Math. J. Okayama Univ. **50** (2008), 113–125

# STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS BY VISCOSITY APPROXIMATION METHODS IN BANACH SPACES

Xiaolong QIN, Yongfu SU and Changqun WU

Abstract. In this paper, we introduce a modified Ishikawa iterative process for a pair of nonexpansive mappings and obtain a strong convergence theorem in the framework of uniformly Banach spaces. Our results improve and extend the recent ones announced by Kim and Xu [T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal. 61 (2005) 51-60], Xu [H.K. Xu, Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. 298 (2004) 279-291] and some others.

## 1. Introduction and Preliminaries

Let $E$ be a real Banach space and let $J$ denotes the normalized duality mapping from $E$ into $2^{E^*}$ given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E,$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Recall that a self mapping $f : C \to C$ is a contraction on $C$ if there esists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad x, y \in C.$$

We use $\Pi_C$ to denote the collection of all contractions on $C$. That is, $\Pi_C = \{f|f : C \to C \text{ a contraction}\}$. Note that each $f \in \Pi_C$ has a unique fixed point in $C$. Also, recall that $T$ is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in C.$$

A point $x \in C$ is a fixed point of $T$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{x \in C : Tx = x\}$. Given a real number $t \in (0, 1)$ and a contraction $f \in \Pi_C$. We define a mapping $T_t x = tf(x) + (1 - t)Tx,\ x \in C$. It is obviously that $T_t$ is a contraction on

*Mathematics Subject Classification.* Primary 47H09; Secondary 65J15.

*Key words and phrases.* Nonexpansive map; Iteration scheme; Sunny and nonexpansive retraction; viscosity method.

$C$. In fact, for $x, y \in C$, we obtain

$$\begin{aligned}\|T_t x - T_t y\| &\leq \|t(f(x) - f(y)) + (1-t)(Tx - Ty)\| \\ &\leq \alpha t\|x - y\| + (1-t)\|Tx - Ty\| \\ &\leq \alpha t\|x - y\| + (1-t)\|x - y\| \\ &= (1 - t(1-\alpha))\|x - y\|.\end{aligned}$$

Let $x_t$ be the unique fixed point of $T_t$. That is, $x_t$ is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t. \tag{1.1}$$

A special case has been considered by Browder [1] in a Hilbert space as follows. Fix $u \in C$ and define a contraction $S_t$ on $C$ by

$$S_t x = tu + (1-t)Tx, \quad x \in C.$$

If we use $z_t$ to denote the unique fixed point of $S_t$, which yields that $z_t = tu + (1-t)Tz_t$.

In 1967, Browder [1] proved the following theorem.

**Theorem 1.1** *In a Hilbert space, as $t \to 0$, $z_t$ converges strongly to a fixed point of $T$ that is closet to $u$, that is, the nearest point projection of $u$ onto $F(T)$.*

Also, In 1967, Halpern [5] firstly introduced this iteration scheme

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \end{cases} \tag{1.2}$$

which is the special cases of

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n. \end{cases} \tag{1.3}$$

In [9], Moudafi proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. If $H$ is a Hilbert space, $T : C \to C$ is a nonexpansive self-mapping on a nonempty closed convex $C$ of $H$ and $f : C \to C$ is a contraction, he proved the following theorems.

**Theorem 1.2** *(Moudafi [9]). The sequence $\{x_n\}$ generated by the scheme*

$$x_n = \frac{1}{1+\epsilon_n}Tx_n + \frac{\epsilon_n}{1+\epsilon_n}f(x_n)$$

*converges strongly to the unique solution of the variational inequality:*

$$\bar{x} \in F(T), \ \textit{such that} \ \langle (I - f)\bar{x}, \bar{x} - x\rangle \leq 0, \ \forall x \in F(T),$$

*where $\{\epsilon_n\}$ is a sequence of positive numbers tending to zero.*

**Theorem 1.3** *(Moudafi [9]). With and initial $z_0 \in C$ defined the sequence $\{z_n\}$ by*

$$z_{n+1} = \frac{1}{1+\epsilon_n} T z_n + \frac{\epsilon_n}{1+\epsilon_n} f(z_n).$$

*Supposed that* $\lim_{n\to\infty} \epsilon_n = 0$, *and* $\sum_{n=1}^{\infty} \epsilon = \infty$ *and* $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon}| = 0$. *Then $\{z_n\}$ converges strongly to the unique solution of the unique solutions of the variational inequality:*

$$\bar{x} \in F(T) \text{ such that } \langle (I-f)\bar{x}, \bar{x} - x\rangle \leq 0, \ \forall x \in F(T).$$

Recently Xu [14] studied the viscosity approximation methods proposed by Moudafi [9] for nonexpansive mappings in a uniformly smooth Banach space. More precisely, he proved following theorems.

