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A CHARACTERIZATION OF δ-QUASI-BAER RINGS

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Abstract. Let δ be a derivation on R. A ring R is called *δ-quasi-Baer* (resp. *quasi-Baer*) if the right annihilator of every δ-ideal (resp. ideal) of R is generated by an idempotent of R. In this note first we give a positive answer to the question posed in Han et al. [7], then we show that R is δ-quasi-Baer iff the differential polynomial ring $S = R[x;\delta]$ is quasi-Baer iff S is $\overline{\delta}$-quasi-Baer for every extended derivation $\overline{\delta}$ on S of δ. This results is a generalization of Han et al. [7], to the case where R is not assumed to be δ-semiprime.

Throughout this note R denotes an associative ring with unity, $\delta : R \to R$ is derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote $R[x;\delta]$ the skew polynomial ring whose elements are the polynomials $\Sigma_{i=0}^{n} r_i x^i \in R$, $r_i \in R$, where the addition is defined as usual and the multiplication by $xb = bx + \delta(b)$ for any $b \in R$. For a nonempty subset X of a ring R, we write $r_R(X) = \{c \in R | dc = 0 \text{ for any } d \in X\}$ which is called the *right annihilator* of X in R.

Recall from [9] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated by an idempotent. In [9] Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$-regular rings. The class of Baer rings includes the von Neumann algebras. In [6] Clark defines a ring to be *quasi-Baer* if the right annihilator of every ideal is generated, as a right ideal, by an idempotent. Moreover, he shows the left-right symmetry of this condition by proving that R is quasi-Baer if and only if the left annihilator of every left ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Further work on quasi-Baer rings appears in [3, 4, 5, 10, 11]. An ideal I of R is called *δ-ideal* if $\delta(I) \subseteq I$. R is called *δ-quasi-Baer* if the right annihilator of every δ-ideal of R is generated by an idempotent of R. Clearly each quasi-Baer ring is δ-quasi-Baer. But the converse is not true (see [7] Example). R is said to be *reduced* if R has no nonzero nilpotent elements. Note that in a reduced ring R, R is Baer if and only if R is quasi-Baer.

In [1], Armendariz has shown that if R is reduced, then R is Baer if and only if the polynomial ring $R[x]$ is a Baer ring. Han et al. [7], have generalized this result by showing that if R is *δ-semiprime* (i.e., for any δ-ideal I of R, $I^2 = 0$ implies $I = 0$), then R is a δ-quasi-Baer ring if and only if the

Ore extension $R[x;\delta]$ is a quasi-Baer ring.

Han et al. (2000) posed this question: If $e(x) \in R[x;\delta]$ is a left semicentral idempotent, then does there exists a left semicentral idempotent $e_0 \in R$ such that $e(x)R[x;\delta] = e_0R[x;\delta]$? In this note first we give a positive answer to this question, then we show that R is δ-quasi-Baer if and only if the differential polynomial ring $S = R[x;\delta]$ is quasi-Baer if and only if S is $\overline{\delta}$-quasi-Baer for every extended derivation $\overline{\delta}$ on S of δ. This results is a generalization of Han et al. [7], to the case where R is not assumed to be δ-semiprime.

For a ring R with a derivation δ, there exists a derivation on $S = R[x;\delta]$ which extends δ. For example given in [7], consider an inner derivation $\overline{\delta}$ on S by x defined by $\overline{\delta}(f(x)) = xf(x) - f(x)x$ for all $f(x) \in S$. Then $\overline{\delta}(f(x)) = \delta(a_0) + \cdots + \delta(a_n)x^n$ for all $f(x) = a_0 + \cdots + a_nx^n \in S$ and $\overline{\delta}(r) = \delta(r)$ for all $r \in R$, which means that $\overline{\delta}$ is an extension of δ. We call such a derivation $\overline{\delta}$ on S an *extended derivation* of δ. For each $a \in R$ and nonnegative integer n, there exist $t_0, \cdots, t_n \in \mathbb{Z}$ such that $x^na = \sum_{i=0}^{n} t_i\delta^{n-i}(a)x^i$.

Lemma 1. (Han et al. Lemma 1) *Let R be a ring with a derivation δ and $\overline{\delta}$ be an extended derivation of δ on $S = R[x;\delta]$. If I is a δ-ideal of R, then $I[x;\delta]$ is $\overline{\delta}$-ideal of S.*

Proof. By ([8], Lemma 1.3), $I[x;\delta]$ is an ideal of S. Let $f(x) = a_0 + \cdots + a_nx^n \in I[x;\delta]$. For each i, $\overline{\delta}(a_ix^i) = \overline{\delta}(a_i)x^i + a_i\overline{\delta}(x^i) = \delta(a_i)x^i + a_i\overline{\delta}(x^i) \in I[x;\delta]$. Hence $I[x;\delta]$ is a $\overline{\delta}$-ideal of S. □

Now we give a positive answer to the question posed in Han et al. [7].

