the geometrically pro-Σ log fundamental group of $X^{\log}_{(r)}$ is topologically generated by the respective images of the geometrically pro-Σ log fundamental groups of $D^{\log}_{(r)\{i,j\}}$, $D^{\log}_{(r)\{j,k\}}$, and $D^{\log}_{(r)\{i,j,k\}}$.*

Proof. It is immediate that it is enough to show the assertion in the case where Σ is the set of all prime numbers; thus, assume that Σ is the set of all prime numbers. Moreover, it is immediate that it is enough to show the assertion in the case where $\{i, j, k\} = \{1, 2, 3\}$; thus, assume that $\{i, j, k\} = \{1, 2, 3\}$.

It follows that the composite $D^{\log}_{(r)\{2,3\}} \stackrel{\delta^{\log}_{(r)\{2,3\}}}{\rightarrow} X^{\log}_{(r)} \stackrel{p^{\log}_{(r-1)3}}{\rightarrow} X^{\log}_{(r-1)}$ coincides with $\nu^{\log}_{(r)\{2,3\}}$; in particular, this composite is of type $\mathbb{N}$ (cf. Proposition 3.2). Therefore, the composite $\Pi^{\log}_{(r)\{2,3\}} \stackrel{\text{via } \delta^{\log}_{(r)\{2,3\}}}{\rightarrow} \Pi^{\log}_{(r)} \stackrel{\text{via } p^{\log}_{(r-1)3}}{\rightarrow} \Pi^{\log}_{(r-1)}$ is surjective (cf. [3], Lemma 4.5). Thus, it is enough to show that the respective images of $\Pi^{\log}_{(r)\{1,2\}}$ and $\Pi^{\log}_{(r)\{1,2,3\}}$ in $\Pi^{\log}_{(r)}$ topologically generate the kernel of the morphism $\Pi^{\log}_{(r)} \rightarrow \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$. Let $\overline{x}^{\log} \rightarrow X^{\log}_{(r-1)}$ be a strict geometric point of $X^{\log}_{(r-1)}$ whose image lies in $U_{(r-1)\{1,2\}}$. Then it follows from Proposition 2.2 that the kernel of the morphism $\Pi^{\log}_{(r)} \rightarrow \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$ is included in the image of the natural morphism $\pi_1(X^{\log}_{(r)\overline{x}^{\log}}) \rightarrow \Pi^{\log}_{(r)}$, where $X^{\log}_{(r)\overline{x}^{\log}}$ is the fs log scheme obtained as the fiber product of $p^{\log}_{(r-1)3} : X^{\log}_{(r)} \rightarrow X^{\log}_{(r-1)}$ and $\overline{x}^{\log} \rightarrow X^{\log}_{(r-1)}$. On the other hand, it follows from the definition of the open subscheme $U_{(r-1)\{1,2\}} \subseteq D_{(r-1)\{1,2\}}$, together with Proposition 3.1, (v), that the log schemes obtained by equipping the irreducible components of $X_{(r)\overline{x}^{\log}}$ with the log structures induced by the log structure of $X^{\log}_{(r)\overline{x}^{\log}}$ are the log schemes

obtained as the respective fiber products of the diagrams

$$\begin{array}{ccc} & & D^{\log}_{(r)\{1,2\}} \\ & & \Big\downarrow \text{via } p^{\log}_{(r-1)3} \\ \overline{x}^{\log} & \longrightarrow & X^{\log}_{(r-1)} \end{array} \quad ; \quad \begin{array}{ccc} & & D^{\log}_{(r)\{1,2,3\}} \\ & & \Big\downarrow \text{via } p^{\log}_{(r-1)3} \\ \overline{x}^{\log} & \longrightarrow & X^{\log}_{(r-1)} \end{array} .$$

Therefore, the assertion follows from the logarithmic analogue of [13], Example 5.5 (cf. also [13], §5.5). □

Remark. Theorem 4.1 can be regarded as a logarithmic analogue of [8], Remark 1.2.

Definition 6. We shall say that a set of prime numbers Σ is *innocuous in* K if

$$\Sigma = \begin{cases} \text{the set of all prime numbers or } \{l\} & \text{if } p = 0 \\ \{l\} & \text{if } p \geq 2\,, \end{cases}$$

where l is a prime number which is invertible in K.

In the rest of this paper, we assume that

Σ *is innocuous in* K.

Lemma 4.2. *Let r, r' be natural numbers, and I a subset of $\{1, 2, \cdots r\}$ of cardinality $I^{\sharp} \geq 2$.*

(i) *The natural morphism $U_{(r)} \times_K \mathcal{M}_{0,r'+3} \to X^{\log}_{(r)} \times_K \overline{\mathcal{M}}^{\log}_{r'+3}$ induces an isomorphism $\pi_1(U_{(r)} \times_K \mathcal{M}_{0,r'+3})^{(\underline{\Sigma})} \xrightarrow{\sim} \Pi^{\log}_{(r)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{r'+3}}$, where "$\pi_1(-)^{(\underline{\Sigma})}$" is the geometrically pro-Σ fundamental group of "$(-)$". In particular, the natural morphism $U_{(r)} \to X^{\log}_{(r)}$ (respectively, $U_{(r)I} \to X^{\log}_{(r-I^{\sharp}+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^{\sharp}+1}$) induces an isomorphism*

$$\pi_1(U_{(r)})^{(\underline{\Sigma})} \xrightarrow{\sim} \Pi^{\log}_{(r)}$$

(respectively, $\pi_1(U_{(r)I})^{(\underline{\Sigma})} \xrightarrow{\sim} \Pi^{\log}_{(r-I^{\sharp}+1)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{I^{\sharp}+1}}$).

