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INVERSE LIMITS OF SPACES WITH THE WEAK $\mathcal{B}$-PROPERTY

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Abstract. In this paper we show the following facts. Let X be the inverse limit space of an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$, where Λ is a directed set and $|\Lambda| = \kappa \geqslant \omega$. Suppose that each projection $\pi_\alpha : X \to X_\alpha$ is pseudo-open and X is κ-paracompact. If each X_α has the weak $\mathcal{B}$-property, then X has the weak $\mathcal{B}$-property. We also show that the analogous result holds for hereditarily weak $\mathcal{B}$-property.

Recently, on the studies of products of normality or other covering properties, a series of papers related to the following question proposed (see [1, 2, 3, 9, 10]).

Question. *For an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ with $|\Lambda| = \kappa \geqslant \omega$, suppose that the inverse limit space X of $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ is κ-paracompact and each projection $\pi_\alpha : X \to X_\alpha$ is pseudo-open. If each X_α satisfies property $\mathcal{P}$, then what property $\mathcal{P}$ can be preserved by the inverse limit space X?*

On the above question, the major work is by Chiba who shows that many covering properties and some separation properties are preserved by the inverse limit space (see [1, 2, 3, 10]). The aim of this paper is to show that the above also holds under the weak $\mathcal{B}$-property. About the property $\mathcal{B}$, the similar results are announced in [1].

The weak $\mathcal{B}$-property as a generalization of the property $\mathcal{B}$ was first introduced by Yasui [7] who gave results relating to the property $\mathcal{B}$ and countable paracompactness. Subsequently many authors studied this property (see [6]) and in 1985 Rudin [11] renamed this property $\mathcal{D}$. However, in this paper we shall continue to use the term "weak $\mathcal{B}$-property".

Throughout this paper, we assume that all spaces are topological spaces without any separation axiom and all maps are continuous.

Let X be a space and $A \subset G \subset X$, then $\overline{A}$, int$A$ denote the closure, interior of A in X respectively and $\overline{A}^G$, $\mathrm{int}_G(A)$ denote the closure, interior of A in G respectively. ω denotes the first infinite cardinal and $[\Sigma]^{<\omega}$ denotes the collection of all non-empty finite subsets of non-empty set Σ.

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A map f from X onto Y is called *pseudo-open* if $y \in \mathrm{int} f(U)$ holds for each $y \in Y$ and each open set U in X with $f^{-1}(y) \subset U$.

It is easy to see that both open onto maps and closed onto maps are pseudo-open.

Definition 1. Let κ be an arbitrary infinite cardinal. Then the space X is called *κ-paracompact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leqslant \kappa$ has a locally finite open refinement.

The following lemmas are necessary to prove our theorems.

Lemma 1 ([9], Lemma 1.3). *A space X is κ-paracompact if, and only if, for every open cover $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}$ of X with $|\Lambda| = \kappa$, there is a locally finite open cover $\mathcal{V} = \{V_\alpha | \alpha \in \Lambda\}$ of X such that $V_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$.*

Lemma 2 ([2], Lemma 2). *Let X be a κ-paracompact space, Λ a directed set with $|\Lambda| = \kappa$ and $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}$ an open cover of X such that $U_\alpha \subset U_\beta$ for each $\alpha, \beta \in \Lambda$ with $\alpha \leqslant \beta$. Then there is an open cover $\mathcal{V} = \{V_\alpha | \alpha \in \Lambda\}$ of X such that (i) $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$ and (ii) $V_\alpha \subset V_\beta$ if $\alpha \leqslant \beta$.*

Let X be a κ-paracompact space and $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}$ be an open cover of X satisfying the conditions of Lemma 2. Then $\mathcal{U}$ has an open refinement $\mathcal{V} = \{V_\alpha | \alpha \in \Lambda\}$ such that $\overline{V_\alpha} \subset U_\alpha$ $(\alpha \in \Lambda)$. By Lemma 1, $\mathcal{V}$ has a locally finite open refinement $\{W_\alpha | \alpha \in \Lambda\}$ such that $W_\alpha \subset V_\alpha$ $(\alpha \in \Lambda)$. Therefore we have the following Lemma.

