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PRIVALOV SPACE ON THE UPPER HALF PLANE

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Abstract. In this paper, we shall consider Privalov space $N_0^p(D)$ $(p > 1)$ which consists of holomorphic functions f on the upper half plane $D := \{z \in \mathbf{C} \mid \mathtt{Im}z > 0\}$ such that $(\log^+ |f(z)|)^p$ has a harmonic majorant on D. We shall give some properties of $N_0^p(D)$.

1. Introduction

Let U and T denote the unit disk and the unit circle in $\mathbf{C}$, respectively. For $p > 1$, Privalov space $N^p(U)$ is the class of all holomorphic functions f on U such that $(\log^+ |f(z)|)^p$ has a harmonic majorant on U. Letting $p = 1$, we have the Nevanlinna class $N(U)$.

As in [7], for each strongly convex function φ on $(-\infty, \infty)$ we define the Hardy-Orlicz class $H_\varphi(U)$ as the space of all holomorphic functions f on U such that $\varphi(\log^+ |f(z)|)$ has a harmonic majorant on U. Recall that a convex function φ is strongly convex if φ is non-negative, non-decreasing and $\varphi(t)/t \to \infty$ as $t \to \infty$. We define $N_*(U) = \bigcup \{H_\varphi(U) \mid \varphi :$ strongly convex$\}$, which is called the Smirnov class.

For $0 < q < \infty$, the space $H_\varphi(U)$ with $\varphi(t) = e^{qt}$ coincides with the usual Hardy space $H^q(U)$. For each $p > 1$, if we define $\varphi_p(t)$ on $(-\infty, \infty)$ by $\varphi_p(t) = t^p$ for $t \geqq 0$, and $\varphi_p(t) = 0$ for $t < 0$, we obtain $N^p(U)$ as a subspace of $N_*(U)$.

It is well-known that $H^q(U) \subset N^p(U) \subset N_*(U) \subset N(U)$ $(0 < q < \infty, p > 1)$. These including relations are proper. $N^p(U)$ was treated by several authors ([2], [5], [7] and [8]). The spaces $N(U)$, $N_*(U)$, $N^p(U)$ and $H^q(U)$ are called *Nevanlinna-type spaces.*

Let $D := \{z \in \mathbf{C} \mid \mathtt{Im}z > 0\}$. We let the Nevanlinna class $N_0(D)$, as Krylov [4] introduced, consist of all holomorphic functions f on D such that $\log^+ |f(z)|$ has a harmonic majorant on D.

Rosenblum and Rovnyak [6] introduced the Hardy-Orlicz and Smirnov classes on D: for each strongly convex function ϕ on $(-\infty, \infty)$, $H_\phi(D)$ is the set of all holomorphic functions f on D such that $\phi(\log^+ |f(z)|)$ has a harmonic majorant on D. We define $N_0^*(D) = \bigcup \{H_\phi(D) \mid \phi :$ strongly convex$\}$.

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In this paper, we shall define a new class $N_0^p(D)$, analogous to $N^p(U)$; i.e., we denote by $N_0^p(D)\,(p>1)$ the set of all holomorphic functions f on D such that $(\log^+|f(z)|)^p$ has a harmonic majorant on D. First we obtain a factorization theorem for the space $N_0^p(D)$. Moreover, some properties of $N_0^p(D)$ are also given.

2. Preliminaries

Let ν be a real measure on T and $\Psi(z)=(z-i)/(z+i)\quad(z\in\overline{D})$. Then there corresponds a finite real measure μ on $\mathbf{R}$ such that

$$\int_{\mathbf{R}} h(t)\,d\mu(t)=\int_{T^*}(h\circ\Psi^{-1})(\eta)\,d\nu(\eta)\quad(h\in C_c(\mathbf{R})),$$

where $T^*=T\setminus\{1\}$. Let $H(w\,,\eta)=(\eta+w)/(\eta-w)\quad((w\,,\eta)\in U\times T)$. There holds

$$\text{(1)}\qquad \frac{1}{i}\int_{\mathbf{R}}\frac{1+tz}{t-z}\,d\mu(t)=\int_{T^*}H(\Psi(z)\,,\eta)\,d\nu(\eta)$$

$$=\int_T H(\Psi(z)\,,\eta)\,d\nu(\eta)-i\alpha z\quad(z\in D),$$

where $\alpha=-\nu(\{1\})$. We write the Poisson integrals of measures μ on $\mathbf{R}$ and ν on T as follows:

$$P[\mu](z)=\frac{1}{\pi}\int_{\mathbf{R}}\frac{y}{(x-t)^2+y^2}\,d\mu(t)\quad(z=x+iy\in D),$$

$$Q[\nu](w)=\int_T\frac{1-|w|^2}{|\eta-w|^2}\,d\nu(\eta)\quad(w\in U).$$

Taking the real parts in (1), we have

$$P[\pi(1+t^2)d\mu(t)](z)=Q[\nu](\Psi(z))+\alpha\cdot\mathtt{Im}z\quad(z\in D).$$