(ii) *Let $1 \leq i \leq r+1$ (respectively, $1 \leq i \leq r+4$) be an integer, and $\overline{x} \to U_{(r)}$ (respectively, $\overline{x} \to \mathcal{M}_{0,r+3}$) a geometric point of $U_{(r)}$ (respectively, $\mathcal{M}_{0,r+3}$). Then the sequence*

$$1 \longrightarrow \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x})^{(\Sigma)} \xrightarrow{\text{via pr}_1} \Pi^{\log}_{(r+1)} \longrightarrow \Pi^{\log}_{(r)} \longrightarrow 1$$

(respectively,

$$1 \longrightarrow \pi_1(\overline{\mathcal{M}}^{\log}_{0,r+4} \times_{\overline{\mathcal{M}}^{\log}_{0,r+3}} \overline{x})^{(\Sigma)} \xrightarrow{\text{via pr}_1} \Pi^{\log}_{\mathcal{M}_{r+4}} \longrightarrow \Pi^{\log}_{\mathcal{M}_{r+3}} \longrightarrow 1\,)$$

is exact, where the third arrow is the morphism induced by the morphism $p^{\log}_{(r)i}$ (*respectively,* $\overline{\mathcal{M}}^{\log}_{0,r+4} \to \overline{\mathcal{M}}^{\log}_{0,r+3}$ *obtained by forgetting the* i*-th marked point*).

(iii) *For a profinite group* Γ (*respectively, a scheme* S), *we shall denote by* $\mathcal{S}(\Gamma)$ (*respectively,* $S_{\text{ét}}$) *the classifying site of* Γ, *i.e., the site defined by considering the category of finite sets equipped with a continuous action of* Γ (*respectively, the étale site of* S). *Then we have natural morphisms of sites*

$$(U_{(r)} \times_K \mathcal{M}_{0,r'+3})_{\text{ét}} \longrightarrow \mathcal{S}(\pi_1(U_{(r)} \times_K \mathcal{M}_{0,r'+3})^{(\underline{\Sigma})}) \longleftarrow \mathcal{S}(\Pi^{\log}_{(r)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{r'+3}}).$$

Let A *be a finite* $\Pi^{\log}_{(r)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{r'+3}}$*-module whose order is a* Σ*-integer, and* n *an integer. Then the morphisms*

$$H^n_{\text{ét}}(U_{(r)} \times_K \mathcal{M}_{0,r'+3}, \mathcal{F}_A) \longleftarrow H^n(\pi_1(U_{(r)} \times_K \mathcal{M}_{0,r'+3})^{(\underline{\Sigma})}, A) \longrightarrow H^n(\Pi^{\log}_{(r)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{r'+3}}, A)$$

induced by the above morphisms of sites are isomorphisms, where $\mathcal{F}_A$ *is the locally constant sheaf on* $U_{(r)} \times_K \mathcal{M}_{0,r'+3}$ *determined by* A.

Proof. Assertion (i) follows immediately from the log purity theorem (cf. [9], Theorem 3.3, also [3], Remark 1.10), together with [3], Proposition 2.4, (ii).

Next, we prove assertion (ii). To prove assertion (ii), by base-changing, we may assume that K is a separably closed field. Moreover, if Σ is the set of all prime numbers, then this follows from [8], Lemma 2.4, together with assertion (i). Thus, we may assume that $\Sigma = \{l\}$ for a prime number l which is invertible in K. Then, to prove assertion (ii), it follows from [8], Lemma 3.1, (i), the assertion in the case where Σ is the set of all prime numbers, together with [1], Proposition 3, that it is enough to show that the representation

$$\pi_1(U_{(r)}) \longrightarrow \mathrm{Aut}((\pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x})^{(\Sigma)})^{\mathrm{ab}})$$

$$(\text{respectively, } \pi_1(\overline{\mathcal{M}}^{\log}_{0,r+3}) \longrightarrow \mathrm{Aut}((\pi_1(\overline{\mathcal{M}}^{\log}_{0,r+4} \times_{\overline{\mathcal{M}}^{\log}_{0,r+3}} \overline{x})^{(\Sigma)})^{\mathrm{ab}}))$$

determined by the exact sequence appearing in the statement of assertion (ii) in the case where Σ is the set of all prime numbers factors through a pro-Σ quotient of $\pi_1(U_{(r)})$ (respectively, $\pi_1(\overline{\mathcal{M}}^{\log}_{0,r+3})$). On the other hand, it is immediate that this representation is *trivial*. This completes the proof of assertion (ii).

