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ON CENTRAL GAP NUMBERS OF SYMMETRIC GROUPS

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Abstract. $g(G)$ denotes the central gap number of a group G. We show that for $n \geq 8$, $g(S_n) \geq n$ and $g(A_n) \geq n-2$. We give exact values of $g(S_n)$ and $g(A_n)$ for small n's. In particular, $g(S_9) = 9$ and $g(A_9) = 7$. Therefore, for any positive integer $n \neq 1, 3, 5$ there is a group G such that $n = g(G)$. G can be finite or infinite.

1. Introduction

K. Tanaka and others introduced the notion of ladder index of a group related to stability of the logical formula expressing the commutativity of a group [2]. The ladder index of a group is essentially the same as the central gap number introduced by Lennox and Roseblade [3]. K. Tanaka proved that the central gap number of a group cannot be 1, or 3. They are trying to prove that this number cannot be 5, but it seems that they are not successful. K. Tanaka conjectured that the central gap number of a group cannot be odd, and asked what is the central gap number of S_7 in a meeting at RIMS, Kyoto University in March, 2003.

With an aid of a computer, the author found that the central gap number of S_7 is 6. By improving computer programs, the author managed to find that the central gap numbers of S_8, S_9, S_{10}, and S_{11} are 8, 9, 10, and 11 respectively, and those of A_8, A_9, A_{10}, and A_{11} are 6, 7, 8, and 9 respectively.

By looking at logs of computer calculations, the author realized that the central gap number of S_n is at least n for $n \geq 8$, and that of A_n is at least $n-2$ for $n \geq 8$.

In this paper, we prove this fact and calculate the central gap numbers of S_n and A_n for $n \leq 9$. The author has no readable proof for the exact values of the central gap numbers of S_{10}, S_{11}, A_{10}, and A_{11}.

We can see that the central gap number of a direct product of groups is the sum of those of direct components. Since $g(S_3) = 2$ and $g(A_9) = 7$, for any positive integer $n \neq 1, 3, 5$ there is a group with the central gap number n. The groups can be finite or infinite.

2. Preliminaries

Let G be a group. For a subset X of G, we write $C_G(X)$ for the centralizer of X in G. If $X = \{a_1, a_2, \ldots, a_n\}$, we also write $C_G(a_1, a_2, \ldots, a_n)$ for

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$C_G(X)$. If G is known from the context, we just write C for C_G. For a subgroup H of G, we also write $C_H(X)$ for $H \cap C_G(X)$ even if X is not a subset of H.

The following definition is due to Lennox and Roseblade [3].

Definition 2.1. We say that a group G has a *finite central gap number*, or merely a *finite gap number*, if there is a non-negative integer h such that in any chain

$$C_G(H_1) \leq C_G(H_2) \leq \cdots \leq C_G(H_n) \leq \cdots$$

of centralizers of subgroups H_1, H_2, ..., H_n, ... of G, there are at most h strict inclusions. $g(G)$ denotes the least such h. We call $g(G)$ the central gap number of G.

Definition 2.2. Let G be a group. A sequence $(a_1, a_2, \ldots, a_n)$ of elements of G is called a *gap sequence* in G if there is a sequence $(b_1, b_2, \ldots, b_n)$ of elements in G such that for each $i \leq n$, b_i commutes with a_j for every $j < i$ but not with a_i. We call $(b_1, b_2, \ldots, b_n)$ a witness for the gap sequence $(a_1, a_2, \ldots, a_n)$. We often display them in a vertical way as follows:

sequence		witness
a_1	;	b_1
a_2	;	b_2
$\vdots$		$\vdots$
a_n	;	b_n

It is easy to see that $g(G)$ is the length of the longest gap sequence in G.

Definition 2.3. For a natural number n, the *height* of n is k, written $\mathrm{ht}(n) = k$, if $n = p_1 p_2 \cdots p_k$ where each p_i is a prime number.

For a finite group G, the order of a subgroup of G is a divisor of the order of G. Therefore, we have the following:

Lemma 2.4. *If* $\mathrm{ht}(|C_G(a_1, a_2, \ldots, a_k)|) = m$ *then the length of a gap sequence in* G *beginning with* $(a_1, a_2, \ldots, a_k)$ *is at most* $k + m$.

Now, we turn to our notation about permutations.

Let I be a set. $S(I)$ is the symmetric group consisting of all bijections from I to itself using composition as the multiplication. We multiply permutations from right to left. $A(I)$ is the alternating group consisting of all even permutations on I. If $I = \{1, 2, \ldots, n\}$ then $S(I)$ will be written S_n and $A(I)$ will be written A_n. If $\sigma \in S(I)$ and $x \in I$, $\sigma(x)$ is the image of x by σ. If $J \subset I$ then $\sigma(J) = \{\sigma(x) : x \in J\}$. If $\tau \in S(I)$, $\sigma^\tau = \tau\sigma\tau^{-1}$. We say that σ is conjugate to σ' over π by τ if (1) $\pi^\tau = \pi$, and (2) $\sigma = \sigma'^\tau$ or $\sigma^\tau = \sigma'$.

Definition 2.5. For a permutation $\sigma \in S_n$, the *type* (*cycle type*) of σ is

$$(n^{m_n}, \ldots, 2^{m_2}, 1^{m_1})$$

where m_k is the number of k-cycles in the cycle decomposition of σ. We usually omit k^{m_k} if $m_k = 0$. We often omit 1^{m_1} also. For example, if $\sigma = (1\ 2)(3\ 4)(5\ 6\ 7)(8)(9) \in S_9$ then the type of σ is $(3^1, 2^2, 1^2)$, or $(3^1, 2^2)$. We call a cycle in the cycle decomposition of σ a *cycle component* of σ.

Let U be a subset of $S(I)$. Then we define the *support* and the set of *fixed points* of U by

$$\mathrm{supp}_I(U) = \{x \in I : \sigma(x) \neq x \text{ for some } \sigma \in U\}$$

and

$$\mathrm{fix}_I(U) = \{x \in I : \sigma(x) = x \text{ for all } \sigma \in U\}.$$

The following lemma is an easy fact but useful for checking if two permutations are commuting.

Lemma 2.6. (1) *Two permutations σ and τ are commuting if and only if $\sigma^\tau = \sigma$. In particular, if σ and τ are commuting and θ is a cycle component of σ then so is θ^τ.*

(2) *Suppose two permutations σ and τ act on a set Ω and $\sigma\tau = \tau\sigma$. If $I \subset \Omega$ is σ-invariant then so is $\tau(I)$. In particular, $\mathrm{fix}_\Omega(\sigma)$ and $\mathrm{supp}_\Omega(\sigma)$ are τ-invariant.*

Lemma 2.7. *If $X \subset S(J)$ and $I = \mathrm{supp}_J(X)$ then*

$$C_{S(J)}(X) = C_{S(I)}(X) \times S(J - I).$$

Proof. It is clear that the right hand side is a subset of the left hand side.

Suppose $\tau \in C_{S(J)}(X)$. Then $I = \mathrm{supp}_J(X)$ is τ-invariant by Lemma 2.6 (2) and $J - I$ is also τ-invariant. Therefore, $\tau \in S(I) \times S(J - I)$ and hence $\tau \in C_{S(I)}(X) \times S(J - I)$. $\square$

The following fact is useful for analysis of A_n.

Lemma 2.8. *If G is a subgroup of S_n containing an odd permutation then $(G : G \cap A_n) = 2$.*

3. Lower Bounds

In this section, we show that $n \leq g(S_n)$ and $n - 2 \leq g(A_n)$ for any $n \geq 8$. We calculate the exact values of $g(S_n)$ and $g(A_n)$ for small n in later sections. Note that $g(S_n)$ and $g(A_n)$ are less than $n \log_2 n$.

Theorem 3.1. (1) $g(S_3) \geq 2$.

(2) $g(S_5) \geq g(S_4) \geq 4$.

(3) $g(S_7) \geq g(S_6) \geq 6$.

(4) *$g(S_n) \geq n$ for $n \geq 8$.*

Proof. The following tables of gap sequences show the theorem:
(1)

sequence		witness
(1 2)	;	(1 2 3)
(1 2 3)	;	(1 2)

(2)

sequence		witness
(1 2)(3 4)	;	(2 3)
(1 2)	;	(1 3)(2 4)
(1 3)(2 4)	;	(1 2)
(2 3)	;	(1 2)(3 4)

(3)

sequence		witness
(1 2)	;	(2 3)
(3 4)(5 6)	;	(4 5)
(3 4)	;	(3 5)(4 6)
(1 3)(2 4)	;	(1 2)
(2 3)	;	(1 2)(3 4)
(4 5)	;	(5 6)

(4)

sequence		witness
(1 2)(3 4)	;	(1 3)
(1 2)	;	(1 3)(2 4)
(1 3)(2 4)	;	(1 2)
(2 3)	;	(1 2)(3 4)
(4 5)	;	(5 6)
(5 6)	;	(6 7)
$\vdots$		$\vdots$
$(n-5\ n-4)$	;	$(n-4\ n-3)$
$(n-3\ n-2)(n-1\ n)$	;	$(n-2\ n-1)$
$(n-3\ n-2)$	;	$(n-3\ n-1)(n-2\ n)$
$(n-3\ n-1)(n-2\ n)$	;	$(n-3\ n-2)$
$(n-2\ n-1)$	;	$(n-3\ n-2)(n-1\ n)$

□

Theorem 3.2. (1) *$g(A_5) \geq g(A_4) \geq 2$.*
(2) *$g(A_7) \geq g(A_6) \geq 4$.*

(3) $g(A_n) \geq n-2$ *for* $n \geq 8$.

Proof. The following tables of gap sequences show the theorem:

(1)

sequence		witness
$(1\ 2)(3\ 4)$	;	$(1\ 2\ 3)$
$(1\ 2\ 3)$	;	$(1\ 2)(3\ 4)$

(2)

sequence		witness
$(1\ 2)(3\ 4)$	;	$(1\ 2\ 3)$
$(1\ 3)(2\ 4)$	;	$(1\ 2)(5\ 6)$
$(1\ 2)(5\ 6)$	;	$(1\ 3)(2\ 4)$
$(1\ 2\ 3)$	;	$(1\ 2)(3\ 4)$

(3)

sequence		witness
$(1\ 2)(3\ 4)$	;	$(1\ 3\ 2)$
$(1\ 3)(2\ 4)$	;	$(1\ 2)(5\ 6)$
$(1\ 3\ 2)$	;	$(1\ 2)(3\ 4)$
$(1\ 2)(4\ 5)$	;	$(5\ 7\ 6)$
$(1\ 2)(5\ 6)$	;	$(6\ 8\ 7)$
$\vdots$		$\vdots$
$(1\ 2)(n-5\ n-4)$	;	$(n-4\ n-2\ n-3)$
$(n-3\ n-2)(n-1\ n)$	;	$(n-3\ n-1\ n-2)$
$(1\ 2)(n-3\ n-2)$	;	$(n-3\ n-1)(n-2\ n)$
$(1\ 2)(n-4\ n-3)$	;	$(n-3\ n-2)(n-1\ n)$

□

4. Exact Values

We begin with an evaluation of upper bounds of $g(S_n)$. The following lemma is well-known.