Lemma 3 . *Let X be a κ-paracompact space, Λ a directed set with $|\Lambda| = \kappa$ and $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}$ an open cover of X such that $U_\alpha \subset U_\beta$ for each $\alpha, \beta \in \Lambda$ with $\alpha \leqslant \beta$. Then there is a locally finite open cover $\mathcal{V} = \{V_\alpha | \alpha \in \Lambda\}$ of X such that $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$.*

Definition 2. [10] Let κ be an arbitrary infinite cardinal. A space X has the *property $\mathcal{B}^*(\kappa)$* if for any decreasing collection $\{F_\alpha | \alpha < \kappa\}$ of closed subsets in X with $\cap\{F_\alpha | \alpha < \kappa\} = \emptyset$, there exists a collection $\{G_\alpha | \alpha < \kappa\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha < \kappa$ and $\cap\{\overline{G_\alpha} | \alpha < \kappa\} = \emptyset$.

A space X has the *weak $\mathcal{B}$-property* if X has the property $\mathcal{B}^*(\kappa)$ for every infinite cardinal κ.

It is easy to see that:

Proposition 1. *For a space X, the following are equivalent:*

(1) *X has the property $\mathcal{B}^*(\kappa)$.*

(2) [5] *Every increasing open cover $\{U_\alpha|\alpha < \kappa\}$ of X has an open cover $\{V_\alpha|\alpha < \kappa\}$ of X such that $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha < \kappa$.*

Theorem 1. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X be its inverse limit space $\varprojlim\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$. Suppose that each projection $\pi_\alpha : X \to X_\alpha$ is a psuedo-open map and X is a κ-paracompact space, where $|\Lambda| = \kappa$. If each X_α has the weak $\mathcal{B}$-property, then X also has the weak $\mathcal{B}$-property.*

Proof. Let τ be an arbitrary infinite cardinal and $\mathcal{G} = \{G_\xi|\xi < \tau\}$ be an increasing open cover of X. For each $\alpha \in \Lambda$ and $\xi < \tau$, we put $U_{\alpha,\xi} = \cup\{U|U\text{open in } X_\alpha,\ \pi_\alpha^{-1}(U) \subset G_\xi\}$ and $U_\alpha = \cup\{U_{\alpha,\xi}|\xi < \tau\}$, then the collection $\{\pi_\alpha^{-1}(U_\alpha)|\alpha \in \Lambda\}$ satisfies:

(1) $\cup\{\pi_\alpha^{-1}(U_\alpha)|\alpha \in \Lambda\} = X$.

(2) $\pi_\alpha^{-1}(U_\alpha) \subset \pi_\beta^{-1}(U_\beta)$ if $\alpha \leqslant \beta$.

Since X is κ-paracompact, by Lemma 2, there is an open cover $\{W_\alpha|\alpha \in \Lambda\}$, such that

(3) $\overline{W_\alpha} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$.

(4) If $\alpha \leqslant \beta$, then $W_\alpha \subset W_\beta$.

Now, for each $\alpha \in \Lambda$, we define the closed subset $T_\alpha = X_\alpha \backslash \mathrm{int}\pi_\alpha(X \setminus \overline{W_\alpha})$ of X_α. Since each projection $\pi_\alpha : X \to X_\alpha$ is a psuedo-open map, we have

(5) $T_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$.

Put $C_\alpha = \mathrm{int}\pi_\alpha^{-1}(T_\alpha)$ for each $\alpha \in \Lambda$, then

(6) $\{C_\alpha|\alpha \in \Lambda\}$ is an open cover of X.

Because, for each $x \in X$, some W_α contains x. Hence there are $\beta \in \Lambda$ and an open subset V in X_β such that $x \in \pi_\beta^{-1}(V) \subset W_\alpha$. Then there is $\gamma \in \Lambda$ with $\alpha, \beta \leqslant \gamma$, and $x \in \pi_\beta^{-1}(V) \subset \pi_\gamma^{-1}(T_\gamma)$. Hence $x \in C_\gamma$.

Since X is κ-paracompact, by Lemma 1, there is a locally finite open cover $\{O_\alpha|\alpha \in \Lambda\}$ of X such that $O_\alpha \subset C_\alpha$ for each $\alpha \in \Lambda$. For each $\alpha \in \Lambda$ and $\xi < \tau$, we put $U'_{\alpha,\xi} = U_{\alpha,\xi} \cup (X_\alpha \setminus T_\alpha)$, then $\{U'_{\alpha,\xi}|\xi < \tau\}$ is an increasing open cover of X_α since $\{U_{\alpha,\xi}|\xi < \tau\}$ is an increasing open cover of U_α. Thus we have

(7) $\{X_\alpha \setminus U'_{\alpha,\xi}|\xi < \tau\}$ is a decreasing collection of closed subsets of X_α satisfying that $\cap\{X_\alpha \setminus U'_{\alpha,\xi}|\xi < \tau\} = \emptyset$.