Finally, we prove assertion (iii). The assertion that the second morphism in question is an isomorphism follows immediately from assertion (i); thus,

we prove the assertion that the first morphism in question is an isomorphism. We prove this assertion by induction on $r + r'$. If $r + r' = 0$, then the assertion is well-known. If $r \neq 0$, then by considering the partial compactification $U_{(r-1)} \times_K X \times_K \mathcal{M}_{0,r'+3} \overset{(\mathrm{pr}_1,\mathrm{pr}_3)}{\to} U_{(r-1)} \times_K \mathcal{M}_{0,r'+3}$ of $(p_{(r-1)r}, \mathrm{id})|_{U_{(r)} \times_K \mathcal{M}_{0,r'+3}} : U_{(r)} \times_K \mathcal{M}_{0,r'+3} \to U_{(r-1)} \times_K \mathcal{M}_{0,r'+3}$, it follows from [2], Corollary 10.3, that the sheaf

$$\mathcal{F}' \overset{\text{def}}{=} \mathbb{R}^q((p_{(r-1)r}, \mathrm{id})|_{U_{(r)} \times_K \mathcal{M}_{0,r'+3}})_* \mathcal{F}_A$$

is locally constant and constructible; moreover, it follows from [2], Theorem 7.3, that the $\Pi^{\log}_{(r-1)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{r'+3}}$-module $\mathcal{F}'_{\overline{x}}$ is naturally isomorphic to $H^q(U, \mathcal{F}_A|_U)$, where $\overline{x} \to U_{(r-1)} \times_K \mathcal{M}_{0,r'+3}$ is a geometric point of $U_{(r-1)} \times_K \mathcal{M}_{0,r'+3}$, and $U \overset{\text{def}}{=} U_{(r)} \times_{U_{(r-1)}} \overline{x}$. On the other hand, the natural morphism $H^n(\pi, A) \to H^n_{\text{ét}}(U, \mathcal{F}_A|_U)$ is an isomorphism, where $\pi \overset{\text{def}}{=} \pi_1(U)^{(\Sigma)}$. Indeed, one then verifies immediately that it is enough to verify that every étale cohomology class of U with coefficients in $\mathcal{F}_A|_U$ vanishes upon pull-back to some connected finite étale Σ-covering $V \to U$. Moreover, by passing to an appropriate V, we may assume that $\mathcal{F}_A|_U$ is trivial. Then the vanishing assertion in question is immediate (respectively, a tautology) for $n = 0$ (respectively, $n = 1$). Moreover, the vanishing assertion in question is immediate for $n \geq 3$ by [2], Theorem 9.1. If U is *affine*, then since $H^n_{\text{ét}}(U, \mathcal{F}_A|_U)$ vanishes for $n = 2$ (cf. [2], Theorem 9.1), the assertion is immediate. If U is proper, then it is enough to take $V \to U$ so that the degree of $V \to U$ annihilates A (cf. e.g., the discussion at the bottom of [2], p. 136).

Therefore, by considering the Hochschild-Serre spectral sequence (cf. [12], Theorem 2.1.5) associated to the exact sequence

$$1 \longrightarrow \pi \longrightarrow \Pi^{\log}_{(r)} \times_{G_K} \Pi_{\mathcal{M}_{r'+3}} \overset{\text{via } (p^{\log}_{(r-1)r}, \mathrm{id})}{\longrightarrow} \Pi^{\log}_{(r-1)} \times_{G_K} \Pi_{\mathcal{M}_{r'+3}} \longrightarrow 1$$

obtained by assertion (ii) and the Leray spectral sequence associated to the morphism $(p_{(r-1)r}, \mathrm{id})|_{U_{(r)} \times_K \mathcal{M}_{0,r'+3}}$, it follows that it is enough to show the assertion in the case where the pair of natural numbers "(r, r')" in the statement of assertion (iii) is $(r - 1, r')$. Moreover, if $r' \neq 0$, then by a similar argument to the above argument, it is verified that it is enough to show the assertion in the case where the pair of natural numbers "(r, r')" in the statement of assertion (iii) is $(r, r' - 1)$. This completes the proof of assertion (iii). □

Remark. Let $r \geq 2$ be an integer, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^\sharp \geq 2$. Then by Lemma 4.2, (i), (iii), it follows from Proposition

3.2, together with similar arguments to the arguments used in the proves of [10], Lemmas 4.3; 4.4, and [3], Proposition 4.22, that the morphism $\nu^{\log}_{(r)I}$ determines an exact sequence

$$1 \longrightarrow \Lambda \longrightarrow \Pi^{\log}_{(r)I} \overset{\text{via } \nu^{\log}_{(r)I}}{\longrightarrow} \Pi^{\log}_{(r-I^\sharp+1)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{I^\sharp+1}} \longrightarrow 1\,.$$

Lemma 4.3. *Let $r \geq 2$; $1 \leq i \leq r$ be integers, I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^\sharp \geq 2$, and $I^{[i]}$ the subset of $\{1, 2, \cdots, r-1\}$ defined in the statement of Proposition 3.2. Then the following hold:*

(i) *If $i \in I$, then the diagram*

$$\begin{array}{ccc} \Pi^{\log}_{(r)I} & \overset{\text{via } \nu^{\log}_{(r)I}}{\longrightarrow} & \Pi^{\log}_{(r-I^\sharp+1)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{I^\sharp+1}} \\ {\scriptstyle \text{via } p^{\log}_{(r-1)i}}\downarrow & & \downarrow \\ \Pi^{\log}_{(r-1)I^{[i]}} & \underset{\text{via } \nu^{\log}_{(r-1)I^{[i]}}}{\longrightarrow} & \Pi^{\log}_{(r-I^\sharp+1)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{I^\sharp}} \end{array}$$

is cartesian, where if $I = \{i_1, i_2, \cdots, i_{I^\sharp}\}$, $i_1 < i_2 < \cdots < i_{I^\sharp}$, and $i = i_j$, then the right-hand vertical arrow is the morphism induced by the morphism $\overline{\mathcal{M}}^{\log}_{0,I^\sharp+1} \to \overline{\mathcal{M}}^{\log}_{0,I^\sharp}$ obtained by forgetting the j-th marked point.