Lemma 4.1. *If* $\sigma \in S_n$ *has type* $(1^{m_1}, 2^{m_2}, \dots n^{m_n})$ *then*

$$|C_{S_n}(\sigma)| = 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!.$$

The following lemma is due to K. Tanaka [2]. We give a proof for convenience.

Lemma 4.2 (K. Tanaka)**.** $g(G) \neq 3$ *for any group* G.

Proof. Suppose $g(G) \geq 3$. Let (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively be a gap sequence and its witness in G.

If a_1 and a_2 are commuting then (a_1, a_2, b_2, b_1) is a gap sequence in G with witness (b_1, b_2, a_2, a_1).

If a_1 and a_2 are not commuting then (b_3, a_1, a_2, a_3) is a gap sequence in G with witness (a_3, a_2, a_1, b_3).

Therefore, $g(G) \geq 4$ in both cases. □

Proposition 4.3. *$g(S_3) = 2$ and $g(A_3) = 0$.*

Proof. We work in S_3. Any nontrivial element of S_3 is conjugate to $(1\ 2)$ or $(1\ 2\ 3)$. By Lemma 4.1, $|C((1\ 2))| = 2$ and $|C((1\ 2\ 3))| = 3$. Since both orders have height 1, $g(S_3) \leq 2$ by Lemma 2.4. $g(S_3) \geq 2$ by Theorem 3.1 (1).

$g(A_3) = 0$ since A_3 is abelian. □

Proposition 4.4. *$g(S_4) = g(S_5) = 4$ and $g(A_4) = g(A_5) = 2$.*

Proof. We work in S_5. For any non-trivial element σ of S_5, Table 1 obtained by Lemma 4.1 shows that $\mathrm{ht}(|C(\sigma)|) \leq 3$.

TABLE 1. Orders of centralizers in S_5

type of σ	$\lvert C(\sigma)\rvert$	$\mathrm{ht}(\lvert C(\sigma)\rvert)$
$(2^1, 1^3)$	$2^2 \cdot 3$	3
$(2^2, 1^1)$	2^3	3
$(3^1, 1^2)$	$2 \cdot 3$	2
$(3^1, 2^1)$	$2 \cdot 3$	2
$(4^1, 1^1)$	2^2	2
(5^1)	5	1

We have $g(S_4) = g(S_5) = 4$ by Lemma 2.4 and Theorem 3.1 (2).

By Lemma 2.8, Table 1 shows that $\mathrm{ht}(|C_{A_5}(\sigma)|) \leq 2$ for any non-trivial element σ in A_5. Hence, $g(A_5) \leq 3$ by Lemma 2.4. We have $g(A_5) \neq 3$ by Lemma 4.2. Therefore, $g(A_4) = g(A_5) = 2$ by Theorem 3.2 (1). □

Proposition 4.5. *$g(S_6) = g(S_7) = 6$.*

Proof. We work in S_7. We have Table 2 for S_7 by Lemma 4.1.

Let $(\sigma_1, \sigma_2, \ldots)$ be a gap sequence in S_7. We show that $\mathrm{ht}(|C(\sigma_1)|) \leq 5$ or $\mathrm{ht}(|C(\sigma_1, \sigma_2)|) \leq 4$. Then we have $g(S_6) = g(S_7) = 6$ by Lemma 2.4 and Theorem 3.1 (3).

Suppose $\mathrm{ht}(|C(\sigma_1)|) > 5$. Table 2 shows that σ_1 has type (2^1). If σ_2 has a type other than (2^1), (2^2), (2^3), and (3^1), then we have $\mathrm{ht}(|C(\sigma_1, \sigma_2)|) \leq 4$. If σ_2 has type (2^2), (2^3), or (3^1), we can find $\tau \in S_7$ such that τ commutes

TABLE 2. Orders of centralizers in S_7

type of σ	$\lvert C(\sigma)\rvert$	$\mathrm{ht}(\lvert C(\sigma)\rvert)$	type of σ	$\lvert C(\sigma)\rvert$	$\mathrm{ht}(\lvert C(\sigma)\rvert)$
$(2^1, 1^5)$	$2^4 \cdot 3 \cdot 5$	6	$(4^1, 1^3)$	$2^3 \cdot 3$	4
$(2^2, 1^3)$	$2^4 \cdot 3$	5	$(4^1, 2^1, 1^1)$	2^3	3
$(2^3, 1^1)$	$2^4 \cdot 3$	5	$(4^1, 3^1)$	$2^2 \cdot 3$	3
$(3^1, 1^4)$	$2^3 \cdot 3^2$	5	$(5^1, 1^2)$	$2 \cdot 5$	2
$(3^1, 2^1, 1^2)$	$2^2 \cdot 3$	3	$(5^1, 2^1)$	$2 \cdot 5$	2
$(3^1, 2^2)$	$2^3 \cdot 3$	4	$(6^1, 1^1)$	$2 \cdot 3$	2
$(3^2, 1^1)$	$2 \cdot 3^2$	3	(7^1)	7^1	1

with σ_2 but not with σ_1. This means that in these cases, $C(\sigma_1, \sigma_2)$ is a proper subgroup of $C(\sigma_2)$ and thus its order has a height at most 4. Hence σ_2 must have type (2^1). So, the pair (σ_1, σ_2) is conjugate to $((1\ 2), (2\ 3))$ or $((1\ 2), (3\ 4))$.

By Lemma 2.7, $C((1\ 2), (2\ 3)) = S(\{4, 5, 6, 7\})$, and it has order 24 with $\mathrm{ht}(24) = 4$. Again by Lemma 2.7,

$$C((1\ 2), (3\ 4)) = S(\{1, 2\}) \times S(\{3, 4\}) \times S(\{5, 6, 7\}),$$

and it has order 24 with $\mathrm{ht}(24) = 4$. □

Proposition 4.6. $g(S_9) = 9$.

Proof. We work in S_9. We have Table 3 for S_9 by Lemma 4.1.

Let $(\sigma_1, \sigma_2, \ldots)$ be a gap sequence in S_9. We show that $\mathrm{ht}(|C(\sigma_1)|) \leq 8$ or $\mathrm{ht}(|C(\sigma_1, \sigma_2)|) \leq 7$. Then we have the statement by Lemma 2.4 and Theorem 3.1 (4).

If $\mathrm{ht}(|C(\sigma_1)|) > 8$ then σ_1 has type (2^1). If σ_2 has a type other than (2^1), (2^2), (2^4), and (3^1), then $\mathrm{ht}(|C(\sigma_1, \sigma_2)|) \leq 7$.

If σ_1 has type (2^1) and σ_2 has type (2^2), (2^4) or (3^1), we can find $\tau \in S_9$ such that $\tau \in C(\sigma_2)$ but $\tau \notin C(\sigma_1)$. Hence $C(\sigma_1, \sigma_2)$ is a proper subgroup of $C(\sigma_2)$ and thus the height of its order is strictly less than 8.

If σ_1 and σ_2 have the same type (2^1), then the pair (σ_1, σ_2) is conjugate to $((1\ 2), (2\ 3))$ or $((1\ 2), (3\ 4))$. $C_{S_9}((1\ 2), (2\ 3)) = S(\{4, \ldots, 9\})$ has order $6!$ with $\mathrm{ht}(6!) = 7$.

$$C_{S_9}((1\ 2), (3\ 4)) = S(\{1, 2\}) \times S(\{3, 4\}) \times S(\{5, \ldots, 9\})$$

has order $2 \cdot 2 \cdot 5!$ with $\mathrm{ht}(2 \cdot 2 \cdot 5!) = 7$. □

5. GAP NUMBERS OF S_8, A_6, A_7, A_8 AND A_9

We calculate $g(S_8)$, $g(A_9)$, $g(A_6)$, $g(A_7)$, and $g(A_8)$ in this order.

TABLE 3. Orders of centralizers in S_9

type of σ	$\lvert C(\sigma)\rvert$	$\mathrm{ht}(\lvert C(\sigma)\rvert)$	type of σ	$\lvert C(\sigma)\rvert$	$\mathrm{ht}(\lvert C(\sigma)\rvert)$
$(2^1, 1^7)$	$2^5 \cdot 3^2 \cdot 5 \cdot 7$	9	$(4^1, 3^1, 2^1)$	$2^3 \cdot 3$	4
$(2^2, 1^5)$	$2^6 \cdot 3 \cdot 5$	8	$(4^2, 1^1)$	2^5	5
$(2^3, 1^3)$	$2^5 \cdot 3^2$	7	$(5^1, 1^4)$	$2^3 \cdot 3 \cdot 5$	5
$(2^4, 1^1)$	$2^7 \cdot 3$	8	$(5^1, 2^1, 1^2)$	$2^2 \cdot 5$	3
$(3^1, 1^6)$	$2^4 \cdot 3^3 \cdot 5$	8	$(5^1, 2^2)$	$2^3 \cdot 5$	4
$(3^1, 2^1, 1^4)$	$2^4 \cdot 3^2$	6	$(5^1, 3^1, 1^1)$	$3 \cdot 5$	2
$(3^1, 2^2, 1^2)$	$2^4 \cdot 3$	5	$(5^1, 4^1)$	$2^2 \cdot 5$	3
$(3^1, 2^3)$	$2^4 \cdot 3^2$	6	$(6^1, 1^3)$	$2^2 \cdot 3^2$	4
$(3^2, 1^3)$	$2^2 \cdot 3^3$	5	$(6^1, 2^1, 1^1)$	$2^2 \cdot 3$	3
$(3^2, 2^1, 1^1)$	$2^2 \cdot 3^2$	4	$(6^1, 3^1)$	$2 \cdot 3^2$	3
(3^3)	$2^1 3^4$	5	$(7^1, 1^2)$	$2 \cdot 7$	2
$(4^1, 1^5)$	$2^5 \cdot 3 \cdot 5$	7	$(7^1, 2^1)$	$2 \cdot 7$	2
$(4^1, 2^1, 1^3)$	$2^4 \cdot 3$	5	$(8^1, 1^1)$	2^3	3
$(4^1, 2^2, 1^1)$	2^5	5	(9^1)	3^2	2
$(4^1, 3^1, 1^2)$	$2^3 \cdot 3$	4			

Lemma 5.1. *Let $G_8 = C_{S_4}((1\ 2)(3\ 4))$. $|G_8| = 8$ and consists of the identity, $(1\ 2)$, $(3\ 4)$ (type (2^1)), $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, $(1\ 4)(2\ 3)$ (type (2^2)), $(1\ 3\ 2\ 4)$ and $(1\ 4\ 2\ 3)$ (type (4^1)).*

Let G_4 be the subgroup of G_8 consists of the identity, and the three elements of type (2^2). Then $G_4 = G_8 \cap A_4$ and $G_4 = C_{S_4}(\sigma, \sigma')$ for any two distinct elements of type (2^2) in S_4.