When $f\in L^1(\mathbf{R},(1+t^2)^{-1}dt)$ and $g\in L^1(T)$, we write $P[f]$ and $Q[g]$ instead of $P[f(t)dt]$ and $Q[g\sigma]$, respectively. If $g\in L^1(T)$, then we have $g\circ\Psi\in L^1(\mathbf{R},(1+t^2)^{-1}dt)$ and

$$\text{(2)}\qquad P[g\circ\Psi](z)=Q[g](\Psi(z)).$$

3. Some properties of $N_0(D)$, $N_0^*(D)$ and $N^p(U)$

In this section, we shall summarize some properties of $N_0(D)$, $N_0^*(D)$ and $N^p(U)$ $(p > 1)$. For the following results, the reader refers to [2], [3] and [6].

Recall that an *outer function on D* is of the form

$$d(z) = \exp\left(\frac{1}{\pi i}\int_{\mathbf{R}} \frac{1+tz}{t-z}\frac{1}{1+t^2}\log h(t)\,dt\right),$$

where $h(t) \geqq 0$, $\log h \in L^1\left(\mathbf{R}, \dfrac{dt}{1+t^2}\right)$.

Proposition 3.1. *Let $f \in N_0(D)$, $f \neq 0$. Then $f^*(x) = \lim\limits_{z\to x} f(z)$ exists nontangentially a.e. for $x \in \mathbf{R}$.*

Proposition 3.2. *Let $H^\infty(D)$ be the class of all bounded holomorphic functions on D.*

(i) $N_0(D) = \left\{\dfrac{g}{h} : g, h \in H^\infty(D)\,,\ h \neq 0\right\}$.

(ii) $N_0^*(D) = \left\{\dfrac{g}{h} : g, h \in H^\infty(D)\,,\ h \textit{ is outer}\right\}$.

Proposition 3.3. *Let f be holomorphic on D.*

(i) *$f \in N_0(D)$ if and only if*

$$\sup_{y>0}\int_{\mathbf{R}} \frac{\log^+|f(x+iy)|}{x^2+(y+1)^2}\,dx < \infty.$$

(ii) *If ϕ is a strongly convex function, then $f \in N_0^*(D)$ if and only if*

$$\sup_{y>0}\int_{\mathbf{R}} \frac{\phi(\log^+|f(x+iy)|)}{x^2+(y+1)^2}\,dx < \infty.$$

Proposition 3.4. *Let $p > 1$ and f be holomorphic on U. Then the following are equivalent:*

(i) $f \in N^p(U)$.

(ii) $\sup\limits_{0<r<1}\displaystyle\int_0^{2\pi}\left(\log^+|f(re^{i\theta})|\right)^p d\theta < \infty$.

(iii) *$f \in N(U)$ and the condition*

$$\int_0^{2\pi}\left(\log^+|f^*(e^{i\theta})|\right)^p d\theta < \infty$$

holds, where $f^*(e^{i\theta}) = \lim\limits_{r\to 1^-} f(re^{i\theta})$ (a.e. $e^{i\theta} \in T$).

Proposition 3.5. *Let* $f \in N^p(U)$ $(p > 1)$, $f \neq 0$. *Then,* $\log|f^*| \in L^1(T)$ *and* $\log(1 + |f^*|) \in L^p(T)$. *Furthermore,* f *can be uniquely factored as follows,*

$$f(z) = aB(z)F(z)S(z) \quad (z \in U), \tag{3}$$

where the factors above have the following properties.

(i) $a \in T$ *is a constant.*

(ii) $B(z) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$ $(z \in U)$ *is a Blaschke product with respect to the zeros of* f.

(iii) $F(z) = \exp\left(\int_T \frac{\zeta + z}{\zeta - z} \log|f^*(\zeta)|\, d\sigma(\zeta)\right)$, *where* σ *denotes normalized Lebesgue measure on* T.