(ii) *If $i \notin I$, then the diagram*

$$\begin{array}{ccc} \Pi^{\log}_{(r)I} & \overset{\text{via } \nu^{\log}_{(r)I}}{\longrightarrow} & \Pi^{\log}_{(r-I^\sharp+1)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{I^\sharp+1}} \\ {\scriptstyle \text{via } p^{\log}_{(r-1)i}}\downarrow & & \downarrow \\ \Pi^{\log}_{(r-1)I^{[i]}} & \underset{\text{via } \nu^{\log}_{(r-1)I^{[i]}}}{\longrightarrow} & \Pi^{\log}_{(r-I^\sharp)} \times_{G_K} \Pi^{\log}_{\mathcal{M}_{I^\sharp+1}} \end{array}$$

is cartesian, where if $\{1, 2, \cdots, r\} \setminus I = \{i_1, i_2, \cdots, i_{r-I^\sharp}\}$, $i_1 < i_2 < \cdots < i_{r-I^\sharp}$, and $i = i_j$, then the right-hand vertical arrow is the morphism induced by $p^{\log}_{(r-I^\sharp)j} : X^{\log}_{(r-I^\sharp+1)} \to X^{\log}_{(r-I^\sharp)}$.

Proof. This follows from Proposition 3.2; Remark following Lemma 4.2, together with the fact that the restriction of $D^{\log}_{(r)I} \to D^{\log}_{(r-1)I^{[i]}}$ to the generic point of $D^{\log}_{(r)I}$ is *strict*. □

Lemma 4.4.

(i) *Let $r \geq 2$ be an integer, and $I = \{i, i+1\}$, where $i = 1, 2$. Then the following diagram is cartesian:*

$$\begin{array}{ccc} \Pi^{\log}_{(r)I} & \xrightarrow{\text{via } \mathrm{pr}^{\log}_{(r)I}} & \Pi^{\log}_{(2)\{1,2\}} \\ {\scriptstyle \text{via } \nu^{\log}_{(r)I}}\downarrow & & \downarrow{\scriptstyle \text{via } \nu^{\log}_{(2)\{1,2\}}} \\ \Pi^{\log}_{(r-1)} & \xrightarrow[\text{via } \mathrm{pr}^{\log}_{(r-1)\{i\}}]{} & \Pi_X\,. \end{array}$$

(ii) *Let $r \geq 3$ be an integer. Then the following diagram is cartesian:*

$$\begin{array}{ccccc} \Pi^{\log}_{(r)\{1,2,3\}} & \xrightarrow{\text{via } \mathrm{pr}^{\log}_{(r)\{1,2\}}} & & & \Pi^{\log}_{(2)\{1,2\}} \\ {\scriptstyle \text{via } \nu^{\log}_{(r)\{1,2,3\}}}\downarrow & & & & \downarrow{\scriptstyle \text{via } \nu^{\log}_{(2)\{1,2\}}} \\ \Pi^{\log}_{(r-2)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} & \xrightarrow[\mathrm{pr}_1]{} & \Pi^{\log}_{(r-2)} & \xrightarrow[\text{via } \mathrm{pr}^{\log}_{(r-2)\{1\}}]{} & \Pi_X\,. \end{array}$$

Proof. This follows from Lemma 4.3 by induction on r. □

Definition 7.

(i) Let $r \geq 2$ be an integer, and $I = \{i, i+1\}$, where $i = 1, 2$. Then we shall write

$$\Pi^{\mathcal{G}}_{X_{(r)I}} \stackrel{\text{def}}{=} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}\,,$$

where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r-1)}} \to \Pi_X$ (respectively, $\Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi_X$) is the morphism induced by $\mathrm{pr}^{\log}_{(r-1)\{i\}}$ (respectively, $\nu^{\log}_{X_{(2)}\{1,2\}}$). On the other hand, by Lemma 4.4, (i), the morphism $\Pi^{\log}_{X_{(r)I}} \to \Pi^{\log}_{X_{(r-1)}}$ induced by $\nu^{\log}_{X_{(r)I}}$ and the morphism $\Pi^{\log}_{X_{(r)I}} \to \Pi^{\log}_{X_{(2)}\{1,2\}}$ induced by $\mathrm{pr}^{\log}_{X_{(r)I}}$ induce an isomorphism

$$\Pi^{\mathcal{G}}_{X_{(r)I}} \xrightarrow{\sim} \Pi^{\log}_{X_{(r)I}}\,.$$

We shall denote this isomorphism by $\alpha^{\log}_{X_{(r)I}}$.