Since X_α has the weak $\mathcal{B}$-property, there exists an open collection $\{V_{\alpha,\xi}|\xi < \tau\}$ of X_α such that

(8) $X_\alpha \setminus U'_{\alpha,\xi} \subset V_{\alpha,\xi}$ for each $\xi < \tau$.

(9) $\cap\{\overline{V_{\alpha,\xi}}|\xi < \tau\} = \emptyset$.

To show that X has the weak $\mathcal{B}$-property, by Proposition 1, it is sufficient to construct an open cover $\{A_\xi|\xi<\tau\}$ of X satisfying $\overline{A_\xi}\subset G_\xi$ for each $\xi<\tau$. Let $A_\xi=\cup\{\pi_\alpha^{-1}(X_\alpha\setminus\overline{V_{\alpha,\xi}})\cap O_\alpha|\alpha\in\Lambda\}$ for each $\xi<\tau$. Then

(10) $\{A_\xi|\xi<\tau\}$ is an open cover of X.

In the fact, for each $x\in X$, $x\in O_{\alpha_0}$ for some $\alpha_0\in\Lambda$. Since $\pi_{\alpha_0}(x)\in X_{\alpha_0}$ and $\cap\{\overline{V_{\alpha_0,\xi}}|\xi<\tau\}=\emptyset$, we have $x_{\alpha_0}\in X_{\alpha_0}\setminus\overline{V_{\alpha_0,\xi_0}}$ for some $\xi_0<\tau$. Hence $x\in\pi_{\alpha_0}^{-1}(X_{\alpha_0}\setminus\overline{V_{\alpha_0,\xi_0}})\cap O_{\alpha_0}\subset A_{\xi_0}$.

Last, we show that $\overline{A_\xi}\subset G_\xi$ for each $\xi<\tau$. In the fact, for each $\xi<\tau$, observe that the collection $\{\pi_\alpha^{-1}(X_\alpha\setminus\overline{V_{\alpha,\xi}})\cap O_\alpha|\alpha\in\Lambda\}$ is locally finite in X. Therefore for each $\xi<\tau$,

$$\begin{aligned}\overline{A_\xi}&=\cup\Big\{\overline{\pi_\alpha^{-1}(X_\alpha\setminus\overline{V_{\alpha,\xi}})\cap O_\alpha}|\alpha\in\Lambda\Big\}\subset\cup\Big\{\overline{\pi_\alpha^{-1}(X_\alpha\setminus V_{\alpha,\xi})}\cap\overline{O_\alpha}|\alpha\in\Lambda\Big\}\\&=\cup\{\pi_\alpha^{-1}(X_\alpha\setminus V_{\alpha,\xi})\cap\overline{O_\alpha}|\alpha\in\Lambda\}\subset\cup\{\pi_\alpha^{-1}(U'_{\alpha,\xi})\cap\overline{O_\alpha}|\alpha\in\Lambda\}\\&\subset\cup\{\pi_\alpha^{-1}(U'_{\alpha,\xi})\cap\pi_\alpha^{-1}(T_\alpha)|\alpha\in\Lambda\}\\&=\cup\{\pi_\alpha^{-1}\big((U_{\alpha,\xi}\cup(X_\alpha\setminus T_\alpha))\cap T_\alpha\big)|\alpha\in\Lambda\}\\&=\cup\{\pi_\alpha^{-1}(U_{\alpha,\xi}\cap T_\alpha)|\alpha\in\Lambda\}\subset\cup\{\pi_\alpha^{-1}(U_{\alpha,\xi})|\alpha\in\Lambda\}\subset G_\xi\end{aligned}$$

The proof of Theorem 1 is completed.

We describe inverse limit spaces of the hereditarily weak $\mathcal{B}$-properties

A space X has the *hereditarily weak $\mathcal{B}$-property* if every subspace of X has the weak $\mathcal{B}$-property.

