Math. J. Okayama Univ. **49** (2007), 149–161

HIGHER WEIGHTS OF CODES FROM PROJECTIVE PLANES AND BIPLANES

Steven T. DOUGHERTY and Reshma RAMADURAI

Abstract. We study the higher weights of codes formed from planes and biplanes. We relate the higher weights of the Hull and the code of a plane and biplane. We determine all higher weight enumerators of planes and biplanes of order less or equal to 4.

1. Introduction

Some of the most interesting open questions in combinatorics are about the existence and classification of projective planes and biplanes. Codes have often been useful in examining these questions. For example, the proof of the non-existence of the plane of order 10 consisted primarily in showing its corresponding code does not exist [6]. Central to this proof was determining the weight enumerator of the putative code's extension to a self-dual code. In this work we shall study the codes formed from designs by examining their higher weights. We do this so that the structure of the code of the design can be better understood which would aid in the classification of planes and biplanes.

A projective plane Π is $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is a set of points, $\mathcal{L}$ is a set of lines, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$, such that through any two points in $\mathcal{P}$ there is a unique line; any two lines in $\mathcal{L}$ meet in a unique point; and there exists at least 4 points, no 3 collinear.

It follows immediately that $|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$ and that there are $n+1$ points on a line and $n+1$ lines through a point. The number n is said to be the order of the plane.

A biplane $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is a set of points, $\mathcal{L}$ is a set of lines, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$, such that through any two points in $\mathcal{P}$ there are two lines and any two lines in $\mathcal{L}$ meet exactly twice. It follows immediately that $|\mathcal{P}| = |\mathcal{L}| = \frac{n^2+3n+4}{2}$ and that there are $n+2$ points on a line and $n+2$ lines through a point. The number n is said to be the order of the biplane.

Key words and phrases. Projective plane, biplane, codes of designs, higher weights.

The second author is grateful for the hospitality of the University of Scranton where she stayed while this paper was written.

A linear code is a subspace of $\mathbb{F}_p^n$, where $\mathbb{F}_p$ is a field with p elements. We attach the standard inner product: $[v,w] = \sum v_i w_i$, and for a code C define $C^\perp = \{v \in \mathbb{F}_p^n \mid [v,w] = 0 \ \ \forall w \in C\}$. As usual, if $C \subseteq C^\perp$ we say that C is self-orthogonal, and if $C = C^\perp$ then C is self-dual. The Hamming weight enumerator of a code C is $H_C(x,y) = \sum_{c \in C} x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)}$, where $\mathrm{wt}(c)$ is the number of the non-zero coordinates in c. Often we set $x = 1$ and simply write $H_C(y)$.

Let $D \subseteq \mathbb{F}_p^n$ be a linear subspace, then $||D|| = |\mathrm{Supp}(D)|$, where $\mathrm{Supp}(D) = \{i \mid \exists v \in D, \ v_i \neq 0\}$. For a linear code C we define $d_r(C) = \min\{||D|| \mid D \subseteq C, \ \dim(D) = r\}$. The minimum Hamming weight of a code C is $d_1(C)$. We know that $d_i < d_j$ when $i < j$ (Proposition 3.1 in [7]).

We define the higher weight spectrum as $A_i^r = |\{D \subseteq C \mid \dim(D) = r, \ ||D|| = i\}|$. This naturally allows us to define the higher weight enumerators by $W^r(C;y) = W^r(C) = \sum A_i^r y^i$. It is immediate that if C is a code with dimension k over $\mathbb{F}_p$ then $W^r(C;1) = \frac{(p^k-1)(p^k-p)\dots(p^k-p^{r-1})}{(p^r-1)(p^r-p)\dots(p^r-p^{r-1})}$.

We use throughout that $|\mathrm{Supp}(\langle v,w \rangle)| = \mathrm{wt}(v) + \mathrm{wt}(w) - |v \wedge w|$, where $|v \wedge w|$ is the number of coordinates where v and w and are both non-zero.