(ii) Let $r \geq 3$ be an integer. Then we shall write

$$\Pi^{\mathcal{G}}_{X_{(r)}\{1,2,3\}} \stackrel{\text{def}}{=} \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-2)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}\,,$$

where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r-2)}} \to \Pi_X$ (respectively, $\Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi_X$) is the morphism induced by $\mathrm{pr}^{\log}_{(r-2)\{1\}}$

(respectively, $\nu^{\log}_{X_{(2)}\{1,2\}}$). On the other hand, by Lemma 4.4, (ii), the morphism $\Pi^{\log}_{X_{(r)}\{1,2,3\}} \to \Pi^{\log}_{X_{(r-2)}} \times_{G_K} \Pi^{\log}_{\mathbb{P}_K}$ induced by $\nu^{\log}_{X_{(r)}\{1,2,3\}}$ and the morphism $\Pi^{\log}_{X_{(r)}\{1,2,3\}} \to \Pi^{\log}_{X_{(2)}\{1,2\}}$ induced by $\mathrm{pr}^{\log}_{X_{(r)}\{1,2\}}$ induce an isomorphism

$$\Pi^{\mathcal{G}}_{X_{(r)}\{1,2,3\}} \xrightarrow{\sim} \Pi^{\log}_{X_{(r)}\{1,2,3\}}\,.$$

We shall denote this isomorphism by $\alpha^{\log}_{X_{(r)}\{1,2,3\}}$.

5. Partial reconstruction of the fundamental groups of higher dimensional log configuration schemes

We continue with the notation of the preceding Section; in particular, we assume that the set of prime numbers Σ is innocuous in K. In this Section, we show that the fundamental group of the configuration space can be partially reconstructed from a collection of data concerning the fundamental groups of the configuration spaces of lower dimension.

Definition 8. Let $r \geq 3$ be an integer. Then we shall denote by $\Pi^{\mathcal{G}}_{X_{(r)}}$ the profinite group obtained as the free profinite product of

$$\Pi^{\log}_{X_{(r)}\{1,2\}}\ ;\ \Pi^{\log}_{X_{(r)}\{2,3\}}\ ;\ \Pi^{\log}_{X_{(r)}\{1,2,3\}}\,.$$

We shall denote by

$$f_{X_{(r)}} : \Pi^{\mathcal{G}}_{X_{(r)}} \longrightarrow \Pi^{\log}_{X_{(r)}}$$

the morphism determined by the morphisms $D^{\log}_{X_{(r)}I} \overset{\delta^{\log}_{X_{(r)}I}}{\to} X^{\log}_{(r)}$ where $I = \{1,2\}$, $\{2,3\}$, and $\{1,2,3\}$. Note that it follows from Theorem 4.1 that $f_{X_{(r)}}$ is *surjective*. Let $I = \{1,2\}$, $\{2,3\}$, or $\{1,2,3\}$. Then by the definition of $\Pi^{\mathcal{G}}_{X_{(r)}}$, we have a natural morphism

$$\Pi^{\log}_{X_{(r)}I} \longrightarrow \Pi^{\mathcal{G}}_{X_{(r)}}\,.$$

We shall denote this morphism by $\delta^{\mathcal{G}}_{X_{(r)}I}$.

Next, we define the collection of data used in the *partial reconstruction of the log fundamental groups of higher dimensional log configuration schemes* performed in Theorem 5.2 below.

Definition 9. Let $r \geq 2$ be an integer.

(i) We shall denote by $\mathcal{D}_X(\Sigma)$, or $\mathcal{D}_{X_{(1)}}(\Sigma)$, the collection of data consisting of

(i-1) the profinite groups

$$\Pi^{\log}_{X_{(2)}},\ \Pi^{\log}_{X_{(2)}\{1,2\}},\ \Pi_X,\ G_K,\ \Pi^{\log}_{\mathbb{P}_K}\,;$$

(i-2) the morphisms

$$\Pi^{\log}_{X_{(2)}} \overset{\text{via } p^{\log}_{X_{(1)^i}}}{\longrightarrow} \Pi_X\ \ (i = 1,\, 2)\,,$$

and the morphisms induced by the respective structure morphisms

$$\Pi_X \longrightarrow G_K\ ;\ \Pi^{\log}_{\mathbb{P}_K} \longrightarrow G_K\,;$$

(i-3) the morphism

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\text{via } \delta^{\log}_{X_{(2)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(2)}}\,.$$

(ii) We shall denote by $\mathcal{D}_{X_{(r)}}(\Sigma)$ the collection of data consisting of

(ii-1) the profinite groups

$$\Pi^{\log}_{X_{(k)}}\ (1 \leq k \leq r+1),\ \Pi^{\log}_{X_{(2)}\{1,2\}},\ G_K,\ \Pi^{\log}_{\mathbb{P}_K}\,;$$

(ii-2) the morphisms

$$\Pi^{\log}_{X_{(k)}} \overset{\text{via } p^{\log}_{X_{(k-1)^i}}}{\longrightarrow} \Pi^{\log}_{X_{(k-1)}}\ (2 \leq k \leq r+1,\ 1 \leq i \leq k),$$

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\text{via } \nu^{\log}_{X_{(2)}\{1,2\}}}{\longrightarrow} \Pi_X\,,$$

and the morphisms induced by the respective structure morphisms

$$\Pi_X \longrightarrow G_K\ \ ;\ \Pi^{\log}_{\mathbb{P}_K} \longrightarrow G_K\,;$$

(ii-3) the composites

$$\Pi^{\mathcal{G}}_{X_{(r+1)I}} \overset{\alpha^{\log}_{X_{(r+1)^I}}}{\overset{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)I}} \overset{\text{via } \delta^{\log}_{X_{(r+1)^I}}}{\longrightarrow} \Pi^{\log}_{X_{(r+1)}}\,,$$

where $I = \{1,2\}$, $\{2,3\}$, and $\{1,2,3\}$.