(iv) $S(z) = \exp\left(-\int_T \frac{\zeta + z}{\zeta - z}\, d\nu(\zeta)\right)$, *where* ν *is a positive singular measure on* T.

If f *is expressed in the form* (3), then $f \in N^p(U)$.

Proposition 3.6. *Let* $f \in N^p(U)$, $p > 1$. *Then* $(\log^+|f|)^p$ *has the least harmonic majorant* $Q[(\log^+|f^*|)^p]$.

4. A factorization theorem for the space $N_0^p(D)$

Theorem 4.1. *Let* $p > 1$. $f \in N_0^p(D)$, $f \neq 0$, *is factorized in the form*

$$f(z) = ae^{i\alpha z} b(z) d(z) g(z) \quad (z \in D), \tag{4}$$

with the following properties.

(i) $a \in T$, $\alpha \geqq 0$.

(ii) $b(z)$ *is the Blaschke product with respect to the zeros of* f.

(iii) $d(z) = \exp\left(\frac{1}{\pi i} \int_{\mathbf{R}} \frac{1 + tz}{t - z} \frac{1}{1 + t^2} \log h(t)\, dt\right)$, *where* $h(t) \geqq 0$, $\log h \in L^1(\mathbf{R}, (1 + t^2)^{-1} dt)$ *and* $\log^+ h \in L^p(\mathbf{R}, (1 + t^2)^{-1} dt)$.

(iv) $g(z) = \exp\left(-\frac{1}{i} \int_{\mathbf{R}} \frac{1 + tz}{t - z}\, d\mu(t)\right)$, *where* μ *is a finite real measure* on $\mathbf{R}$, *singular with respect to Lebesgue measure.*

If f *is expressed in the form* (4), then $f \in N_0^p(D)$.

Proof. Suppose $f \in N_0^p(D)$, $f \neq 0$. Then $f \circ \Psi^{-1} \in N^p(U)$, and Proposition 3.5 implies $(f \circ \Psi^{-1})(w) = aB(w)F(w)S(w) \quad (w \in U)$. In the factorization $f(z) = aB(\Psi(z))F(\Psi(z))S(\Psi(z)) \quad (z \in D)$, $b(z) := B(\Psi(z))$ is the Blaschke product formed from the zeros of f, and the change of the variables $\eta = \Psi(t) \quad (t \in \mathbf{R})$ shows that

$$d(z) := F(\Psi(z)) = \exp\left(\frac{1}{\pi i}\int_{\mathbf{R}} \frac{1+tz}{t-z}\frac{1}{1+t^2}\log|f^*(t)|\,dt\right).$$

This is of the form (iii). Since $\log|(f \circ \Psi^{-1})^*| \in L^1(T)$, we have $\log|f^*| \in L^1(\mathbf{R}, (1+t^2)^{-1}dt)$ by (2). Next $\log^+|f(\Psi^{-1}(\eta))| \in L^p(T)$ implies $\log^+|f^*| \in L^p(\mathbf{R}, (1+t^2)^{-1}dt)$. Setting $\alpha = \nu\{1\}$, we have $S(\Psi(z)) = g(z)e^{i\alpha z}$, where g is of the form (iv).

Suppose, conversely, that f is of the form (4). Then

$$|f(z)| = |e^{i\alpha z}||b(z)|\exp(P[\log h - \pi(1+t^2)d\mu(t)](z)) \leqq \exp(P[\log h](z)).$$

Since $\log^+|(f \circ \Psi^{-1})(w)| \leqq Q[\log^+|(h \circ \Psi^{-1})|](w)$, we have $f \circ \Psi^{-1} \in N^p(U)$. Letting $y \to 0^+$ in $|f(x+iy)|$, we have $|f^*(x)| = h(x)$ a.e. for $x \in \mathbf{R}$. Furthermore, $(\log^+|f \circ \Psi^{-1}|)^p$ has the least harmonic majorant $v' = Q[(\log^+|(f \circ \Psi^{-1})^*|)^p]$ by Proposition 3.6, hence $v := v' \circ \Psi$ is the least harmonic majorant of $(\log^+|f|)^p$; i.e., $(\log^+|f(z)|)^p \leqq P[(\log^+|f^*|)^p](z)$. Integrating the both sides, we have $f \in N_0^p(D)$. $\square$

5. Some theorems for the space $N_0^p(D)$

In this section, we prove some theorems for the space $N_0^p(D)$.