There exists MacWilliams type identities for higher weight enumerators, see [3], [7], namely

$$\sum_{r=0}^{s} [s]_r W^r(C^\perp; y) = p^{-sk}(1 + (p^s - 1)y)^n \sum_{r=0}^{s} W^r(C; \frac{1-y}{1+(p^s-1)y}) \tag{1}$$

where the code has dimension k in $\mathbb{F}_p^n$, and $[s]_r = \prod_{j=0}^{r-1}(p^s - p^j)$.

2. Codes of Planes and Biplanes

For a line L, we define the characteristic function of the line by

$$v_L(q) = \begin{cases} 1 & \text{if } q \text{ is incident with } L \\ 0 & \text{if } q \text{ is not incident with } L \end{cases} \tag{2}$$

where q is a point in the design. We denote by v_L the vector in $\mathbb{F}_p^{|\mathcal{P}|}$ that corresponds to the characteristic function of the line.

The code of the design Π over $\mathbb{F}_p$ is defined by $C_p(\Pi) = \langle v_L \mid L \in \mathcal{L} \rangle$. The Hull of a design is defined as $\mathrm{Hull}_p(\Pi) = C_p(\Pi) \cap C_p(\Pi)^\perp$. It is a self-orthogonal code.

As usual we study those codes over $\mathbb{F}_p$ where p is a prime that divides the order of the design. The following result is well known, see [1] for the result for projective planes. We prove a similar result for biplanes. Throughout we denote the all one vector by $\mathbf{1}$.

Theorem 2.1. *Let* Π *be a projective plane, then* $\mathrm{Hull}_p(\Pi)$ *is of codimension 1 in* $C_p(\Pi)$, $C_p(\Pi) = \langle \mathrm{Hull}_p(\Pi), \mathbf{1}\rangle$ *and* $\mathrm{Hull}_p(\Pi) = \langle v_L - v_M \mid L, M \in \mathcal{L}\rangle$. *If* p *sharply divides* n, *then* $\mathrm{Hull}_p(\Pi) = C_p(\Pi)^{\perp}$.

Theorem 2.2. *Let* Π *be a biplane of order* n, *with* p *an odd prime dividing* n, *then* $\mathrm{Hull}_p(\Pi) = \langle v_L - v_M | \ L, M \in \mathcal{L}\rangle$, *and* $\mathrm{Hull}_p(\Pi)$ *is of codimension 1 in* $C_p(\Pi)$.

Proof. Let $D = \langle v_L - v_M | \ L, M \in \mathcal{L}\rangle$. Let L, L' and M be lines in Π then $[v_M, v_L - v_{L'}] = 2 - 2 = 0$. Hence the code D is contained in $C_p(\Pi)^{\perp}$. Of course, D is naturally contained in $C_p(\Pi)$. Let L be any line in the biplane, then $\langle D, v_L\rangle = C_p(\Pi)$. Hence the code D is at most codimension 1 in $C_p(\Pi)$. However, since p is an odd prime we know that $C_p(\Pi)$ is not self-orthogonal since $[v_L, v_L] = 2 \neq 0$ for all lines in the biplane. Hence $D \subseteq \mathrm{Hull}_p(\Pi) \subset C_p(\Pi)$ and D is of codimension 1 in $C_p(\Pi)$ and we have the result. □

3. Higher Weights

In this section we shall relate the minimum higher weights of the Hull and the code of a plane and biplane.

Lemma 3.1. *Let* Π *be a projective plane or a biplane of odd order. Let* V *be a* k*-dimensional subspace of* $C_p(\Pi)$ *then* $V \cap \mathrm{Hull}_p(\Pi)$ *is a* k *or* $k-1$ *dimensional subspace of* $\mathrm{Hull}_p(\Pi)$.