Theorem 2. *Let I be a δ-ideal of R and $S = R[x;\delta]$. If $r_S(I[x;\delta]) = e(x)S$ for some idempotent $e(x) = e_0+e_1x+\cdots+e_nx^n \in S$, then $r_S(I[x;\delta]) = e_0S$.*

Proof. Since $Ie(x) = 0$, we have $Ie_i = 0$ for each $i = 0, \cdots, n$. Hence $0 = \delta(Ie_i) = \delta(I)e_i + I\delta(e_i)$ for $i = 0, \cdots, n$. Since I is δ-ideal and $Ie_i = 0$, so $I\delta(e_i) = 0$ for each $i = 0, \cdots, n$. By a similar argument we can show that $I\delta^k(e_i) = 0$ for each $i = 0, \cdots, n$ and $k \geq 0$. Hence $\delta^k(e_i) \in r_S(I[x;\delta])$ for each $i = 0, \cdots, n$ and $k \geq 0$. Thus $\delta^k(e_i) = e(x)\delta^k(e_i)$ and that $e_n\delta^k(e_i) = 0$ for each $i = 0, \cdots, n$ and $k \geq 0$. Hence $\delta^k(e_i) = (e_0 + e_1x + \cdots + e_{n-1}x^{n-1})\delta^k(e_i)$ and that $e_{n-1}\delta^k(e_i) = 0$ for each $i \geq 0, k \geq 0$. Continuing in this way, we have $e_j\delta^k(e_i) = 0$ for each $i \geq 0$, $k \geq 0$, $j = 1, \cdots, n$. Thus $\delta^k(e_i) = e_0\delta^k(e_i)$ for each $i \geq 0$, $k \geq 0$. Therefore $e(x) = e_0e(x)$ and that $r_S(I[x;\delta]) = e(x)S \subseteq e_0S$. Since $\delta^k(e_0) \in r_R(I)$, so $e_0 \in r_S(I[x;\delta])$ and that $e_0S \subseteq r_S(I[x;\delta])$. Therefore $r_S(I[x;\delta]) = e_0S$. □

Proposition 3. *Let R be a δ-quasi-Baer ring. Then $S = R[x;\delta]$ is a quasi-Baer ring.*

Proof. Let J be an arbitrary ideal of S. Consider the set J_0 of leading coefficients of polynomials in J. Then J_0 is a δ-ideal of R. Since R is δ-quasi-Baer, $r_R(J_0) = eR$ for some idempotent $e \in R$. Since $J_0 e = 0$ and J_0 is δ-ideal of R, we have $J_0\delta^k(e) = 0$ for each $k \geq 0$. Hence $\delta^k(e) = e\delta^k(e)$ and $eS \subseteq r_S(J_0[x;\delta])$. Clearly $r_S(J_0[x;\delta]) \subseteq eS$. Thus $r_S(J_0[x;\delta]) = eS$. We claim that $r_S(J) = eS$. Let $f(x) = a_0 + \cdots + a_n x^n \in J$. Then $a_n \in J_0$ and that $a_n\delta^k(e) = 0$ for each $k \geq 0$. Hence $f(x)e = (a_0 + \cdots + a_{n-1}x^{n-1})e = \cdots + a_{n-1}ex^{n-1}$. Thus $a_{n-1}e \in J_0$, and $a_{n-1}\delta^k(e) = a_{n-1}e\delta^k(e) = 0$ for each $k \geq 0$. Hence $a_{n-1}x^{n-1}e = 0$. Continuing in this way, we can show that $a_i x^i e = 0$, for each $i = 0, \cdots, n$. Hence $f(x)e = 0$ and so $eS \subseteq r_S(J)$. Now, let $g(x) = b_0 + \cdots + b_m x^m \in r_S(J)$ and $f(x) = a_0 + \cdots + a_n x^n \in J$. First, we will show that $a_i x^i b_j x^j = 0$, for $i = 0, \cdots, n$, $j = 0, \cdots, m$. Since $f(x)g(x) = 0$, we have $a_n b_m = 0$. Hence $b_m \in r_R(J_0)$. Since J_0 is δ-ideal of R, $\delta^k(b_m) \in J_0$ for each $k \geq 0$ and that $b_m \in r_S(J_0[x;\delta])$. Thus $b_m = eb_m$ and $a_n x^n b_m x^m = 0$. Since $f(x)e = (a_0 + \cdots + a_n x^n)e = (a_0 + \cdots + a_{n-1}x^{n-1})e$, we have $a_{n-1}e \in J_0$ and $a_{n-1}\delta^k(e) = a_{n-1}e\delta^k(e) = 0$, for each $k \geq 0$. There exist $t_0, \cdots, t_{n-1} \in \mathbb{Z}$ such that, $a_{n-1}x^{n-1}b_m x^m = a_{n-1}x^{n-1}eb_m x^m = a_{n-1}(\sum_{j=0}^{n-1} t_j \delta^{n-1-j}(e)x^j)b_m x^m = (\sum_{j=0}^{n-1} t_j a_{n-1}\delta^{n-1-j}(e)x^j)b_m x^m$. Hence $a_{n-1}x^{n-1}b_m x^m = 0$. Continuing in this way, we have $a_i x^i b_j x^j = 0$ for each i, j. Therefore $b_j \in r_S(J_0[x;\delta]) = eS$, for each $j \geq 0$. Consequently, $g(x) = eg(x)$ and $r_S(J) = eS$. Therefore S is a quasi-Baer ring. □

Theorem 4. *Let R be a ring and $S = R[x;\delta]$. Then the following are equivalent:*

(1) *R is δ-quasi-Baer;*
(2) *S is quasi-Baer;*
(3) *S is $\overline{\delta}$-quasi-Baer for every extended derivation $\overline{\delta}$ on S of δ.*

Proof. (1)⇒(2). It follows from Proposition 3.

(2)⇒(3). It is clear.

(3)⇒(1). Suppose that R is $\overline{\delta}$-quasi-Baer for every extended derivation $\overline{\delta}$ on S of δ. Let I be any δ-ideal of R. Then by Lemma 1, $I[x;\delta]$ is $\overline{\delta}$-ideal of S. Since S is $\overline{\delta}$-quasi-Baer, $r_S(I[x;\delta]) = e(x)S$ for some idempotent $e(x) \in S$. Hence $r_S(I[x;\delta]) = e_0 S$ for some idempotent $e_0 \in R$, by Theorem 2. Since $r_R(I) = r_S(I[x;\delta]) \cap R = e_0 R$, R is δ-quasi-Baer. □

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