(iii) We shall denote by $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ the collection of data consisting of

(iii-1) the data obtained by replacing $\Pi^{\log}_{X_{(r+1)}}$ in (ii-1) by $\Pi^{\mathcal{G}}_{X_{(r+1)}}$;

(iii-2) the data obtained by replacing $\Pi^{\log}_{X_{(r+1)}} \overset{\text{via } p^{\log}_{X_{(r)}i}}{\to} \Pi^{\log}_{X_{(r)}}$ $(1 \leq i \leq r+1)$ in (ii-2) by $\Pi^{\mathcal{G}}_{X_{(r+1)}} \overset{f_{X_{(r+1)}}}{\to} \Pi^{\log}_{X_{(r+1)}} \overset{\text{via } p^{\log}_{X_{(r)}i}}{\to} \Pi^{\log}_{X_{(r)}}$ $(1 \leq i \leq r+1)$;

(iii-3) the data obtained by replacing $\Pi^{\log}_{X_{(r+1)}}$ (respectively, "via $\delta^{\log}_{X_{(r+1)}I}$") in (ii-3) by $\Pi^{\mathcal{G}}_{X_{(r+1)}}$ (respectively, "via $\delta^{\mathcal{G}}_{X_{(r+1)}I}$").

In the following, let Y be a smooth, proper, geometrically connected curve of genus ≥ 2 over a field L, and Σ_Y a non-empty set of prime numbers which is innocuous in L. We shall fix a separable closure L^{sep} of L and denote by G_L the absolute Galois group $\mathrm{Gal}(L^{\mathrm{sep}}/L)$ of L.

Definition 10. Let $r \geq 2$ be an integer.

(i) We shall refer to isomorphisms

$$\Pi^{\log}_{X_{(k)}} \overset{\sim}{\longrightarrow} \Pi^{\log}_{Y_{(k)}} \ (k = 1, 2) \ ; \ \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\sim}{\longrightarrow} \Pi^{\log}_{Y_{(2)}\{1,2\}} \ ;$$

$$G_K \overset{\sim}{\longrightarrow} G_L \ ; \ \Pi^{\log}_{\mathbb{P}_K} \overset{\sim}{\longrightarrow} \Pi^{\log}_{\mathbb{P}_L}$$

which are compatible with the morphisms given in the definitions of $\mathcal{D}_X(\Sigma)$ and $\mathcal{D}_Y(\Sigma_Y)$ as an *isomorphism of* $\mathcal{D}_X(\Sigma)$ *with* $\mathcal{D}_Y(\Sigma_Y)$.

(ii) We shall refer to isomorphisms

$$\Pi^{\log}_{X_{(k)}} \overset{\sim}{\longrightarrow} \Pi^{\log}_{Y_{(k)}} \ (1 \leq k \leq r+1) \ ; \ \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\sim}{\longrightarrow} \Pi^{\log}_{Y_{(2)}\{1,2\}} \ ;$$

$$G_K \overset{\sim}{\longrightarrow} G_L \ ; \ \Pi^{\log}_{\mathbb{P}_K} \overset{\sim}{\longrightarrow} \Pi^{\log}_{\mathbb{P}_L}$$

which are compatible with the morphisms given in the definitions of $\mathcal{D}_{X_{(r)}}(\Sigma)$ and $\mathcal{D}_{Y_{(r)}}(\Sigma_Y)$ as an *isomorphism of* $\mathcal{D}_{X_{(r)}}(\Sigma)$ *with* $\mathcal{D}_{Y_{(r)}}(\Sigma_Y)$.

(iii) We shall refer to isomorphisms

$$\Pi^{\mathcal{G}}_{X_{(r+1)}} \overset{\sim}{\longrightarrow} \Pi^{\mathcal{G}}_{Y_{(r+1)}} \ ; \ \Pi^{\log}_{X_{(k)}} \overset{\sim}{\longrightarrow} \Pi^{\log}_{Y_{(k)}} \ (1 \leq k \leq r) \ ;$$

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\sim}{\longrightarrow} \Pi^{\log}_{Y_{(2)}\{1,2\}} \ ; \ G_K \overset{\sim}{\longrightarrow} G_L \ ; \ \Pi^{\log}_{\mathbb{P}_K} \overset{\sim}{\longrightarrow} \Pi^{\log}_{\mathbb{P}_L}$$

which are compatible with the morphisms given in the definitions of $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ and $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ as an *isomorphism of* $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ *with* $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$.

(iv) Let $\phi^{\mathcal{G}}_{(r)} : \mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma) \overset{\sim}{\to} \mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ be an isomorphism of $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ with $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$. Then by forgetting the isomorphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \overset{\sim}{\to} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$ in $\phi^{\mathcal{G}}_{(r)}$, we obtain an isomorphism of $\mathcal{D}_{X_{(r-1)}}(\Sigma)$ with $\mathcal{D}_{Y_{(r-1)}}(\Sigma_Y)$.

We shall denote this isomorphism by $F_{-1}(\phi^{\mathcal{G}}_{(r)})$. Note that it is immediate that the correspondence

$$\phi^{\mathcal{G}}_{(r)} \mapsto F_{-1}(\phi^{\mathcal{G}}_{(r)})$$

is *functorial*.