Proof. If V is contained in $\mathrm{Hull}_p(\Pi)$ then $V \cap \mathrm{Hull}_p(\Pi)$ is a k dimensional subspace of $\mathrm{Hull}_p(\Pi)$. Otherwise we have $V \oplus \mathrm{Hull}_p(\Pi) = C_p(\Pi)$ which gives

$$\begin{aligned}
\dim\,(V) + \dim\,(\mathrm{Hull}_p(\Pi)) - \dim\,(V \cap \mathrm{Hull}_p(\Pi)) &= \dim(C_p(\Pi)) \\
k + \dim\, C_p(\Pi) - 1 - \dim\,(V \cap \mathrm{Hull}_p(\Pi)) &= (\dim C_p(\Pi)) \\
k - 1 &= \dim\,(V \cap \mathrm{Hull}_p(\Pi)).
\end{aligned}$$

□

Theorem 3.2. *Let* Π *be a projective plane or a biplane of odd order. Then for* $1 \leq k \leq \dim(C_p(\Pi))$, *we have*

$$d_k(C_p(\Pi)) \geq d_{k-1}(\mathrm{Hull}_p(\Pi)). \tag{3}$$

Proof. If V is a k dimensional subspace of $C_p(\Pi)$ then V is either a k dimensional subspace of $\text{Hull}_p(\Pi)$ or $V \cap \text{Hull}_p(\Pi)$ is a $k-1$ dimensional subspace of $\text{Hull}_p(\Pi)$. We know that $d_k(\text{Hull}_p(\Pi)) > d_{k-1}(\text{Hull}_p(\Pi))$ and if $V \cap \text{Hull}_p(\Pi)$ is a $k-1$ dimensional subspace we know

$$|\text{Supp}(V)| \geq |\text{Supp}(V \cap \text{Hull}_p(\Pi)|,$$

which gives $d_k(C_p(\Pi)) \geq d_{k-1}(\text{Hull}_p(\Pi))$. □

It is possible that $d_k(C_p(\Pi)) = d_{k-1}(\text{Hull}_p(\Pi))$. For instance $d_6(C_p(\Pi)) = d_5(\text{Hull}_p(\Pi)) = 12$ for the projective plane of order 3.

Using the MacWilliams relations given in Equation 1 and the bounds given in Equation 3, we can compute the higher weight enumerators of the projective plane of order 3. We use the known Hamming weight enumerators of the plane of order 3 in this computation. We require a bit more to determine the higher weight enumerators of the biplane of order 3, but we compute it later. The higher weights of the projective plane are given in Table 1 and for the biplane are given Table 8.

TABLE 1. Higher Weight Enumerators of the code and the Hull of the Projective Plane of Order 3

Weight	4	5	6	7	8	9	10	11	12	13
$W^1_{C_3(\Pi)}$	13	0	78	312	0	247	390	0	39	14
$W^1_{H_3(\Pi)}$	0	0	78	0	0	247	0	0	39	0
$W^2_{C_3(\Pi)}$	0	0	0	78	819	4030	11310	26910	34710	21606
$W^2_{H_3(\Pi)}$	0	0	0	0	117	286	1404	3042	3705	2457
$W^3_{C_3(\Pi)}$	0	0	0	0	0	715	8580	64350	283140	568986
$W^3_{H_3(\Pi)}$	0	0	0	0	0	13	234	2340	10296	20997
$W^4_{C_3(\Pi)}$	0	0	0	0	0	0	286	8580	125125	791780
$W^4_{H_3(\Pi)}$	0	0	0	0	0	0	0	78	1417	9516
$W^5_{C_3(\Pi)}$	0	0	0	0	0	0	0	78	4576	94809
$W^5_{H_3(\Pi)}$	0	0	0	0	0	0	0	0	13	351
$W^6_{C_3(\Pi)}$	0	0	0	0	0	0	0	0	13	1080
$W^6_{H_3(\Pi)}$	0	0	0	0	0	0	0	0	0	1
$W^7_{C_3(\Pi)}$	0	0	0	0	0	0	0	0	0	1

Theorem 3.3. *Let* Π *be a projective plane or a biplane with* N *points and* $\dim(C_p(\Pi)) = r$. *Then* $W^{r-1}(y) = Ny^{N-1} + (\frac{p^r-1}{p-1} - N)y^N$.

Proof. We know that $W^{r-1}(1) = \frac{p^r-1}{p-1}$ so we need only to determine the number of $r-1$ dimensional subspaces that have support size $N-1$ and show that there are none with support size less than $N-1$.