**Theorem 1.4** *(Xu [14]). Let $E$ be a uniformly smooth Banach space, $C$ a closed convex subset of $E$ and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Pi_C$. Then the path $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$, $t \in (0,1)$, converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by $Q(f) = lim_{t\to 0} x_t$, the $Q(f)$ solves the variational inequality*

$$\langle (I-f)Q(f), j(Q(f)-x)\rangle, \quad f \in \Pi_C, \ x \in F(T).$$

**Theorem 1.5** *(Xu [14]). Let $E$ be a uniformly smooth Banach space, $C$ a closed convex subset of $E$ and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Assume that $\alpha_n \in (0,1)$ satisfies the following conditions*
*(i)* $\lim_{n\to\infty} \alpha_n = 0$;
*(ii)* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
*(iii) either* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ *or* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty$. *Then the sequence $\{x_n\}$ generated by*

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n = 0, 1, 2, \ldots$$

*converges strongly to a fixed point of $T$.*

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [8] and is defined as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.4}$$

where the initial guess $x_0$ is taken in $C$ arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval [0, 1].

The second iteration process is referred to as Ishikawa's iteration process [6] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \end{cases} \tag{1.5}$$

where the initial guess $x_0$ is taken in $C$ arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1]. But both (1.4) and (1.5) have only weak convergence, in general (see [4] for an example). For example, Reich [11], shows that if $E$ is a uniformly convex and has a *Fréhet* differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by processes (1.4) converges weakly to a point in $F(T)$. (An extension of this result to processes (1.5) can be found in [13].) Therefore, many authors attempt to modify (1.4) and (1.5) to have strong convergence. Recently, Kim and Xu [7] introduced the following iteration process in the framework of Banach spaces.

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n. \end{cases} \tag{1.6}$$

More precisely, they proved the following theorem:

**Theorem 1.6** *(Kim and Xu [7]). Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$ and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Give a point $u \in C$ and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$, the following conditions are satisfied:*
*(i) $\alpha_n \to 0$, $\beta_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,*
*(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.*
*Define a sequence $\{x_n\}$ in $C$ by (1.6). Then $\{x_n\}$ strongly to converges to a fixed point of $T$.*

In this paper, we use viscosity approximation methods to study strong convergence of a pair of nonexpansive mappings in the framework of uniformly smooth Banach spaces. We introduce the composite iteration process as follows:

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)T_2 x_n, \\ y_n = \beta_n x_n + (1-\beta_n)T_1 z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \end{cases} \tag{1.7}$$

where the sequence $\{\alpha_n\}$ in (0,1) and $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in [0,1]. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, that $\{x_n\}$ defined by (1.7) converges to a common fixed point of $T_1$ and $T_2$, which solves some variational inequality.

If $\{\gamma_n\} = 1$ in (1.7) this can be viewed as a modified Mann iteration process

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)T_1 x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n. \end{cases} \tag{1.8}$$

If $\{\gamma_n\} = 1$ and $\{\beta_n\} = 0$ in (1.7), then (1.7) reduces to (1.3) which considered by Xu [14].

It is our purpose in this paper is to introduce this composite iteration scheme for approximating a common fixed point of two nonexpansive mappings by using viscosity methods in the framework of uniformly smooth Banach spaces. we establish the strong convergence of the sequence $\{x_n\}$ defined by (1.7). Our results improve and extend the ones announced by Kim and Xu [7], Xu [14] and some others.

We need the following definitions and lemmas for the proof of our main results.

The norm of $E$ is said to be Gâteaux differentiable (and $E$ is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.9}$$

exists for each $x, y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth ) if the limit in (1.9) is attained uniformly for $(x, y) \in U \times U$.