Lemma 5.2. *Suppose $\sigma, \sigma' \in S_8$ have the same type (2^4) and $\sigma \neq \sigma'$.*

(1) *The pair (σ, σ') is conjugate to a pair of $(1\ 2)(3\ 4)(5\ 6)(7\ 8)$ and one of the following:*
   (i) $(1\ 3)(2\ 4)(5\ 6)(7\ 8)$;
   (ii) $(1\ 3)(2\ 5)(4\ 6)(7\ 8)$;
   (iii) $(1\ 3)(2\ 4)(5\ 7)(6\ 8)$;
   (iv) $(1\ 3)(2\ 7)(4\ 5)(6\ 8)$.

(2) *The order of $C_{S_8}(\sigma, \sigma')$ is 32, 12, 32, and 8 respectively for cases (i), (ii), (iii) and (iv) in (1).*

(3) *Let G_8 and G_4 be as in Lemma 5.1 and G'_8 and G'_4 respectively be their conjugates by $(1\ 5)(2\ 6)(3\ 7)(4\ 8)$ (G'_8 and G'_4 are subgroups of $S(\{5, 6, 7, 8\})$).*

   In case (i) in (1), $C_{S_8}(\sigma, \sigma') = G_4 \times G'_8$.

   In case (iii) in (1), $G_4 \times G'_4$ is a normal subgroup of $G_{32} = C_{S_8}(\sigma, \sigma')$ of index 2.

Proof. (1) Up to conjugacy, we can assume that $\sigma = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$. If σ and σ' have 3 or more cycle components in common then $\sigma = \sigma'$.

Assume that σ and σ' have exactly 2 cycle components in common. Up to conjugacy over σ, we can assume $(5\ 6)$ and $(7\ 8)$ are the common cycle components. Then $\sigma' = (1\ a)(2\ b)(5\ 6)(7\ 8)$. Since σ is fixed under conjugation by $(3\ 4) = (a\ b)$, σ' is conjugate to $(1\ 3)(2\ 4)(5\ 6)(7\ 8)$ over σ. This is (i).

Suppose σ and σ' have exactly one cycle component in common. Up to conjugacy over σ, we can assume that $(7\ 8)$ is the common cycle component of σ and σ'. Then $\sigma' = (1\ a)(2\ b)(c\ d)(7\ 8)$ and $\{c, d\} \neq \{3, 4\}$. Since σ is fixed under the conjugation by $(3\ 4)$, we can assume that a or b is 3, and since σ is also fixed under the conjugation by $(1\ 2)$, we can assume that $\sigma' = (1\ 3)(2\ b)(c\ d)(7\ 8)$. Since $\{c, d\} \neq \{5, 6\}$, b must be 5 or 6. Also, σ is fixed under the conjugation by $(5\ 6)$. Therefore, σ' is conjugate to $(1\ 3)(2\ 5)(4\ 6)(7\ 8)$ over σ.

Suppose σ and σ' have no cycle components in common. Since $(1\ 2)$ is not a cycle component of σ', σ' is $(1\ a)(2\ b)(*\ *)(*\ *)$. Since σ has 3 orbits other than $\{1, 2\}$, there is an orbit $\{c, d\}$ of σ such that $a, b \notin \{c, d\}$. Since $(c\ d)$ is not a cycle component of σ', $\sigma' = (1\ *)(2\ *)(c\ *)(d\ *)$. Hence, σ' is conjugate to $(1\ *)(2\ *)(5\ *)(6\ *)$ over σ. Consider how 3 and 4 occur. σ' is conjugate to $(1\ 3)(2\ 4)(5\ *)(6\ *)$ or $(1\ 3)(2\ *)(5\ 4)(6\ *)$ over σ. Therefore, σ' is conjugate to

$$(1\ 3)(2\ 4)(5\ 7)(6\ 8) \text{ or } (1\ 3)(2\ 7)(4\ 5)(6\ 8)$$

over σ. These are (iii) and (iv).

(2) Recall G_4 and G_8 in Lemma 5.1. Let G_8' and G_4' be conjugates of G_8 and G_4 by $(1\ 5)(2\ 6)(3\ 7)(4\ 8)$. So, $G_8' = C_{S(\{5,6,7,8\})}((5\ 6)(7\ 8))$ and $G_4' = C_{S(\{5,6,7,8\})}((5\ 6)(7\ 8), (5\ 7)(6\ 8))$.

Let $\sigma = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$.

Case (i). $\sigma' = (1\ 3)(2\ 4)(5\ 6)(7\ 8)$. If τ commutes with both σ and σ' then $(5\ 6)^\tau$ and $(7\ 8)^\tau$ are cycle components of both σ and σ'. Hence, $\{5, \ldots, 8\}$ is τ-invariant and thus $\tau \in S_4 \times S(\{5, \ldots, 8\})$. Therefore,

$$C_{S_8}(\sigma, \sigma') = C_{S_4 \times S(\{5,\ldots,8\})}(\sigma, \sigma') = G_4 \times G_8'.$$

Case (ii). $\sigma' = (1\ 3)(2\ 5)(4\ 6)(7\ 8)$. Let $\tau \in S_8$ be an element commuting with both σ and σ'. Then $(7\ 8)^\tau$ is a cycle component of both σ and σ', and thus $(7\ 8)^\tau = (7\ 8)$. Hence, $\{1, \ldots, 6\}$ and $\{7, 8\}$ are τ-invariant.

Suppose $\tau(1) = 1$. $(1\ 2)^\tau = (1\ \tau(2))$ is a cycle component of σ and thus $\tau(2) = 2$. $(2\ 5)^\tau = (2\ \tau(5))$ is a cycle component of σ' and thus $\tau(5) = 5$. $(5\ 6)^\tau = (5\ \tau(6))$ is a cycle component of σ and thus $\tau(6) = 6$. $(4\ 6)^\tau = (\tau(4)\ 6)$ is a cycle component of σ' and thus $\tau(4) = 4$. Similarly, if

$\tau(1) \in \{1, \ldots, 6\}$ then τ on $\{1, \ldots, 6\}$ is uniquely determined depending on the value of $\tau(1)$. Hence there are 6 possibilities for τ on $\{1, \ldots, 6\}$.

We conclude that $C_{S_8}(\sigma, \sigma')$ has exactly 12 elements and contains $(7\ 8)$.

Case (iii). $\sigma' = (1\ 3)(2\ 4)(5\ 7)(6\ 8)$. Let $\tau \in S_8$ be an element commuting with both σ and σ'.

Suppose $\tau(1) = 3$. $(1\ 2)^\tau = (3\ \tau(2))$ is a cycle component of σ and $(1\ 3)^\tau = (3\ \tau(3))$ is a cycle component of σ'. Thus, $\tau(2) = 4$ and $\tau(3) = 1$. $(3\ 4)^\tau = (1\ \tau(4))$ is a cycle component of σ. Thus, $\tau(4) = 2$. Hence, $\tau = (1\ 3)(2\ 4)\tau'$ for some $\tau' \in S(\{5, 6, 7, 8\})$. This τ' must commute with $(5\ 6)(7\ 8)$ and $(5\ 7)(6\ 8)$, and thus $\tau' \in G'_4$.

With similar arguments, we can see that if $\tau(1) \in \{1, 2, 3, 4\}$ then $\tau \in G_4 \times G'_4$. There are 16 elements of this form.

If $\tau(1) \in \{5, 6, 7, 8\}$, we can see that τ on $\{1, 2, 3, 4\}$ is represented by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 8 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 7 & 8 & 5 & 6 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{pmatrix},$$

and τ on $\{5, 6, 7, 8\}$ is represented by

$$\begin{pmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Hence, $C_{S_8}(\sigma, \sigma')$ has an order at most 32. Since $G_4 \times G'_4$ is a subgroup of $C_{S_8}(\sigma, \sigma')$ and $(1\ 5)(2\ 6)(3\ 7)(4\ 8) \in C_{S_8}(\sigma, \sigma')$, $C_{S_8}(\sigma, \sigma')$ has order 32.

$G_4 \times G'_4$ has exactly two cosets in $C_{S_8}(\sigma, \sigma')$, namely, a coset including $(1\ 5)(2\ 6)(3\ 7)(4\ 8)$ and itself. Since $G_4 \times G'_4$ is invariant under conjugation by $(1\ 5)(2\ 6)(3\ 7)(4\ 8)$, $G_4 \times G'_4$ is a normal subgroup of $C_{S_8}(\sigma, \sigma')$.

Case (iv). $\sigma' = (1\ 3)(2\ 4)(5\ 7)(6\ 8)$. Let $\tau \in S_8$ be an element commuting with both σ and σ'. We can see then $\tau \in S_8$ is uniquely determined depending on the value of $\tau(1) \in \{1, \ldots, 8\}$ by considering conjugates of cycle components of σ and σ' by τ. Therefore, $C_{S_8}(\sigma, \sigma')$ has order 8. ☐

With Lemma 5.2, we can calculate $g(S_8)$.

Proposition 5.3. $g(S_8) = 8$.

Proof. We have Table 4 below by Lemma 4.1.

For any gap sequence $(\sigma_1, \sigma_2, \ldots)$, we show that $\mathrm{ht}(|C_{S_8}(\sigma_1)|) \leq 7$ or $\mathrm{ht}(|C_{S_8}(\sigma_1, \sigma_2)|) \leq 6$. Then we have the statement by Lemma 2.4.

Suppose $\mathrm{ht}(|C_{S_8}(\sigma_1)|) > 7$. Then the type of σ_1 is (2^1) or (2^4). If the type of σ_2 is neither (2^1), (2^4), nor (2^2) then $\mathrm{ht}(|C_{S_8}(\sigma_1, \sigma_2)|) \leq 6$.