It is not difficult to show the following lemma.

Proposition 2. *A space X has the hereditarily weak $\mathcal{B}$-property i, and only if, every open subspace of X has the weak $\mathcal{B}$-property.*

Theorem 2. *Let $\{X_\alpha,\pi_\beta^\alpha,\Lambda\}$ be an inverse system and $X=\varprojlim\{X_\alpha,\pi_\beta^\alpha,\Lambda\}$. Suppose that G is a $\kappa-$paracompact open subspace of X. If each X_α has the hereditarily weak $\mathcal{B}$-property, then G has the weak $\mathcal{B}$-property.*

Proof. Let τ be an arbitrary infinite cardinal and $\mathcal{G}=\{G_\xi|\xi<\tau\}$ be an increasing open cover of G. For each $\alpha\in\Lambda$ and $\xi<\tau$, we put $U_{\alpha,\xi}=\cup\{U|U\text{open in }X_\alpha,\ \pi_\alpha^{-1}(U)\subset G_\xi\}$ and $U_\alpha=\cup\{U_{\alpha,\xi}|\xi<\tau\}$, then similar to the case of Theorem 1, the collection $\{\pi_\alpha^{-1}(U_\alpha)|\alpha\in\Lambda\}$ satisfies

(1) $\cup\{\pi_\alpha^{-1}(U_\alpha)|\alpha\in\Lambda\}=G$ and

(2) $\pi_\alpha^{-1}(U_\alpha)\subset\pi_\beta^{-1}(U_\beta)$ if $\alpha\leqslant\beta$.

Since G is a κ-paracompact open subspace of X and $\{\pi_\alpha^{-1}(U_\alpha)|\alpha\in\Lambda\}$ satisfies the condition of Lemma 3, there is a locally finite open refinement

$\{O_\alpha | \alpha \in \Lambda\}$ of $\{\pi_\alpha^{-1}(U_\alpha) | \alpha \in \Lambda\}$ in G such that $\overline{O_\alpha}^G \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$.

Note that the collection $\mathcal{U}_\alpha = \{U_{\alpha,\xi} | \xi < \tau\}$ is an increasing open cover of U_α because $\mathcal{G}$ is an increasing open cover of G. Therefore $\{U_\alpha \setminus U_{\alpha,\xi} | \xi < \tau\}$ is a decreasing closed collection of U_α satisfying that $\cap\{U_\alpha \setminus U_{\alpha,\xi} | \xi < \tau\} = \emptyset$. Since U_α has the weak $\mathcal{B}$-property, there exists an open collection $\{V_{\alpha,\xi} | \xi < \tau\}$ in U_α such that

(3) $U_\alpha \setminus U_{\alpha,\xi} \subset V_{\alpha,\xi}$ for each $\xi < \tau$ and

(4) $\cap\big\{\overline{V_{\alpha,\xi}}^{U_\alpha} | \xi < \tau\big\} = \emptyset$.

To show that G has the weak $\mathcal{B}$-property, by Proposition 1, it is sufficient to construct an open cover $\{A_\xi | \xi < \tau\}$ of G satisfying $\overline{A_\xi}^G \subset G_\xi$ for each $\xi < \tau$.

Now, for each $\xi < \tau$, we put $A_\xi = \cup\big\{\pi_\alpha^{-1}(U_\alpha \setminus \overline{V_{\alpha,\xi}}^{U_\alpha}) \cap O_\alpha |\ \alpha \in \Lambda\big\}$. Then similar to the proof of Theorem 1, $\{A_\xi | \xi < \tau\}$ is an open cover of G. Moreover we have

(5) $\overline{A_\xi}^G \subset G_\xi$ for each $\xi < \tau$

Indeed, for each $\xi < \tau$, since the collection $\big\{\pi_\alpha^{-1}(U_\alpha \setminus \overline{V_{\alpha,\xi}}^{U_\alpha}) \cap O_\alpha |\ \alpha \in \Lambda\big\}$ is locally finite in G, we have