**Lemma 1.1** *A Banach space $E$ is uniformly smooth if and only if the duality map $J$ is single-valued and norm-to-norm uniformly continuous on bounded sets of $E$.*

**Lemma 1.2** *In a Banach space $E$, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E$$

*where $j(x+y) \in J(x+y)$.*

**Lemma 1.3** *( Xu [15], [16]). Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \gamma_n \sigma_n, \quad n \ge 0,$$

*where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that*
*(i) $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$,*
*(ii) either $\limsup_{n\to\infty} \sigma_n \le 0$ or $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$.*
*Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.*

Recall that if $C$ and $D$ are nonempty subsets of a Banach space $E$ such that $C$ is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny ([2], [12]) provided $Q(x+t(x-Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction,which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [2, 3, 12]: if $E$ is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0 \quad \text{for all} \quad x \in C \quad \text{and} \quad y \in D.$$

Reich [10] showed that if $E$ is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from $C$ into itself, then there is a sunny nonexpansive retraction from $C$ onto $D$ and it can be constructed as follows.

**Lemma 1.4** *(Reich [10]). Let $E$ be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converging strongly as $t \to 0$ to a fixed point of $T$. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t \to 0} x_t$. Then $Q$ is the unique sunny nonexpansive retract from $C$ onto $F(T)$; that is, $Q$ satisfies the property*

$$\langle u - Qu, J(z - Qu)\rangle \leq 0, u \in C, \qquad z \in F(T).$$

**Lemma 1.5** *(Xu [14]). Let $E$ be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of $T$. Define $Q : \Pi_C \to F(T)$ by*

$$Qf = s - \lim_{t \to 0} x_t, \ f \in \Pi_C. \tag{1.10}$$

*Then $Q(f)$ solves the variational inequality*

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad f \in \Pi_C, p \in F(T). \tag{1.11}$$

*In particular, if $f = u$ is a constant, then (1.10) is reduced to the sunny nonexpansive retract from $C$ onto $F(T)$:*

$$\langle u - Qu, J(p - Qu)\rangle \leq 0, u \in C, p \in F(T). \tag{1.12}$$

## 2. Main Results

**Theorem 2.1** Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$ and let $T_1, T_2 : C \to C$ be a pair of nonexpansive mappings such that $F(T_1T_2) = F(T_1) \bigcap F(T_2) \neq \emptyset$. The initial guess $x_0 \in C$ is chosen

arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ in (0,1) and $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in [0,1], the following conditions are satisfied
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \alpha_n \to 0$;
(ii) $\beta_n \to 0,\ \gamma_n \to 0$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.
Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)T_2 x_n, \\ y_n = \beta_n x_n + (1-\beta_n)T_1 z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some common fixed point $p \in F(T_1) \cap F(T_2)$ which solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C, p \in F(T_1) \cap F(T_2). \tag{2.1}$$

**Proof.** First we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, taking a fixed point $p$ of $F(T_1) \cap F(T_2)$, we note that

$$\|z_n - p\| \le \gamma_n \|x_n - p\| + (1-\gamma_n)\|T_2 x_n - p\| \le \|x_n - p\|. \tag{2.2}$$

It follows that

$$\begin{aligned} \|y_n - p\| &\le \beta_n \|x_n - p\| + (1-\beta_n)\|T_1 z_n - p\| \\ &\le \beta_n \|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{2.3}$$

It follows from (2.3) that

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|y_n - p\| \\ &\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\le \max\{\frac{1}{1-\alpha}\|f(p) - p\|, \|x_n - p\|\}. \end{aligned}$$

Now, an induction yields

$$\|x_n - p\| \le \max\{\frac{1}{1-\alpha}\|f(p) - p\|, \|x_0 - p\|\}.\ \ n \ge 0, \tag{2.4}$$

which implies that $\{x_n\}$ is bounded, so are $\{T_2 x_n\}$, $\{f(x_n)\}$ $\{y_n\}$, $\{z_n\}$ and $\{T_1 z_n\}$.
Since condition (i), we obtain

$$\|x_{n+1} - y_n\| = \alpha_n \|f(x_n) - y_n\| \to 0, \quad as\ n \to \infty. \tag{2.6}$$