Remark. If there exists an isomorphism of $\mathcal{D}_X(\Sigma)$ (respectively, $\mathcal{D}_{X_{(r)}}(\Sigma)$; respectively, $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$) with $\mathcal{D}_Y(\Sigma_Y)$ (respectively, $\mathcal{D}_{Y_{(r)}}(\Sigma_Y)$; respectively, $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$), then $\Sigma = \Sigma_Y$. Indeed, this follows from the fact that the abelianization of the kernel of $\Pi_X \to G_K$ (respectively, $\Pi_Y \to G_L$) is a *non-trivial module which is free over* $\mathbb{Z}^{(\Sigma)}$ (respectively, $\mathbb{Z}^{(\Sigma_Y)}$).

Proposition 5.1. *Let $r \geq 2$ be an integer, Σ a set of prime numbers that is innocuous in K and L, and $\phi^{\mathcal{G}}_{(r)} : \mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma) \xrightarrow{\sim} \mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma)$ an isomorphism. Then if the isomorphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$ in $\phi^{\mathcal{G}}_{(r)}$ induces an isomorphism of the kernel of the morphism $f_{X_{(r+1)}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ with the kernel of the morphism $f_{Y_{(r+1)}} : \Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r+1)}}$, then there exists an isomorphism $F(\phi^{\mathcal{G}}_{(r)}) : \mathcal{D}_{X_{(r)}}(\Sigma) \xrightarrow{\sim} \mathcal{D}_{Y_{(r)}}(\Sigma)$. Moreover, the correspondence*

$$\phi^{\mathcal{G}}_{(r)} \mapsto F(\phi^{\mathcal{G}}_{(r)})$$

is functorial.

Proof. Since the morphism $f_{X_{(r+1)}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ (respectively, $f_{Y_{(r+1)}} : \Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r+1)}}$) is surjective (cf. Theorem 4.1), by the assumption, we obtain an isomorphism $\underline{\phi} : \Pi^{\log}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r+1)}}$ induced by the isomorphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$ in $\phi^{\mathcal{G}}_{(r)}$. Therefore, by replacing the isomorphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$ in $\phi^{\mathcal{G}}_{(r)}$ by $\underline{\phi}$, we obtain an isomorphism $F(\phi^{\mathcal{G}}_{(r)})$ of the desired type. □

Theorem 5.2. *Let $r \geq 2$ be an integer, Σ a set of prime numbers which is innocuous in K and L, and $\phi_{(r-1)} : \mathcal{D}_{X_{(r-1)}}(\Sigma) \xrightarrow{\sim} \mathcal{D}_{Y_{(r-1)}}(\Sigma)$ an isomorphism. Then there exists an isomorphism $F_{+1}(\phi_{(r-1)}) : \mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma) \xrightarrow{\sim} \mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma)$ such that*

$$F_{-1}(F_{+1}(\phi_{(r-1)})) = \phi_{(r-1)}\,.$$

Moreover, the correspondence

$$\phi_{(r-1)} \mapsto F_{+1}(\phi_{(r-1)})$$

is functorial.

Proof. First, we define an isomorphism $\phi^{\mathcal{G}_{\{1,2\}}}$ (respectively, $\phi^{\mathcal{G}_{\{2,3\}}}$; respectively, $\phi^{\mathcal{G}_{\{1,2,3\}}}$) of $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}}$, (respectively, $\Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}}$; respectively, $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}}$) with $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}}$, (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}}$; respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}}$) as follows:

(i) We define an isomorphism $\phi^{\mathcal{G}_{\{1,2\}}} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}}$ as the isomorphism induced by the isomorphisms $\Pi^{\log}_{X_{(r)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r)}}$, $\Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}}$, and $\Pi_X \xrightarrow{\sim} \Pi_Y$ in $\phi_{(r-1)}$.

(ii) We define an isomorphism $\phi^{\mathcal{G}_{\{2,3\}}} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}}$ as the isomorphism induced by the isomorphisms $\Pi^{\log}_{X_{(r)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r)}}$, $\Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}}$, and $\Pi_X \xrightarrow{\sim} \Pi_Y$ in $\phi_{(r-1)}$.

(iii) We define an isomorphism $\phi^{\mathcal{G}_{\{1,2,3\}}} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}}$ as the isomorphism induced by the isomorphisms $\Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$, $\Pi^{\log}_{X_{(r-1)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r-1)}}$, $\Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}}$, $G_K \xrightarrow{\sim} G_L$, and $\Pi_X \xrightarrow{\sim} \Pi_Y$ in $\phi_{(r-1)}$.

Then these isomorphisms $\phi^{\mathcal{G}_{\{1,2\}}}$, $\phi^{\mathcal{G}_{\{2,3\}}}$, and $\phi^{\mathcal{G}_{\{1,2,3\}}}$ induce an isomorphism of the profinite group $\Pi^{\mathcal{G}}_{X_{(r+1)}}$ obtained as the free profinite product of

$$\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} \ ; \ \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} \ ; \ \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}}$$

(cf. Definitions 7; 8) with the profinite group $\Pi^{\mathcal{G}}_{Y_{(r+1)}}$ obtained as the free profinite product of

$$\Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}} \ ; \ \Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}} \ ; \ \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}} .$$

Now we denote this isomorphism by $\phi^{\mathcal{G}}$.