If σ_2 has type (2^2) then it is easy to see that $C_{S_8}(\sigma_1, \sigma_2)$ is a proper subgroup of $C_{S_8}(\sigma_2)$. Hence, $\mathrm{ht}(|C_{S_8}(\sigma_1, \sigma_2)|) \leq 6$ in this case. Therefore, the types of σ_1 and σ_2 are among (2^1) and (2^4).

Now, we have only three cases to consider.

TABLE 4. Orders of centralizers in S_8

type of σ	$\|C(\sigma)\|$	ht($\|C(\sigma)\|$)	type of σ	$\|C(\sigma)\|$	ht($\|C(\sigma)\|$)
$(2^1,1^6)$	$2^5\cdot 3^2\cdot 5$	8	$(4^1,2^2)$	2^5	5
$(2^2,1^4)$	$2^6\cdot 3$	7	$(4^1,3^1,1^1)$	$2^2\cdot 3$	3
$(2^3,1^2)$	$2^5\cdot 3$	6	(4^2)	2^5	5
(2^4)	$2^7\cdot 3$	8	$(5^1,1^3)$	$2^1\cdot 3\cdot 5$	3
$(3^1,1^5)$	$2^3\cdot 3^2\cdot 5$	6	$(5^1,2^1,1^1)$	$2\cdot 5$	2
$(3^1,2^1,1^3)$	$2^2\cdot 3^2$	4	$(5^1,3^1)$	$3\cdot 5$	2
$(3^1,2^2,1^1)$	$2^3\cdot 3$	4	$(6^1,1^2)$	$2^2\cdot 3$	3
$(3^2,1^2)$	$2^2\cdot 3^2$	4	$(6^1,2^1)$	$2^2\cdot 3$	3
$(3^2,2^1)$	$2^2\cdot 3^2$	4	$(7^1,1^1)$	7	1
$(4^1,1^4)$	$2^5\cdot 3$	6	(8^1)	2^3	3
$(4^1,2^1,1^2)$	2^4	4			

Case 1. σ_1 and σ_2 have type (2^1). If $|\mathrm{supp}(\sigma_1,\sigma_2)|=3$ then $C_{S_8}(\sigma_1,\sigma_2)\cong S_5$. Hence, $\mathrm{ht}(|C_{S_8}(\sigma_1,\sigma_2)|)=5$. If $|\mathrm{supp}(\sigma_1,\sigma_2)|=4$ then $C_{S_8}(\sigma_1,\sigma_2)\cong S_2\times S_2\times S_4$. Hence, $\mathrm{ht}(|C_{S_8}(\sigma_1,\sigma_2)|)=6$.

Case 2. σ_1 and σ_2 have type (2^4). In this case, $\mathrm{ht}(|C_{S_8}(\sigma_1,\sigma_2)|)\le 5$ by Lemma 5.2.

Case 3. σ_1 has type (2^1) and σ_2 has type (2^4), or vice versa. The order of $C_{S_8}(\sigma_1,\sigma_2)$ is a common divisor of $2^5\cdot 3^2\cdot 5$ and $2^7\cdot 3$, hence a divisor of $2^5\cdot 3$. Therefore, $\mathrm{ht}(|C_{S_8}(\sigma_1,\sigma_2)|)\le 6$. □

Lemma 5.4. *Suppose permutations $\sigma,\sigma'\in S_9$ have the same type (2^4) and $\mathrm{supp}(\sigma,\sigma')=\{1,\dots,9\}$. Then $\mathrm{ht}(|C_{S_9}(\sigma,\sigma')|)\le 5$.*

Proof. Up to conjugacy, we can assume that $\sigma=(1\ 2)(3\ 4)(5\ 6)(7\ 8)$, and $(1\ 9)$ is a cycle component of σ'.

Let τ be an element of S_9 commuting with both σ and σ'. Since 9 is the only fixed point of σ, $\tau(9)=9$. Since $(1\ 9)^\tau=(\tau(1),9)$ is a cycle component of σ', we have $\tau(1)=1$. Since $(1\ 2)^\tau=(1,\tau(2))$ is a cycle component of σ, we have $\tau(2)=2$. Thus, $\tau\in S(\{3,\dots,8\})$ and τ commutes with $(3\ 4)(5\ 6)(7\ 8)$. Hence, $C_{S_9}(\sigma,\sigma')$ is isomorphic to a subgroup of $C_{S_6}((1\ 2)(3\ 4)(5\ 6))$, which has order $2^4\cdot 3$. Therefore, $\mathrm{ht}(|C_{S_9}(\sigma,\sigma')|)\le 5$. □

Lemma 5.5. *Suppose two permutations σ and σ' have the same type (3^1). Let $I=\mathrm{supp}(\sigma,\sigma')$. If $|\mathrm{supp}(\sigma,\sigma')|=3$ then $C_{S(I)}(\sigma,\sigma')=C_{S(I)}(\sigma)$, if $|\mathrm{supp}(\sigma,\sigma')|=4$ or 5 then $|C_{S(I)}(\sigma,\sigma')|=1$, and if $|\mathrm{supp}(\sigma,\sigma')|=6$ then $|C_{S(I)}(\sigma,\sigma')|=9$.*

Proof. Easy. □

Lemma 5.6. *Suppose two permutations σ and σ' have the same type (2^2) and $\sigma \neq \sigma'$.*

(1) *The pair (σ, σ') is conjugate to a pair of* $(1\ 2)(3\ 4)$ *and one of the following: (i)* $(1\ 3)(2\ 4)$*; (ii)* $(1\ 5)(3\ 4)$*; (iii)* $(1\ 5)(2\ 3)$*; (iv)* $(1\ 2)(5\ 6)$*; (v)* $(1\ 3)(5\ 6)$*; (vi)* $(1\ 5)(2\ 6)$*; (vii)* $(1\ 5)(3\ 6)$*; (viii)* $(1\ 7)(5\ 6)$*; and (ix)* $(5\ 6)(7\ 8)$*.*

(2) *Let $I = \mathrm{supp}(\sigma, \sigma')$. Then $|C_{S(I)}(\sigma, \sigma')|$ and the size of* $\mathrm{supp}(\sigma, \sigma')$ *are given as follows according to cases in (1): (i) 4 ($|\mathrm{supp}| = 4$), (ii) 2 ($|\mathrm{supp}| = 5$), (iii) 1 ($|\mathrm{supp}| = 5$), (iv) 8 ($|\mathrm{supp}| = 6$), (v) 4 ($|\mathrm{supp}| = 6$), (vi) 4 ($|\mathrm{supp}| = 6$), (vii) 2 ($|\mathrm{supp}| = 6$), (viii) 4 ($|\mathrm{supp}| = 7$), and (ix) 64 ($|\mathrm{supp}| = 8$).*

Proof. (1) Up to conjugacy, we can assume that $\sigma = (1\ 2)(3\ 4)$. Suppose $|\mathrm{supp}(\sigma, \sigma')| = 4$. Then σ' belong to S_4. Since $\sigma' \neq \sigma$, $\sigma' = (1\ a)(2\ b)$ where $\{a, b\} = \{3, 4\}$. Since σ is fixed under the conjugation by $(3\ 4)$, σ is conjugate to $(1\ 3)(2\ 4)$ over $(1\ 2)(3\ 4)$. This is (i).

Suppose $|\mathrm{supp}(\sigma, \sigma')| = 5$. We can assume that $|\mathrm{supp}(\sigma, \sigma')| = \{1, \ldots, 5\}$. Since 5 is moved by σ', σ' is conjugate to $(1\ 5)(a\ b)$ over σ where a and b belong to $\{2, 3, 4\}$. If $(1\ 5)(a\ b)$ fixes 2 then it is $(1\ 5)(3\ 4)$. This is (ii). If it moves 2, then it is conjugate to $(1\ 5)(2\ 3)$ over σ. This is (iii).

Suppose $|\mathrm{supp}(\sigma, \sigma')| = 6$. We can assume that $|\mathrm{supp}(\sigma, \sigma')| = \{1, \ldots, 6\}$. Then 5 and 6 are moved by σ'. Therefore, $\sigma' = (a\ b)(5\ 6)$ or $\sigma' = (a\ 5)(b\ 6)$ for some a and b in $\{1, 2, 3, 4\}$. If $\sigma' = (a\ b)(5\ 6)$ then it is conjugate to $(1\ 2)(5\ 6)$ or $(1\ 3)(5\ 6)$ over σ. These are (iv) and (v). If $\sigma' = (a\ 5)(b\ 6)$ then it is conjugate to $(1\ 5)(2\ 6)$ or $(1\ 5)(3\ 6)$ over σ. These are (vi) and (vii).

Suppose $|\mathrm{supp}(\sigma, \sigma')| = 7$. We can assume that $|\mathrm{supp}(\sigma, \sigma')| = \{1, \ldots, 7\}$. Then $\mathrm{supp}(\sigma') = \{a, 5, 6, 7\}$ where a is 1, 2, 3 or 4. Therefore σ' is conjugate to $(1\ 7)(5\ 6)$ over σ. This is (viii).

If $|\mathrm{supp}(\sigma, \sigma')| = 8$, $\mathrm{supp}(\sigma)$ and $\mathrm{supp}(\sigma')$ are disjoint. Therefore, σ' is conjugate to $(5\ 6)(7\ 8)$ over σ. This is (ix).

(2) Let G_4, G_8, and G_8' be as in Lemma 5.2. $C_{S(I)}(\sigma, \sigma')$ is isomorphic to G_4 in case (i), and to $G_8 \times G_8'$ in case (ix).

Case (ii). $\sigma = (1\ 2)(3\ 4)$ and $\sigma' = (1\ 5)(3\ 4)$. Let τ be an element of S_5 commuting with both σ and σ'. Since 5 is the only fixed point of σ in $\{1, \ldots, 5\}$, $\tau(5) = 5$. Since $(1\ 5)^\tau = (\tau(1)\ 5)$ is a cycle component of σ', $\tau(1) = 1$. Since $(1\ 2)^\tau = (1\ \tau(2))$ is a cycle component of σ, $\tau(2) = 2$. Hence, $\tau \in S(\{3, 4\})$. Therefore, $C_{S_5}(\sigma, \sigma') = S(\{3, 4\})$.

Case (iii). $\sigma = (1\ 2)(3\ 4)$ and $\sigma' = (1\ 5)(2\ 3)$. Let τ be an element of S_5 commuting with both σ and σ'. As in case (ii), starting from $\tau(5) = 5$, we get $\tau(i) = i$ for $i = 1, \ldots, 5$.