Next, we claim that

$$\|x_{n+1} - x_n\| \to 0. \tag{2.6}$$

In order to prove (2.6) from

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \\ x_n = \alpha_{n-1} f(x_n) + (1-\alpha_{n-1})y_n. \end{cases}$$

We have

$$\begin{aligned} x_{n+1} - x_n =& (1-\alpha_n)(y_n - y_{n-1}) \\ &+ (\alpha_{n-1} - \alpha_n)(y_{n-1} - f(x_{n-1})) + \alpha_n(f(x_n) - f(x_{n-1})). \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x_n\| \le& (1-\alpha_n)\|y_n - y_{n-1}\| \\ &+ |\alpha_{n-1} - \alpha_n|\|y_{n-1} - f(x_{n-1})\| + \alpha\alpha_n\|x_n - x_{n-1}\|. \end{aligned} \tag{2.7}$$

Similarly, Since

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)T_1 z_n, \\ y_{n-1} = \beta_{n-1} x_{n-1} + (1-\beta_{n-1})T_1 z_{n-1}. \end{cases}$$

We obtain

$$\begin{aligned} y_n - y_{n-1} =& (1-\beta_n)(T_1 z_n - T_1 z_{n-1}) + \beta_n(x_n - x_{n-1}) \\ &+ (T_1 z_{n-1} - x_{n-1})(\beta_{n-1} - \beta_n). \end{aligned}$$

It follow that

$$\begin{aligned} \|y_n - y_{n-1}\| \le& (1-\beta_n)\|T_1 z_n - T_1 z_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\ &+ \|T_1 z_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n| \\ \le& (1-\beta_n)\|z_n - z_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\ &+ \|T_1 z_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n|. \end{aligned} \tag{2.8}$$

On the other hand, from

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)T_2 x_n, \\ z_{n-1} = \gamma_{n-1} x_{n-1} + (1-\gamma_{n-1})T_2 z_{n-1}, \end{cases}$$

we also can obtain

$$\begin{aligned} z_n - z_{n-1} =& (1-\gamma_n)(T_2 x_n - T_2 x_{n-1}) + \gamma_n(x_n - x_{n-1}) \\ &+ (\gamma_{n-1} - \gamma_n)(T_2 x_{n-1} - x_{n-1}), \end{aligned}$$

which yields that

$$\|z_n - z_{n-1}\| \le \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|T_2 x_{n-1} - x_{n-1}\|. \tag{2.9}$$

Substituting (2.9) into (2.8), we get

$$\begin{aligned} \|y_n - y_{n-1}\| \le& (1-\beta_n)(\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|T_2 x_{n-1} - x_{n-1}\|) \\ &+ \beta_n\|x_n - x_{n-1}\| + \|T_1 z_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n|. \end{aligned} \tag{2.10}$$

That is,

$$\begin{aligned}\|y_n - y_{n-1}\| \le \|x_n - x_{n-1}\| &+ |\gamma_{n-1} - \gamma_n|\|T_2 x_{n-1} - x_{n-1}\| \\ &+ \|T_1 z_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n|.\end{aligned} \tag{2.11}$$

Similarly, substitute (2.11) into (2.7) yields that

$$\begin{aligned}\|x_{n+1} - x_n\| &\le (1-\alpha_n)(\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|T_2 x_{n-1} - x_{n-1}\| \\ &\quad + \|T_1 z_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n|) \\ &\quad + |\alpha_{n-1} - \alpha_n|\|y_{n-1} - f(x_{n-1})\| + \alpha\alpha_n\|x_n - x_{n-1}\| \\ &\le (1-(1-\alpha)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + M_1(|\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n| + |\gamma_{n-1} - \gamma_n|),\end{aligned} \tag{2.12}$$

where $M_1$ is a constant such that

$$M_1 \ge \max\{\|y_{n-1} - f(x_{n-1})\|, \|x_{n-1} - T_2 x_{n-1}\|, \|x_{n-1} - T_1 z_{n-1}\|\}$$

for all $n$. By assumptions (i)-(iii), we have that

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty}(1-\alpha)\alpha_n = \infty,$$

and

$$\sum_{n=1}^{\infty}(|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|) < \infty.$$