Theorem 5.1. *Let f be holomorphic on D. Then, for $p > 1$,*

$$N_0^p(D) = \left\{ \frac{k_1}{k_2} : k_1,\, k_2 \in H^\infty(D),\ k_2 \textit{ is invertible in } N_0^p(D) \right\}.$$

Proof. Let $f \in N_0^p(D)$. Then $f(z) = ae^{i\alpha z}b(z)d(z)g(z) \quad (z \in D)$ by Theorem 4.1. Now d takes the form

$$d(z) = \exp\left(\frac{1}{\pi i}\int_{\mathbf{R}} \frac{1+tz}{t-z}\frac{1}{1+t^2}\log|f^*(t)|\,dt\right) = \frac{d_1(z)}{d_2(z)},$$

where

$$d_1(z) = \exp\left(-\frac{1}{\pi i}\int_{\mathbf{R}} \frac{1+tz}{t-z}\frac{1}{1+t^2}\log^-|f^*(t)|\,dt\right)$$

and

$$d_2(z) = \exp\left(-\frac{1}{\pi i}\int_{\mathbf{R}} \frac{1+tz}{t-z}\frac{1}{1+t^2}\log^+|f^*(t)|\,dt\right).$$

We note that d_1 and d_2 both belong to $H^\infty(D)$ and are outer functions on D. Moreover, we find d_2 , $1/d_2 \in N_0^p(D)$. Therefore we have $f = ae^{i\alpha z}bdg = ae^{i\alpha z}bd_1g/d_2$, where $k_1 := ae^{i\alpha z}bd_1g$ and $k_2 := d_2$ are both in $H^\infty(D)$ and k_2 is invertible in $N_0^p(D)$.

On the other hand, let $f = k_1/k_2$, where k_1 , $k_2 \in H^\infty(D)$ and k_2 is invertible in $N_0^p(D)$. Since $N_0^p(D)$ is an algebra, it follows that $f \in N_0^p(D)$. □

Theorem 5.2. *Let f be holomorphic on D. Then, for $p > 1$, $f \in N_0^p(D)$ if and only if*

$$\sup_{y>0} \int_{\mathbf{R}} \frac{(\log^+ |f(x+iy)|)^p}{x^2+(y+1)^2}\, dx < \infty.$$

Proof. $(\log^+ |f(x+iy)|)^p$ is non-negative and subharmonic on D. Therefore we prove the result by the theorem of Flett and Kuran [6, p.89]. □

Theorem 5.3. *Let $f(z) \in N_0(D)$ and $p > 1$. Then f belongs to $N_0^p(D)$ if and only if*

$$\frac{1}{\pi} \int_{\mathbf{R}} \frac{(\log^+ |f^*(x)|)^p}{1+x^2}\, dx < \infty. \tag{5}$$

Proof. The function $f(z)$ is in $N_0^p(D)$ if and only if $F(z) = f(\Psi^{-1}(z))$ is in $N^p(U)$. By Proposition 3.4, this is the case if and only if

$$\int_0^{2\pi} (\log^+ |f^*(e^{i\theta})|)^p\, d\theta < \infty.$$

This is the same as condition (5). □

Theorem 5.4. *Let $p > 1$. If $f \in N_0^p(D)$, then*

$$\lim_{y \to 0^+} \int_{\mathbf{R}} \left|\log^+ |f(x+iy)| - \log^+ |f^*(x)|\right|^p\, dx = 0.$$

Proof. Let $f \in N_0^p(D)$. Then $F(z) = f(\Psi^{-1}(z))$ belongs to $N^p(U)$. By [7, Proposition 4.1], we have $(\log^+ |f(z)|)^p \geqq P[(\log^+ |f^*|)^p](z)$. Integrating the both sides, it follows that

$$\int_{\mathbf{R}} \left(\log^+ |f(x+iy)|\right)^p\, dx \geqq \int_{\mathbf{R}} \left(\log^+ |f^*(x)|\right)^p\, dx.$$

Using Fatou's lemma, we obtain

$$\lim_{y \to 0^+} \int_{\mathbf{R}} \left(\log^+ |f(x+iy)|\right)^p\, dx = \int_{\mathbf{R}} \left(\log^+ |f^*(x)|\right)^p\, dx.$$

Applying [1, Lemma 1, p.21], we have the desired result. □

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