On the other hand, for an integer $1 \leq i \leq r+1$, we define a projection $q_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r)}}$ as follows:

(i) If $i = 1$ or 2, then we define a morphism $q^{\{1,2\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$ as the first projection. If $i \geq 3$, then we define a morphism $q^{\{1,2\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$ as the composite

$$\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} \xrightarrow{\text{via } p^{\log}_{X_{(r-1)}{}^{i-2}} \times \mathrm{id}_{D^{\log}_{X_{(2)}\{1,2\}}}} \Pi^{\mathcal{G}}_{X_{(r)}\{1,2\}}$$

$$\overset{\alpha^{\log}_{X_{(r)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(r)}\{1,2\}} \overset{\text{via } \delta^{\log}_{X_{(r)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(r)}}.$$

(ii) We define a morphism $q^{\{2,3\}}_{X_{(r)}1} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} \to \Pi^{\log}_{X_{(r)}}$ as the morphism obtained by replacing $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}}$ (respectively, $p^{\log}_{X_{(r-1)}i-2}$) by $\Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}}$ (respectively, $p^{\log}_{X_{(r-1)}1}$) in the definition of "$q^{\{1,2\}}_{X_{(r)}i}$ for $i \geq 3$". If $i = 2$ or 3, then we define a morphism $q^{\{2,3\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$ as the first projection. If $i \geq 4$, then we define a morphism $q^{\{2,3\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} \to \Pi^{\log}_{X_{(r)}}$ as the morphism obtained by replacing $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}}$ by $\Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}}$ in the definition of "$q^{\{1,2\}}_{X_{(r)}i}$ for $i \geq 3$".

(iii) If $i = 1$, 2, or 3, then we define a morphism $q^{\{1,2,3\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} \to \Pi^{\log}_{X_{(r)}}$ as the composite

$$\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} \overset{\text{pr}_{2,3}}{\longrightarrow} \Pi^{\mathcal{G}}_{X_{(r)}\{1,2\}} \overset{\alpha^{\log}_{X_{(r)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(r)}\{1,2\}} \overset{\text{via } \delta^{\log}_{X_{(r)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(r)}}.$$

If $i \geq 4$, then we define a morphism $q^{\{1,2,3\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} \to \Pi^{\log}_{X_{(r)}}$ as the composite

$$\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} \overset{\text{via } \text{id}_{\mathbb{P}^{\log}_K} \times p^{\log}_{X_{(r-2)}i-3} \times \text{id}_{D^{\log}_{X_{(2)}\{1,2\}}}}{\longrightarrow} \Pi^{\mathcal{G}}_{X_{(r)}\{1,2,3\}} \overset{\alpha^{\log}_{X_{(r)}\{1,2,3\}}}{\longrightarrow} \Pi^{\log}_{X_{(r)}\{1,2,3\}} \overset{\text{via } \delta^{\log}_{X_{(r)}\{1,2,3\}}}{\longrightarrow} \Pi^{\log}_{X_{(r)}}.$$

Moreover, by a similar procedure to the procedure that we applied to define the morphism $q^{\{1,2\}}_{X_{(r)}i}$ (respectively, $q^{\{2,3\}}_{X_{(r)}i}$; respectively, $q^{\{1,2,3\}}_{X_{(r)}i}$), we define a morphism $q^{\{1,2\}}_{Y_{(r)}i}$ (respectively, $q^{\{2,3\}}_{Y_{(r)}i}$; respectively, $q^{\{1,2,3\}}_{Y_{(r)}i}$).

Then these morphisms $q^{\{1,2\}}_{X_{(r)}i}$, $q^{\{2,3\}}_{X_{(r)}i}$, and $q^{\{1,2,3\}}_{X_{(r)}i}$ (respectively, $q^{\{1,2\}}_{Y_{(r)}i}$, $q^{\{2,3\}}_{Y_{(r)}i}$, and $q^{\{1,2,3\}}_{Y_{(r)}i}$) induce a morphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r)}}$). We denote this morphism by $q_{X_{(r)}i}$ (respectively, $q_{Y_{(r)}i}$).

Then by Lemma 4.4, together with constructions, for any $1 \leq i \leq r+1$, the morphism $q_{X_{(r)}i}$ (respectively, $q_{Y_{(r)}i}$) factors as the composite $\Pi^{\mathcal{G}}_{X_{(r+1)}} \overset{f_{X_{(r+1)}}}{\to} \Pi^{\log}_{X_{(r+1)}} \overset{\text{via } p^{\log}_{X_{(r)}i}}{\to} \Pi^{\log}_{X_{(r)}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \overset{f_{Y_{(r+1)}}}{\to} \Pi^{\log}_{Y_{(r+1)}} \overset{\text{via } p^{\log}_{Y_{(r)}i}}{\to} \Pi^{\log}_{Y_{(r)}}$);

moreover, the diagram

$$\begin{array}{ccc} \Pi^{\mathcal{G}}_{X_{(r+1)}} & \overset{\phi^{\mathcal{G}}}{\overset{\sim}{\longrightarrow}} & \Pi^{\mathcal{G}}_{X_{(r+1)}} \\ {\scriptstyle q_{X_{(r)}}{}^{i}}\downarrow & & \downarrow{\scriptstyle q_{Y_{(r)}}{}^{i}} \\ \Pi^{\log}_{X_{(r)}} & \overset{\sim}{\longrightarrow} & \Pi^{\log}_{Y_{(r)}} \end{array}$$

commutes, where the bottom horizontal arrow is the isomorphism in $\phi_{(r-1)}$. Therefore, by equipping $\phi_{(r-1)}$ with the isomorphism $\phi^{\mathcal{G}}$, we obtain an isomorphism $F_{+1}(\phi_{(r-1)})$ of $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ with $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma)$ of the desired type. □

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