Hence, Lemma 1.3 is applicable to (2.12) and we obtain (2.6) holds. Observe that

$$\begin{aligned}&\|T_1T_2x_n - x_n\| \\ &\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T_1 z_n\| + \|T_1 z_n - T_1T_2 x_n\| \\ &\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - T_1 z_n\| + \|z_n - T_2 x_n\| \\ &\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - T_1 z_n\| + \gamma_n\|x_n - T_2 x_n\|.\end{aligned} \tag{2.13}$$

Since assumption $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\gamma_n = 0$, (2.5) and (2.6), we know

$$\|T_1T_2x_n - x_n\| \to 0. \tag{2.14}$$

Put $T = T_1T_2$. Since $T_1$ and $T_2$ are nonexpansive, we have $T$ is also nonexpansive. Next, we claim that

$$\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \le 0, \tag{2.15}$$

where $q = Qf = s-\lim_{t\to 0} x_t$ with $x_t$ being the fixed point of the contraction $x \mapsto tf(x) + (1-t)Tx$, where $T = T_1T_2$. From $x_t$ solves the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t.$$

Thus we have

$$\|x_t - x_n\| = \|(1-t)(Tx_t - x_n) + t(f(x_t) - x_n)\|.$$

It follows from Lemma 1.2 that

$$\begin{aligned} \|x_t - x_n\|^2 &\le (1-t)^2\|Tx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\ &\le (1-2t+t^2)\|x_t - x_n\|^2 + f_n(t) \\ &\quad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2, \end{aligned} \tag{2.16}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - Tx_n\|)\|x_n - Tx_n\| \to 0, \text{ as } n \to 0. \tag{2.17}$$

It follows that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{2.18}$$

Let $n \to \infty$ in (2.18) and note (2.17) yields

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M_2, \tag{2.19}$$

where $M_2 > 0$ is a constant such that $M_2 \ge \|x_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 1$. Taking $t \to 0$ from (2.19), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0.$$

So, for any $\epsilon > 0$, there exists a positive number $\delta_1$ such that, for $t \in (0, \delta_1)$, we get

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{\epsilon}{2}. \tag{2.20}$$

On the other hand, since $x_t \to q$ as $t \to 0$, from Lemma 1.1, there exists $\delta_2 > 0$ such that, for $t \in (0, \delta_2)$ we have

$$\begin{aligned} & |\langle f(q) - q, J(x_n - q)\rangle - \langle x_t - f(x_t), J(x_t - x_n)\rangle| \\ \le\ & |\langle f(q) - q, J(x_n - q)\rangle - \langle f(q) - q, J(x_n - x_t)\rangle| \\ & + |\langle f(q) - q, J(x_n - x_t)\rangle - \langle x_t - f(x_t), J(x_t - x_n)\rangle| \\ \le\ & |\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle| \\ & + |\langle f(q) - f(x_t) - q + x_t, J(x_n - q)\rangle| \\ \le\ & \|f(q) - q\|\|J(x_n - q) - J(x_n - x_t)\| \\ & + \|f(q) - f(x_t) - q + x_t\|\|x_n - q\| \\ <\ & \tfrac{\epsilon}{2}. \end{aligned}$$

Picking $\delta = \min\{\delta_1, \delta_2\}, \forall t \in (0, \delta)$, we have

$$\langle f(q) - q, J(x_n - q)\rangle \le \langle x_t - f(x_t), J(x_t - x_n)\rangle + \frac{\epsilon}{2}.$$

That is,

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le \limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t-x_n)\rangle + \frac{\epsilon}{2}.$$