Case (iv). $\sigma = (1\ 2)(3\ 4)$ and $\sigma' = (1\ 2)(5\ 6)$. Let τ be an element of S_6 commuting with both σ and σ'. On $\{1, \dots, 6\}$, $\mathrm{fix}(\sigma) = \{5, 6\}$, $\mathrm{fix}(\sigma') = \{3, 4\}$ and they are τ-invariant. Thus, $\{1, 2\}$ is also τ-invariant. Therefore,

$$C_{S_6}(\sigma, \sigma') = S_2 \times S(\{3, 4\}) \times S(\{5, 6\}).$$

Case (v). $\sigma = (1\ 2)(3\ 4)$ and $\sigma' = (1\ 3)(5\ 6)$. Let τ be an element of S_6 commuting with both σ and σ'. $\mathrm{supp}(\sigma) \cap \mathrm{supp}(\sigma') = \{1, 3\}$ is τ-invariant. If $\tau(1) = 1$, considering conjugates by τ of cycle components of σ and σ', we have $\tau(i) = i$ for $i = 1, \dots, 4$ and $\{5, 6\}$ is τ-invariant. If $\tau(1) = 3$, we have $\tau = (1\ 3)(2\ 4)\tau'$ with $\tau' \in S(\{5, 6\})$. Therefore, $C_{S_6}(\sigma, \sigma')$ has order 4.

Case (vi). $\sigma = (1\ 2)(3\ 4)$ and $\sigma' = (1\ 5)(2\ 6)$. Let τ be an element of S_6 commuting with both σ and σ'. $\mathrm{fix}(\sigma) = \{5, 6\}$ and $\mathrm{fix}(\sigma') = \{3, 4\}$ are τ-invariant. Since $(1\ 5)^\tau$ and $(2\ 6)^\tau$ are cycle components of σ', τ on $\{1, 2\}$ is uniquely determined by τ on $\{5, 6\}$. Therefore, $C_{S_6}(\sigma, \sigma')$ has order 4.

Case (vii). $\sigma = (1\ 2)(3\ 4)$ and $\sigma' = (1\ 5)(3\ 6)$. Let τ be an element of S_6 commuting with both σ and σ'. $\mathrm{supp}(\sigma) \cap \mathrm{supp}(\sigma') = \{1, 3\}$ is τ-invariant. If $\tau(1) = 1$ then τ is the identity on $\{1, \dots, 6\}$. If $\tau(1) = 3$ then $\tau = (1\ 3)(2\ 4)(5\ 6)$. Therefore, $C_{S_6}(\sigma, \sigma')$ has order 2.

Case (viii). $\sigma = (1\ 2)(3\ 4)$ and $\sigma' = (1\ 7)(5\ 6)$. Let τ be an element of S_6 commuting with both σ and σ'. $\mathrm{supp}(\sigma) \cap \mathrm{supp}(\sigma') = \{1\}$ is τ-invariant. Thus, $\tau(1) = 1$. Then we can show that $\tau(2) = 2$ and $\tau(7) = 7$. Therefore, $C_{S_7}(\sigma, \sigma') = S(\{3, 4\}) \times S(\{5, 6\})$. □

Lemma 5.7. *If $\sigma, \sigma' \in S_9$ have types (2^4) and (2^2) respectively then*

$$\mathrm{ht}(|C_{S_9}(\sigma, \sigma')|) \leq 6; \quad \mathrm{ht}(|C_{A_9}(\sigma, \sigma')|) \leq 5.$$

In particular, if $\sigma, \sigma' \in A_8$ then $\mathrm{ht}(|C_{A_8}(\sigma, \sigma')|) \leq 4$ or $C_{A_8}(\sigma, \sigma')$ is conjugate to $(G_8 \times G'_8) \cap A_8$. Here, G_8 and G'_8 are as in Lemma 5.2.

Proof. Up to conjugacy, we can assume that $\sigma = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$.

Suppose 9 is not a fixed point of σ'. Up to conjugacy, we can also assume that $(1\ 9)$ is a cycle component of σ'. By the same argument as that for Lemma 5.4, $\mathrm{ht}(|C_{S_9}(\sigma, \sigma')|) \leq 5$ in this case.

Now, suppose 9 is a fixed point of σ'. Then we can easily see that σ' is conjugate to one of the following over σ: (i) $(1\ 2)(3\ 4)$; (ii) $(1\ 3)(2\ 4)$; (iii) $(1\ 2)(3\ 5)$; (iv) $(1\ 3)(2\ 5)$; and (v) $(1\ 3)(5\ 7)$.

Case (i). $\sigma' = (1\ 2)(3\ 4)$. In this case, $C_{S_9}(\sigma, \sigma') = G_8 \times G'_8$, and thus $|C_{S_9}(\sigma, \sigma')| = 2^6$. Since $G_8 \times G'_8$ contains a transposition, $C_{A_9}(\sigma, \sigma') = (G_8 \times G'_8) \cap A_8$ has order 2^5.

Case (ii). $\sigma' = (1\ 3)(2\ 4)$. In this case, $C_{S_9}(\sigma, \sigma') = G_4 \times G'_8$ has order 2^5.

Case (iii). $\sigma' = (1\ 2)(3\ 5)$. Let τ be an element of $C_{S_9}(\sigma, \sigma')$. Since $(1\ 2)$ is the only cycle component common to σ and σ', $(1\ 2)^\tau = (1\ 2)$.

Then $(3\ 5)^\tau = (3\ 5)$ by $\sigma'^\tau = \sigma'$. Since $\{3,5\}$ is τ-invariant and $(3\ 4)^\tau$ and $(5\ 6)^\tau$ are cycle components of σ, $\{4\ 6\}$ is also τ-invariant. Hence $\{7,8\}$ is τ-invariant. Therefore,

$$C_{S_9}(\sigma,\sigma') \subset S_2 \times S(\{3,5\}) \times S(\{4,6\}) \times S(\{7,8\})$$

and in fact, both sides are equal for Case (iii).

Case (iv). $\sigma' = (1\ 3)(2\ 5)$. Let τ be an element of $C_{S_9}(\sigma,\sigma')$.

If $\tau(3) = 1$ then $\tau(4) = 2$ since $\sigma^\tau = \sigma$. But in this case, 4 is a fixed point of σ' but $\tau(4) = 2$ is not. Hence, τ and σ' are not commuting.

If $\tau(3) = 2$ then $\tau(1) = 5$ and $\tau(2) = 6$ since $\sigma'^\tau = \sigma'$ and $\sigma^\tau = \sigma$. But in this case, $\tau(2) = 6$ is a fixed point of σ' but $2 = \tau^{-1}(6)$ is not. Hence, τ and σ' are not commuting.

If $\tau(3) = 3$ then $\tau(1) = 1$, $\tau(2) = 2$, $\tau(4) = 4$, $\tau(5) = 5$, and $\tau(6) = 6$. Hence $\tau \in C_{S(\{7,8\})}((7\ 8))$ in this case. There are 2 such τ's.

If $\tau(3) = 5$ then $\tau(1) = 2$, $\tau(2) = 1$, $\tau(4) = 6$, $\tau(5) = 3$, and $\tau(6) = 4$. Hence $\tau|_{\{7,8\}}$ belongs to $C_{S(\{7,8\})}((7\ 8))$ in this case. There are 2 such τ's.

Therefore, $|C_{S_9}(\sigma,\sigma')| = 2^2$ for Case (iv).

Case (v). $\sigma' = (1\ 3)(5\ 7)$. Let τ be an element of $C_{S_9}(\sigma,\sigma')$. Then $\{1,3,5,7\}$ is τ-invariant and there are 8 possibilities for τ on $\{1,3,5,7\}$ corresponding to the elements of G_8. Since $\sigma = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$, τ on $\{2,4,6,8\}$ is uniquely determined by τ on $\{1,3,5,7\}$. Therefore, $C_{S_9}(\sigma,\sigma') \cong G_8$ for Case (v). □

Lemma 5.8. *If $\sigma,\sigma' \in A_9$ have types (2^4) and (3^1) respectively then*

$$\mathrm{ht}(|C_{A_9}(\sigma,\sigma')|) \le 4.$$

Proof. By Lemma 4.1, $|C_{S_9}(\sigma)| = 2^7 \cdot 3$ and $|C_{S_9}(\sigma')| = 2^4 \cdot 3^3 \cdot 5$. Therefore, $|C_{S_9}(\sigma,\sigma')|$ is a divisor of $2^4 \cdot 3$. Since $|\mathrm{supp}(\sigma')| = 3$, there is a cycle component $(a\ b)$ of σ such that $a, b \notin \mathrm{supp}(\sigma')$. Therefore, $|C_{A_9}(\sigma,\sigma')|$ is a divisor of $2^3 \cdot 3$. □

Lemma 5.9. *Suppose two permutations σ and σ' have types (2^2) and (3^1) respectively, and let $I = \mathrm{supp}(\sigma,\sigma')$. Then $C_{S(I)}(\sigma,\sigma')$ has order 1 if $|I| = 4$, at most 2 if $|I| = 5$ or 6, and 24 if $|I| = 7$.*

Proof. Suppose $|I| = 4$. $C_{S(I)}(\sigma) \cong G_8$ has order 8 and $C_{S(I)}(\sigma') \cong A_3$ has order 3. Therefore, $C_{S(I)}(\sigma,\sigma')$ has order 1.

Suppose $|I| = 5$. $C_{S(I)}(\sigma) \cong G_8$ has order 8 and $C_{S(I)}(\sigma') \cong A_3 \times S_2$ has order $3 \cdot 2$. Therefore, $C_{S(I)}(\sigma,\sigma')$ has order at most 2.

Suppose $|I| = 6$. $C_{S(I)}(\sigma) \cong G_8 \times S_2$ has order $8 \cdot 2$ and $C_{S(I)}(\sigma') \cong A_3 \times S_3$ has order $3 \cdot 3!$. Therefore, $C_{S(I)}(\sigma,\sigma')$ has order at most 2.

Suppose $|I| = 7$. Then $\mathrm{supp}(\sigma)$ and $\mathrm{supp}(\sigma')$ have an empty intersection. Therefore, $C_{S(I)}(\sigma,\sigma') \cong G_8 \times A_3$ and it has order 24. □

Lemma 5.10. *Suppose* $(\sigma_1, \sigma_2, \sigma_3)$ *is a gap sequence in* A_9, σ_1 *and* σ_2 *belong to* S_4 *and have the same type* (2^2). *If* $\sigma_3 \in S_9$ *has type* (2^2), (3^1), *or* (2^4) *then* $\mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2, \sigma_3)|) \leq 4$.

Proof. We have $C_{S_4}(\sigma_1, \sigma_2) = G_4$ where G is as in Lemma 5.1. We consider two cases.