$$\begin{aligned}
\overline{A_\xi}^G &= \cup\Big\{\overline{\pi_\alpha^{-1}(U_\alpha \setminus \overline{V_{\alpha,\xi}}^{U_\alpha}) \cap O_\alpha}^G \Big|\ \alpha \in \Lambda\Big\} \\
&\subset \cup\Big\{\overline{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha,\xi})}^G \cap \overline{O_\alpha}^G |\ \alpha \in \Lambda\Big\} \\
&\subset \cup\Big\{\overline{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha,\xi})} \cap G \cap \pi_\alpha^{-1}(U_\alpha) |\ \alpha \in \Lambda\Big\} \\
&\subset \cup\{\pi_\alpha^{-1}(\overline{U_\alpha \setminus V_{\alpha,\xi}} \cap U_\alpha) \cap G |\ \alpha \in \Lambda\} \\
&= \cup\{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha,\xi}) \cap G |\ \alpha \in \Lambda\} \subset \cup\{\pi_\alpha^{-1}(U_{\alpha,\xi}) \cap G |\ \alpha \in \Lambda\} \subset G_\xi
\end{aligned}$$

The proof of Theorem 2 is completed.

By Proposition 2 and Theorem 2, we obtain the following usual statement of inverse limits of the spaces with the hereditarily weak $\mathcal{B}$-property.

Corollary 1. *Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and $X = \varprojlim\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$. Suppose that X is a hereditarily κ-paracompact space, where $|\Lambda| = \kappa$. If each X_α has the hereditarily weak $\mathcal{B}$-property, then X also has the hereditarily weak $\mathcal{B}$-property.*

Finally, we study properties of the product of spaces with the weak $\mathcal{B}$-properties.

Let κ be an infinite cardinal number, $\{X_\alpha | \alpha \in \Sigma\}$ be a collection of spaces with $|\Sigma| = \kappa$ and $X = \prod_{\alpha\in\Sigma} X_\alpha$. We define the relation " $\leqslant$ " of $[\Sigma]^{<\omega}$ as $A \leqslant B$ if and only if $A \subset B$ for $A, B \in [\Sigma]^{<\omega}$ and define the finite subproduct $Z_A = \prod_{\alpha\in A} X_\alpha$ of X for every $A \in [\Sigma]^{<\omega}$. Then $[\Sigma]^{<\omega}$ is a directed set with the relation " $\leqslant$ ". For $A \leqslant B$ $(A, B \in [\Sigma]^{<\omega})$, let $\pi_A^B : Z_B \longrightarrow Z_A$ be the natural projecion. Then π_A^B is an open bonding map from Z_B onto Z_A and hence we obtain the inverse system $\{Z_A, \pi_A^B, [\Sigma]^{<\omega}\}$ induced by the collection $\{X_\alpha | \alpha \in \Sigma\}$, and by Lemma 1 of [2], every projection π_A of this inverse system $\{Z_A, \pi_A^B, [\Sigma]^{<\omega}\}$ is also an open map from the inverse limit space $Z = \varprojlim\{Z_A, \pi_A^B, [\Sigma]^{<\omega}\}$ onto Z_A.

Lemma 4. [8] *Suppose that $\{X_\alpha | \alpha \in \Sigma\}$ is a collection of spaces, $X = \prod_{\alpha\in\Sigma} X_\alpha$ and $Z = \varprojlim\{Z_A, \pi_A^B, [\Sigma]^{<\omega}\}$, where $|\Sigma| \geqslant \omega$.* Then X and Z are homeomorphic .

Theorem 3. *Suppose that $\{X_\alpha | \alpha \in \Sigma\}$ is a collection of Hausdorff spaces and its product space $X = \prod_{\alpha\in\Sigma} X_\alpha$ is κ-paracompact, where $|\Sigma| = \kappa \geqslant \omega$. Then X has the weak $\mathcal{B}$-property if, and only if, for each $A \in [\Sigma]^{<\omega}$ the finite subproduct $Z_A = \prod_{\alpha\in A} X_\alpha$ has the weak $\mathcal{B}$-property.*

Proof. The "only if" part follows from Lemma 4 and Theorem 1. For the "if" part, assume that X has the weak $\mathcal{B}$-property. Then for each $A \in [\Sigma]^{<\omega}$, we choose a fixed point $x_\alpha \in X_\alpha$ for every $\alpha \in \Sigma \setminus A$. Since Z_A is homeomorphic to the closed subspace $\prod_{\alpha\in A} X_\alpha \times \prod_{\alpha\in\Sigma\setminus A}\{x_\alpha\}$ of X, Z_A has the weak $\mathcal{B}$-property.

The proof of Theorem 3 is completed.

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