It follows from (2.21) that

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le \epsilon.$$

Since $\epsilon$ is chosen arbitrarily, we have

$$(2.21). \qquad \limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le 0$$

Finally, we show that $x_n \to q$ strongly and this concludes the proof. Indeed, using Lemma 1.2 again we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)(y_n-q)+\alpha_n(f(x_n)-q)\|^2\\
&\le (1-\alpha_n)^2\|y_n-q\|^2 + 2\alpha_n\langle f(x_n)-q, J(x_{n+1}-q)\rangle\\
&\le (1-\alpha_n)^2\|x_n-q\|^2\\
&\quad + 2\alpha_n\langle f(x_n)-f(q), J(x_{n+1}-q)\rangle + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle\\
&\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n\alpha\|x_n-q\|\|x_{n+1}-q\|\\
&\quad + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle\\
&\le (1-\alpha_n)^2\|x_n-q\|^2 + \alpha_n\alpha(\|x_n-q\|^2+\|x_{n+1}-q\|^2)\\
&\quad + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\le \frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n}\|x_n-q\|^2 - \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(q)-q, J(x_{n+1}-q)\rangle\\
&\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n-q\|^2 - \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(q)-q, J(x_{n+1}-q)\rangle + M_2\alpha_n^2\\
&= (1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n})\|x_n-q\|^2\\
&\quad + \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}(\frac{M_2(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(q)-q), J(x_{n+1}-q)\rangle.
\end{aligned}$$

Now we apply Lemma 1.3 and use (2.21) to see that $\|x_n-q\|\to 0$. This completes the proof.

As corollaries of Theorem 2.1, we have the following.

**Corollary 2.2** Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$ and let $T_1 : C \to C$ be a nonexpansive mapping such that

$F(T_1) \neq \emptyset$. The initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ in (0,1) and $\{\beta_n\}_{n=0}^{\infty}$ in [0,1], the following conditions are satisfied
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty, \ \alpha_n \to 0, ;$
(ii) $\beta_n < a,$ for some $a \in [0, 1);$
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$
Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by (1.8), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some fixed point $p \in F(T_1)$ which $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad f \in \Pi_C, p \in F(T_1).$$

**Proof.** By taking $\{\gamma_n\} = 1$, we can obtain the desired conclusion. This completes the proof.

**Corollary 2.3** *(Xu [14]). Let $E$ be a uniformly smooth Banach space, $C$ a closed convex subset of $E$ and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Pi_C$. Assume that $\alpha_n \in (0, 1)$ satisfies the following conditions*
*(i)* $\lim_{n\to\infty} \alpha_n = 0;$
*(ii)* $\sum_{n=0}^{\infty} \alpha_n = \infty;$
*(iii)* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty$. *Then the sequence $\{x_n\}$ generated by*

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \ldots$$

*converges strongly to $Q(f)$, which solves the variational inequality*

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad f \in \Pi_C, p \in F(T).$$

**Proof.** By taking $\{\gamma_n\} = 1$ and $\{\beta_n\}$=0, we can obtain the desired conclusion. This completes the proof.

## References

[1] F.E. Browder, *Convergence of approximations to fixed points of nonexpansive mappings in Banach spaces*, Arch. Rational Mech. Anal. 24 (1967) 82-90.
[2] R.E. Bruck, *Nonexpansive projections on subsets of Banach spaces,* Pacific J. Math. 47 (1973) 341-355.
[3] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel dekker, New York, 1984.
[4] A. Genel, J. Lindenstrass, *An example concerning fixed points*, Israel J. Math. 22 (1975) 81-86.
[5] B. Halpern, *Fixed points of nonexpansive mpas*, Bull. Amer. Math. Soc. 73 (1967) 957-961.
[6] S. Ishikawa, *Fixed points by a new iteration medthod*, Proc. Am. Math. Soc. 44 (1974) 147-150.

[7] T.H. Kim, H.K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal. 61 (2005) 51-60.

[8] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4 (1953) 506-510.

[9] A. Moudafi, *Viscosity approximation methods for fixed points problems*, J. Math. Anal Appl. 241 (2000) 46-55.

[10] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl.75 (1980) 287-292.

[11] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 67 (1979) 274-276.

[12] S. Reich, *Asymptotic behavior of contractions in banach spaces*, J. Math. Anal. Appl. 44 (1973) 57-70.

[13] K.K. Tan, K.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178 (2) (1993) 301-308.

[14] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. 298 (2004) 279-291.

[15] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. 66 (2002) 240-256.

[16] H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl. 116 (2003) 659-678.

Xiaolong Qin
Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, PR China
*e-mail address*: qxlxajh@163.com

Yongfu Su
Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, PR China
*e-mail address*: suyongfu@tjpu.edu.cn

Changqun Wu
School of Business and Administration, Henan University, Kaifeng 475001, PR China
*e-mail address*: kyls2003@Yahoo.com.cn

(*Received December 7, 2006*)