Case 1. σ_3 has type (2^2) or (3^1).

Suppose the size of $\mathrm{supp}(\sigma_1, \sigma_2, \sigma_3)$ is k. Note that $4 \leq k \leq 9$. Up to conjugacy, we can assume that $\mathrm{supp}(\sigma_1, \sigma_2, \sigma_3) = \{1, \dots, k\}$. Then

$$C_{S_9}(\sigma_1, \sigma_2, \sigma_3) \subset C_{S_k}(\sigma_1, \sigma_2, \sigma_3) \times S(\{k + 1, \dots, 9\})$$

and

$$C_{S_k}(\sigma_1, \sigma_2, \sigma_3) \subset G_4 \times S(\{5, \dots, k\}).$$

Hence,

$$C_{S_9}(\sigma_1, \sigma_2, \sigma_3) \subset G_4 \times S(\{5, \dots, k\}) \times S(\{k + 1, \dots, 9\}).$$

Suppose $k = 4$. If σ_3 has type (2^2) then $\sigma_3 \in G_4$. Hence, $(\sigma_1, \sigma_2, \sigma_3)$ is not a gap sequence in S_9. Thus, σ_3 must have type (3^1). In this case, σ_3 commutes with no elements in G_4 other than the identity. Hence

$$C_{A_9}(\sigma_1, \sigma_2, \sigma_3) = A(\{5, \dots, 9\})$$

and it has order $5!/2$ with $\mathrm{ht}(5!/2) = 4$.

If $k = 5$ then σ_3 commutes with no non-trivial elements in G_4. Therefore, $C_{A_9}(\sigma_1, \sigma_2, \sigma_3)$ is $A(\{6, \dots, 9\})$ which has oder $4!/2$ with $\mathrm{ht}(4!/2) = 3$.

If $k = 6$ then

$$C_{S_9}(\sigma_1, \sigma_2, \sigma_3) \subset G_4 \times S(\{5, 6\}) \times S(\{7, 8, 9\})$$

and If $k = 7$ then

$$C_{S_9}(\sigma_1, \sigma_2, \sigma_3) \subset G_4 \times S(\{5, 6, 7\}) \times S(\{8, 9\}).$$

In either case, $C_{S_9}(\sigma_1, \sigma_2, \sigma_3)$ contains an odd permutation, and therefore $C_{A_9}(\sigma_1, \sigma_2, \sigma_3)$ has an order dividing $4 \cdot 3! \cdot 2/2$. Thus the height of the order is at most 4.

Suppose $k = 8$. Then σ_3 has type (2^2).

$$C_{S_9}(\sigma_1, \sigma_2, \sigma_3) = G_4 \times G'_8$$

and thus

$$C_{A_9}(\sigma_1, \sigma_2, \sigma_3) = G_4 \times G'_4,$$

which has order 16 with $\mathrm{ht}(16) = 4$.

Case 2. σ_3 has type (2^4). $\mathrm{supp}(\sigma_1, \sigma_2, \sigma_3)$ has 8 or 9 elements.

Suppose it has 9 elements. Then σ_3 has a unique fixed point $i \in \{1, \dots, 4\}$. Let

$$\tau \in C_{S_9}(\sigma_1, \sigma_2) = G_4 \times S(\{5, \dots, 9\}).$$

If $\tau|_{\{1,\dots,4\}} \in G_4$ is not identity then τ and σ_3 are not commuting because i is a fixed point of σ_3 while $\tau(i)$ is not. Hence, $C_{S_9}(\sigma_1, \sigma_2, \sigma_3)$ is a subgroup of $S(\{5, \dots, 9\})$. Thus $C_{A_9}(\sigma_1, \sigma_2, \sigma_3)$ is a proper subgroup of $S(\{5, \dots, 9\})$. Therefore, $C_{A_9}(\sigma_1, \sigma_2, \sigma_3)$ has an order of height at most 4.

Finally, suppose $\mathrm{supp}(\sigma_1, \sigma_2, \sigma_3)$ has 8 elements. Up to conjugacy, we can assume that this support is $\{1, \dots, 8\}$. Then $C_{S_9}(\sigma_1, \sigma_2, \sigma_3)$ is a subgroup of $G_4 \times S(\{5, \dots, 8\})$. Since σ_3 has type (2^4), it has a cycle component $(a\ b)$ belonging to $S(\{5, \dots, 8\})$. Choose $c \in \{5, \dots, 8\} - \{a, b\}$. Then $(a\ c) \in S(\{5, \dots, 8\})$ but $(a\ c)$ does not commute with σ_3. Hence, $C_{S_9}(\sigma_1, \sigma_2, \sigma_3)$ is a proper subgroup of $G_4 \times S(\{5, \dots, 8\})$. Also, $C_{S_9}(\sigma_1, \sigma_2, \sigma_3)$ contains (a, b). Thus, $C_{A_9}(\sigma_1, \sigma_2, \sigma_3)$ is a proper subgroup of $C_{S_9}(\sigma_1, \sigma_2, \sigma_3)$. Since $G_4 \times S(\{5, \dots, 8\})$ has an order of height 6, $C_{A_9}(\sigma_1, \sigma_2, \sigma_3)$ has an order of height at most 4. □

Proposition 5.11. *$g(A_9) = 7$.*

Proof. Let $(\sigma_1, \sigma_2, \sigma_3, \dots)$ be a gap sequence in A_9. We show that

$$\mathrm{ht}(|C_{A_9}(\sigma_1)|) \le 6,\ \mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2)|) \le 5,\ \text{or } \mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2, \sigma_3)|) \le 4$$

holds. Then we have the statement by Lemma 2.4 and Theorem 3.2 (3).

By Table 3 in the proof of Proposition 4.6, for non-trivial element $\sigma \in A_9$, $\mathrm{ht}(|C_{A_9}(\sigma)|) = 7$ if σ has type (2^2), (3^1), or (2^4), and $\mathrm{ht}(|C_{A_9}(\sigma)|) \le 4$ otherwise. Therefore, if σ_1, σ_2, or σ_3 has a type other than (2^2), (3^1), and (2^4) then $\mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2, \sigma_3)|) \le 4$.

We can assume that the types of σ_1, σ_2, and σ_3 are among (2^2), (3^1), and (2^4). We consider cases according to the types of σ_1 and σ_2.

Case 1. σ_1 and σ_2 have type (2^4). In this case, $\mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2)|) \le 5$ by Lemmas 5.2 and 5.4.

Case 2. σ_1 and σ_2 have type (3^1). Let $I = \mathrm{supp}(\sigma_1, \sigma_2)$. By Lemma 5.5, $3 < |I| \le 6$. Let $J = \{1, \dots, 9\} - I$. If $|I| = 4$ then

$$C_{S_9}(\sigma_1, \sigma_2) = C_{S(I)}(\sigma_1, \sigma_2) \times S(J) = S(J) \cong S_5$$

by Lemma 5.5. Therefore, $\mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2)|) = \mathrm{ht}(5!/2) = 4$.

Similarly, if $|I| = 5$ then $C_{S_9}(\sigma_1, \sigma_2) = S(J) \cong S_4$ and hence

$$\mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2)|) = \mathrm{ht}(4!/2) = 3.$$

If $|I| = 6$ then $C_{S_9}(\sigma_1, \sigma_2)$ is conjugate to

$$A_3 \times A(\{4, 5, 6\}) \times S(\{7, 8, 9\})$$

and hence $\mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2)|) = \mathrm{ht}(9 \cdot 3!/2) = 3$.

Case 3. σ_1 has type (2^4) and σ_2 has type (2^2), or vice versa. In this case, $\mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2)|) \le 5$ by Lemma 5.7.

Case 4. σ_1 has type (2^4) and σ_2 has type (3^1), or vice versa. In this case, $\mathrm{ht}(|C_{A_9}(\sigma_1, \sigma_2)|) \le 4$ by Lemma 5.8.

Case 5. σ_1 has type (2^2) and σ_2 has type (3^1), or vice versa. By Lemma 5.9, if $|\mathrm{supp}(\sigma_1,\sigma_2)| = 4$ then $|C_{A_9}(\sigma_1,\sigma_2)| = 5!$, if $|\mathrm{supp}(\sigma_1,\sigma_2)| = 5$ then $|C_{A_9}(\sigma_1,\sigma_2)| = 2\cdot 4!$, if $|\mathrm{supp}(\sigma_1,\sigma_2)| = 6$ then $|C_{A_9}(\sigma_1,\sigma_2)| = 2\cdot 3!$, and if $|\mathrm{supp}(\sigma_1,\sigma_2)| = 7$ then $|C_{A_9}(\sigma_1,\sigma_2)| = 24\cdot 2$. Hence, $\mathrm{ht}(|C_{A_9}(\sigma_1,\sigma_2)|) \le 5$.

Case 6. σ_1 and σ_2 have type (2^2). Let $I = \mathrm{supp}(\sigma_1,\sigma_2)$. If $|I| = 5$ then $C_{S_9}(\sigma_1,\sigma_2) \cong C_{S(I)}(\sigma_1,\sigma_2)\times S_4$. Hence, $\mathrm{ht}(|C_{A_9}(\sigma_1,\sigma_2)|) \le 4$ by Lemma 5.6. Similarly, if $|I| > 6$ then $\mathrm{ht}(|C_{A_9}(\sigma_1,\sigma_2)|) \le 5$.

If $|I| = 4$ then $\mathrm{ht}(|C_{A_9}(\sigma_1,\sigma_2,\sigma_3)|) \le 4$ by Lemma 5.10. □

Proposition 5.12. $g(A_6) = g(A_7) = 4$.

Proof. Let G_4 and G_8 be as in Lemma 5.1.

Let $(\sigma_1,\sigma_2,\sigma_3,\ldots)$ be a gap sequence in A_7. We show that

$$\mathrm{ht}(|C_{A_7}(\sigma_1,\sigma_2)|) \le 2, \text{ or } \mathrm{ht}(|C_{A_7}(\sigma_1,\sigma_2,\sigma_3)|) \le 1.$$

Then we have the statement by Lemma 2.4 and Theorem 3.2 (2).

Claim 1. $C_{A_7}(\sigma_1,\sigma_2)$ has an order of height at most 2, or, is conjugate to $G_4 \times A(\{5,6,7\})$.

By looking at the types of even permutations in Table 2 in the proof of Proposition 4.5, $C_{A_7}(\sigma_1,\sigma_2)$ has an order of height at most 2 if the type of σ_1 or σ_2 is neither (2^2) nor (3^1). We have 3 cases to consider.

Case 1. σ_1 has type (2^2) and σ_2 has type (3^1), or vice versa. Let $I = \mathrm{supp}(\sigma_1,\sigma_2)$. Then $C_{S_7}(\sigma_1,\sigma_2)$ is isomorphic to

$$C_{S(I)}(\sigma_1,\sigma_2)\times S_{7-|I|}.$$

If $|I| < 7$, then we can show that the latter group has an order of height at most 2 by Lemma 5.9.

Suppose $|I| = 7$. Then σ_1 and σ_2 have disjoint supports. Hence $C_{S_7}(\sigma_1,\sigma_2)$ is conjugate to $G_8 \times A(\{5,6,7\})$. Therefore, $C_{A_7}(\sigma_1,\sigma_2)$ is conjugate to $G_4 \times A(\{5,6,7\})$.

Case 2. σ_1 and σ_2 have type (3^1). Let $I = \mathrm{supp}(\sigma_1,\sigma_2)$. If the supports of σ_1 and σ_2 are the same then (σ_1,σ_2) cannot be a gap sequence. So, $|I|$ is 4, 5, or 6. Considering the cases according to $|I|$, we can easily check that $\mathrm{ht}(|C_{A_7}(\sigma_1,\sigma_2)|) \le 2$.

Case 3. σ_1 and σ_2 have type (2^2). Let $I = \mathrm{supp}(\sigma_1,\sigma_2)$. If $|I| = 4$ then $C_{S_7}(\sigma_1,\sigma_2)$ is conjugate to $G_4 \times S(\{5,6,7\})$. Since G_4 consists of even permutations, $C_{A_7}(\sigma_1,\sigma_2)$ is conjugate to $G_4 \times A(\{5,6,7\})$. If $|I| > 4$, we can check that $\mathrm{ht}(|C_{A_7}(\sigma_1,\sigma_2)|) \le 2$ using Lemma 5.6. Claim 1 is proved.

Claim 2. Suppose $(\sigma_1,\sigma_2,\sigma_3)$ is a gap sequence in A_7 and $C_{A_7}(\sigma_1,\sigma_2) = G_4 \times A(\{5,6,7\})$. Then $\mathrm{ht}(|C_{A_7}(\sigma_1,\sigma_2,\sigma_3)|) \le 1$.

If $\sigma_3 \notin S_4 \times S(\{5,6,7\})$, we can easily check that $C_{A_7}(\sigma_1,\sigma_2,\sigma_3)$ is a trivial group.

Suppose $\sigma_3 = \tau\tau'$ where $\tau \in S_4$ and $\tau' \in S(\{5,6,7\})$. Then

$$C_{A_7}(\sigma_1, \sigma_2, \sigma_3) = C_{G_4}(\tau) \times C_{A(\{5,6,7\})}(\tau').$$

Since $\sigma_3 \in A_7$, τ and τ' are both even, or both odd.

If τ and τ' are odd, then $C_{A(\{5,6,7\})}(\tau')$ is trivial, and $C_{G_4}(\tau)$ is trivial or has order 2. Hence, $C_{A_7}(\sigma_1, \sigma_2, \sigma_3)$ is trivial or a group of order 2.

If τ and τ' are even, then $C_{A(\{5,6,7\})}(\tau')$ is $A(\{5,6,7\})$, and $C_{G_4}(\tau)$ is G_4 or trivial. Hence, $C_{A_7}(\sigma_1, \sigma_2, \sigma_3)$ is $A(\{5,6,7\})$, which has order 3. $\square$

Lemma 5.13. *Suppose H is a subgroup of a direct product $G \times G'$ and $H_0 = H \cap G'$. Then for any $g_1, g_2 \in G$, $g_1 g_1' H_0 = g_2 g_2' H_0$ for some $g_1', g_2' \in G'$ implies $g_1 = g_2$. Therefore, if H is finite then*

$$|H| = |H_0| \cdot |\{g \in G \,:\, gg' \in H \text{ for some } g' \in G'\}|.$$

Proof. Suppose $g_1, g_2 \in G$, $g_1 g_1' H_0 = g_2 g_2' H_0$ for some $g_1', g_2' \in G'$. Then ${g_1}^{-1} g_2 {g_1'}^{-1} g_2' \in H_0 \subset G'$. Hence ${g_1}^{-1} g_2 \in G'$, and thus ${g_1}^{-1} g_2 \in G \cap G'$. Therefore, ${g_1}^{-1} g_2$ is the identity. $\square$

Proposition 5.14. $g(A_8) = 6$.

Proof. Let G_4, G_4', G_8, G_8', and G_{32} be as in Lemma 5.2.

Suppose $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots)$ is a gap sequence in A_8.

We prove the statement by a sequence of claims. Here is an outline of the proof. In Claim 1, we prove that $C_{A_8}(\sigma_1, \sigma_2)$ has an order of height at most 4 or is conjugate to one of few groups. If $C_{A_8}(\sigma_1, \sigma_2)$ is one of these groups, we show that $C_{A_8}(\sigma_1, \sigma_2, \sigma_3)$ has an order of height at most 3 or it is conjugate to $G_4 \times G_4'$ in Claims 3 to 5. Finally, if $C_{A_8}(\sigma_1, \sigma_2, \sigma_3) = G_4 \times G_4'$ then $C_{A_8}(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ has an order of height at most 2 by Claim 2.

We show Claim 2 before Claim 3 because we need it also in the proof of Claim 3.

Claim 1. $C_{A_8}(\sigma_1, \sigma_2)$ has an order of height at most 4, or, is conjugate to $(G_8 \times G_8') \cap A_8$, G_{32}, or $G_4 \times A(\{5, \ldots, 8\})$.

If the type of $\sigma \in S_8$ is not (7^1), we can easily check that $C_{S_8}(\sigma)$ contains an odd permutation. Therefore, by Table 4 in the proof of Proposition 5.3, if the type of σ_1 is neither (2^2) nor (2^4) then $C_{A_8}(\sigma_1)$ has an order of height at most 5, and hence $C_{A_8}(\sigma_1, \sigma_2)$ has an order of height at most 4.

Suppose that the type of σ_1 is (2^2) or (2^4). If the type of σ_2 is neither (2^2), (2^4) nor (3^1) then $\mathrm{ht}(|C_{A_8}(\sigma_2)|) \leq 4$ by Table 4. Again by Table 4, $|C_{A_8}(\sigma_1)| = 2^i \cdot 3$ with $i = 5$ or 6, if σ_2 has type (3^1) then

$$|C_{A_8}(\sigma_2)| = 2^2 \cdot 3^2 \cdot 5$$

and hence $|C_{A_8}(\sigma_1, \sigma_2)|$ is a divisor of $2^2 \cdot 3$.

Now, we can assume that the types of σ_1 and σ_2 are among (2^2) and (2^4).

Suppose that the types of σ_1 and σ_2 are (2^2) and (2^4) respectively, or vice versa. By Lemma 5.7, $\mathrm{ht}(|C_{A_8}(\sigma_1,\sigma_2)|) \leq 4$ or $C_{A_8}(\sigma_1,\sigma_2)$ is conjugate to $(G_8 \times G_8') \cap A_8$.

Suppose that the types of σ_1 and σ_2 are (2^4). By Lemma 5.2,

$$\mathrm{ht}(|C_{A_8}(\sigma_1,\sigma_2)|) \leq 4$$

or $C_{A_8}(\sigma_1,\sigma_2)$ is conjugate to G_{32}.

Suppose that the types of σ_1 and σ_2 are (2^2). If $5 \leq |\mathrm{supp}(\sigma_1,\sigma_2)| \leq 7$ then $\mathrm{ht}(|C_{A_8}(\sigma_1,\sigma_2)|) \leq 3$ by Lemma 5.6. If $|\mathrm{supp}(\sigma_1,\sigma_2)| = 4$ then $C_{A_8}(\sigma_1,\sigma_2)$ is conjugate to $G_4 \times A(\{5,6,7,8\})$. If $|\mathrm{supp}(\sigma_1,\sigma_2)| = 8$ then $C_{A_8}(\sigma_1,\sigma_2)$ is conjugate to $(G_8 \times G_8') \cap A_8$. Claim 1 is proved.

Claim 2. Let $\sigma_4 \in A_8$. If $C_{G_4\times G_4'}(\sigma_4)$ is a proper subgroup of $G_4 \times G_4'$ then $\mathrm{ht}(|C_{G_4\times G_4'}(\sigma_4)|) \leq 2$.

Assume that $\sigma_4 \notin S_4 \times S(\{5,6,7,8\})$. Then $C_{G_4\times G_4'}(\sigma_3) \cap G_4'$ is trivial. By Lemma 5.13, $|C_{G_4\times G_4'}(\sigma_4)|$ is at most $|G_4| \cdot 1 = 4$. Therefore, $\mathrm{ht}(|C_{G_4\times G_4'}(\sigma_4)|) \leq 2$.

Assume that $\sigma_4 \in S_4 \times S(\{5,6,7,8\})$. Let $\sigma_4 = \tau\tau'$ with $\tau \in S_4$ and $\tau' \in S(\{5,6,7,8\})$. Then $C_{G_4\times G_4'}(\sigma_4) = C_{G_4}(\tau) \times C_{G_4'}(\tau')$. Since σ_4 is an even permutation, τ and τ' are both even, or both odd. Suppose that τ and τ' are odd. Then $C_{G_4}(\tau)$ is a proper subgroup of G_4 and $C_{G_4'}(\tau')$ is a proper subgroup of G_4'. Therefore, $\mathrm{ht}(|C_{G_4\times G_4'}(\sigma_4)|) \leq 2$.

Suppose τ and τ' are even. Then $C_{G_4}(\tau)$ is G_4 or trivial and $C_{G_4'}(\tau')$ is G_4' or trivial. Therefore, If $C_{G_4\times G_4'}(\sigma_4)$ is a proper subgroup of $G_4 \times G_4'$ then $\mathrm{ht}(|C_{G_4\times G_4'}(\sigma_4)|) \leq 2$. Claim 2 is proved.

Claim 3. Let $\sigma_3 \in A_8$. If $C_{G_{32}}(\sigma_3)$ is a proper subgroup of G_{32} then $\mathrm{ht}(|C_{G_{32}}(\sigma_3)|) \leq 3$ or $C_{G_{32}}(\sigma_3) = G_4 \times G_4'$.

$G_4 \times G_4' \subset S_4 \times S(\{5,6,7,8\})$ is a normal subgroup of G_{32} of index 2 by Lemma 5.2 (3). Hence, the product of any two elements in $G_{32} - (G_4 \times G_4')$ belongs to $G_4 \times G_4'$. Therefore $C_{G_{32}}(\sigma_3) \cap (G_4 \times G_4')$ has an index at most 2 in $C_{G_{32}}(\sigma_3)$. Since $C_{G_{32}}(\sigma_3) \cap (G_4 \times G_4') = C_{G_4\times G_4'}(\sigma_3)$, it is $G_4 \times G_4'$ or has an order of height at most 2 by Claim 2. Therefore, if $C_{G_{32}}(\sigma_3)$ is a proper subgroup of G_{32} then it is $G_4 \times G_4'$ or it has an order of height at most 3. Claim 3 is proved.

Claim 4. Let $H = (G_8 \times G_8') \cap A_8$, and $\sigma_3 \in A_8$. If $C_H(\sigma_3)$ is a proper subgroup of H then $\mathrm{ht}(|C_H(\sigma_3)|) \leq 3$ or $C_H(\sigma_3) = G_4 \times G_4'$.

We consider $C_{G_8\times G_8'}(\sigma_3)$.

Case 1. $\sigma_3 \notin S_4 \times S(\{5,6,7,8\})$. $C_{G_8 \times G'_8}(\sigma_3) \cap G'_8$ is trivial or consists of the identity and one 2-cycle since G'_8 consists of the identity, two 2-cycles (5 6), (7 8), and five permutations with support $\{5,\dots,8\}$.

We count the number of $\tau_1 \in G_8$ such that $\tau_1\tau_2 \in C_{G_8 \times G'_8}(\sigma_3)$ for some $\tau_2 \in G'_8$ and then use Lemma 5.13.

Up to conjugacy, we can also assume that $\sigma_3(1) = 5$.

Subcase 1a. σ_3 maps $\{1,2,3,4\}$ to $\{5,6,7,8\}$. In this case, $C_{G_8 \times G'_8}(\sigma_3) \cap G'_8$ is trivial and therefore $|C_{G_8 \times G'_8}(\sigma_3)| \le |G_8| = 8$ by Lemma 5.13.

Subcase 1b. σ_3 commutes with $(1\ 3\ 2\ 4)\tau'$ or $(1\ 4\ 2\ 3)\tau'$ for some τ' in G'_8.

Suppose σ_3 commutes with $(1\ 3\ 2\ 4)\tau'$ for some $\tau' \in G'_8$. Then

$$(1\ 3\ 2\ 4)^{\sigma_3}\tau'^{\sigma_3} = (1\ 3\ 2\ 4)\tau'.$$

Since $\sigma_3(1) = 5$, $(1\ 3\ 2\ 4)^{\sigma_3}$ is a cycle component of τ', and hence σ_3 maps $\{1,\dots,4\}$ to $\{5,\dots,8\}$. This is Subcase 1a.

If σ_3 commutes with $(1\ 4\ 2\ 3)\tau'$ for some $\tau' \in G'_8$, the same argument reduces the situation to Subcase 1a.

Subcase 1c. σ_3 commutes with $(1\ 3)(2\ 4)\tau'$ or $(1\ 4)(2\ 3)\tau'$ for some $\tau' \in G'_8$.

Suppose σ_3 commutes with $(1\ 3)(2\ 4)\tau'$ for some $\tau' \in G'_8$. In this case, $(1\ 3)^{\sigma_3} = (5\ \sigma_3(3))$ is a cycle component of τ'. Hence, $\sigma_3(\{1,3\}) \subset \{5,\dots,8\}$. By Subcase 1a, we can assume that $\sigma_3(\{2,4\}) \not\subset \{5,\dots,8\}$. Since $(2\ 4)^{\sigma_3}$ is a cycle component of $(1\ 3)(2\ 4)\tau'$, we have $\sigma_3(\{2,4\}) \subset \{1,\dots,4\}$. Therefore, if $\tau_1\tau_2 \in C_{G_8 \times G'_8}(\sigma_3)$ with $\tau_1 \in G_8$ and $\tau_2 \in G'_8$ then 2-cycles (1 2), (3 4), and (1 4) cannot be a cycle component of τ_1. Hence, τ_1 can be the identity or (1 3)(2 4). By Lemma 5.13, $C_{G_8 \times G'_8}(\sigma_3)$ has order 2 or $4 = 2^2$.

If σ_3 commutes with $(1\ 4)(2\ 3)\tau'$ for some $\tau' \in G'_8$, a similar argument shows that the same statement holds.

Subcase 1d. None of the subcases above hold. In this case, there are at most 4 possibilities for $\tau_1 \in G_8$ such that $\tau_1\tau_2 \in C_{G_8 \times G'_8}(\sigma_3)$ for some $\tau_2 \in G'_8$. Therefore, $C_{G_8 \times G'_8}(\sigma_3)$ has an order at most $8 = 2^3$.

Case 2. $\sigma_3 \in S_4 \times S(\{5,6,7,8\})$. Let $\sigma_3 = \tau\tau'$ with $\tau \in S_4$ and $\tau' \in S(\{5,6,7,8\})$. Then $C_{G_8 \times G'_8}(\sigma_3) = C_{G_8}(\tau) \times C_{G'_8}(\tau')$. Since σ_3 is an even permutation, τ and τ' are both even, or both odd.

Suppose τ and τ' are odd. Then $C_{G_8}(\tau)$ is a proper subgroup of G_8 containing an odd permutation and $C_{G'_8}(\tau')$ is a proper subgroup of G'_8 containing an odd permutation. Therefore, $C_{G_8 \times G'_8}(\sigma_3)$ contains an odd permutation and its order is a divisor of 16. Therefore, $C_H(\sigma_3) = C_{G_8 \times G'_8}(\sigma_3) \cap A_8$ has an order dividing 8.

Suppose τ and τ' are even. Then $C_{G_8}(\tau)$ is G_8, G_4 or trivial, and $C_{G'_8}(\tau')$ is G'_8, G'_4 or trivial. Therefore, if $C_{G_8\times G'_8}(\sigma_3)$ is a proper subset of $G_8 \times G'_8$ then $C_{G_8\times G'_8}(\sigma_3)$ has an order dividing 8 or it is $G_8\times G'_4$, $G_4\times G'_8$, or $G_4\times G'_4$. Therefore, $\mathrm{ht}(|C_H(\sigma_3)|) \le 3$ or $C_H(\sigma_3) = C_{G_8\times G'_8}(\sigma_3)\cap A_8 = G_4\times G'_4$. Claim 4 is proved.

Claim 5. Let $H = G_4 \times A(\{5,6,7,8\}) \subset A_8$ and $\sigma_3 \in A_8$. If $C_H(\sigma_3)$ is a proper subgroup of H then $\mathrm{ht}(|C_H(\sigma_3)|) \le 3$ or $C_H(\sigma_3) = G_4 \times G'_4$.

Assume that $\sigma_3 \notin S_4 \times S(\{5,6,7,8\})$. Then $C_H(\sigma_3) \cap A(\{5,6,7,8\})$ is isomorphic to A_3 or trivial. By Lemma 5.13, $|C_H(\sigma_3)|$ is at most $|G_4|\cdot|A_3| = 12$. Therefore, $\mathrm{ht}(|C_H(\sigma_3)|) \le 3$.

Assume that $\sigma_3 \in S_4 \times S(\{5,6,7,8\})$. Let $\sigma_3 = \tau\tau'$ with $\tau \in S_4$ and $\tau' \in S(\{5,6,7,8\})$. Then $C_H(\sigma_3) = C_{G_4}(\tau) \times C_{A(\{5,\ldots,8\})}(\tau')$. Since σ_3 is an even permutation, τ and τ' are both even, or both odd.

Suppose τ and τ' are odd. Then $C_{G_4}(\tau)$ is a proper subgroup of G_4 and $C_{A(\{5,\ldots,8\})}(\tau')$ is a proper subgroup of $A(\{5,\ldots,8\})$. Therefore, $\mathrm{ht}(|C_H(\sigma_3)|) \le 3$.

Suppose τ and τ' are even. Then $C_{G_4}(\tau)$ is G_4 or trivial; $C_{A(\{5,\ldots,8\})}(\tau')$ is $A(\{5,\ldots,8\})$, G'_4, or a subgroup of $A(\{5,\ldots,8\})$ conjugate to A_3. Therefore, $\mathrm{ht}(|C_H(\sigma_3)|) \le 3$ or $C_H(\sigma_3) = G_4 \times G'_4$. Claim 5 is proved. □

6. Possible Gap Numbers

Lemma 6.1. *$g(G \times G') = g(G) + g(G')$.*

Proof. It is straight forward to show that $g(G \times G') \ge g(G) + g(G')$.

We show that $g(G\times G') \le g(G)+g(G')$. For any $a, b \in G$ and $a', b' \in G'$, aa' and bb' are commuting if and only if a and b are commuting and a' and b' are commuting.

Suppose $(a_1a'_1, \ldots a_ka'_k)$ is a gap sequence in $G \times G'$ with a witness $(b_1b'_1, \ldots b_kb'_k)$ in $G \times G'$, where the a_i's and b_i's are in G and the a'_i's and the b'_i's are in G'.

Let $\{i : a_ib_i \ne b_ia_i\} = \{i_1, \ldots, i_l\}$ where $i_1 < \cdots < i_l$ and let $\{1, \ldots, k\} - \{i_1, \ldots, i_l\} = \{j_1, \ldots, j_m\}$ where $j_1 < \cdots < j_m$. Then $(a_{i_1}, \ldots, a_{i_l})$ is a gap sequence for G with a witness $(b_{i_1}, \ldots, b_{i_l})$ in G and $(a'_{j_1}, \ldots, a'_{j_m})$ is a gap sequence for G' with a witness $(b'_{j_1}, \ldots, b'_{j_m})$ in G'. Therefore, $k = l + m \le g(G) + g(G')$. □

As a corollary, we get the following theorem:

Theorem 6.2. *For any natural number $n \ne 1, 3, 5$ there is a group G such that $n = g(G)$. G can be finite or infinite.*

Proof. If H is an abelian group then $g(H) = 0$. We have $g(S_3) = 2$ and $g(A_9) = 7$. Therefore we have the statement by Lemma 6.1. $\square$

We still do not know whether a group G with $g(G) = 5$ exists.

Finally, we give some questions. Is it true that $g(S_n) = n$ for any $n \geq 8$? Is it true that $g(A_n) = n-2$ for any $n \geq 8$? Is it true that $g(A_n) = g(S_n)-2$ for any $n \geq 3$?

We can see that $g(S_{n+k}) \geq g(S_n) + k$ for $k = 4$ and for any $k \geq 8$. Therefore, if we can find infinitely many n's such that $g(S_n) = n$ then $g(S_n) = n$ for any $n \geq 8$.

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