Let q be any point in Π. Then v_q is the vector that is 1 on q and 0 elsewhere. We note that $v_q \notin C_p(\Pi)^{\perp}$. Let $C_0 = \{w \mid [w, v_q] = 0, w \in C_p(\Pi)\}$. Then C_0 is of codimension 1 in $C_p(\Pi)$ and hence dimension $r-1$. The code C_0 consists precisely of those vectors that are 0 on the coordinate corresponding to the point q. Any other point q' is part of the support of C_0. Simply take a line through the point q' that does not intersect q. Then the characteristic function of this line is in C_0 and has a 1 at the coordinate corresponding to q'. Hence the support size is $N-1$ and there are N such $r-1$ dimensional subspaces.

Assume D is a subspace of $C_p(\Pi)$ with $\dim(D) = r-1$ and $|\mathrm{Supp}(D)| < N-1$. Then there exists a constant vector $v \notin C_p(\Pi)^{\perp}$ with $D = \{w \mid [w, v] = 0, w \in C_p(\Pi)\}$. Let a_1 and a_2 be two points in $\mathrm{Supp}(v)$, i.e. two points not in the support of the subspace D. Each line L through either point must have $[L, v] = 0$ since a_i is not in the support of D. Let L_1 be a line through a_1 that is not through a_2 and let L_1 be a line through a_2 that is not through a_1. Then $\alpha v_{L_1} + \beta v_{L_2} \notin C_p(\Pi)$ for any non-zero α and β. Then D is at least codimension 2 in $C_p(\Pi)$ which is a contradiction. Therefore there are no $r-1$ dimensional subspaces with support size less than $N-1$. □

4. Planes of even order

In this section we shall examine planes of even order and as such we assume $p = 2$ throughout. We note that $\mathrm{Hull}_2(\Pi)$ is a doubly-even code. This can be seen by noticing that it is a self-orthogonal code and that for any lines L and M the vector $v_L - v_M$ has weight $2n$ which is doubly-even when n is even.

Lemma 4.1. *Let* Π *be a projective plane of order* $n \equiv 2 \pmod 4$*, then*

$$H_{\mathrm{Hull}_2(\Pi)}(x, y) + H_{\mathrm{Hull}_2(\Pi)}(y, x) = \frac{1}{|\mathrm{Hull}_2(\Pi)|} H_{\mathrm{Hull}_2(\Pi)}(x+y, x-y). \tag{4}$$

Proof. If $n \equiv 2 \pmod 4$ then 2 sharply divides the order giving that $\mathrm{Hull}_2(\Pi)^{\perp} = C_2(\Pi)$. The left side computes $H_{C_2(\Pi)}(x, y)$ by using the fact that $C_2(\Pi) = \langle \mathrm{Hull}_2(\Pi), \mathbf{1} \rangle$ and the right side computes $H_{C_2(\Pi)}(x, y)$ by using the MacWilliams relations. □

Lemma 4.2. *If* $\alpha = |\text{Supp}\langle v, w\rangle|$, $v, w \in \text{Hull}_2(\Pi)$ *then* α *is even.*

Proof. We know $\text{wt}(v) = 4\beta$, $\text{wt}(w) = 4\gamma$ and they meet in 2δ places for some β, γ, δ. Then we have $|\text{Supp}(\langle v, w\rangle)| = 4\beta + 4\gamma - 2\delta = 2(2\beta + 2\gamma - \delta)$. □

Notice that $|\mathcal{P}| = n^2 + n + 1$ and if $n \equiv 2 \pmod 4$ all weights in $\text{Hull}_2(\Pi)$ are $0 \pmod 4$ and $\mathbf{1}$ has weight $3 \pmod 4$ so all weights in $C_2(\Pi)$ have weight either 3 or $0 \pmod 4$. If $n \equiv 0 \pmod 4$ then all weights in $C_2(\Pi)$ have weight either 1 or $0 \pmod 4$.

Theorem 4.3. *Let* Π *be a projective plane of order* $n \equiv 0 \pmod 2$. *If* $W^2_{C_2(\Pi)}(y) = \sum A_i y^i$ *and* $W^2_{\text{Hull}_2(\Pi)}(y) = \sum B_i y^i$ *then for even* i, $A_i = B_i$.

Proof. The code $\text{Hull}_2(\Pi)$ is a doubly-even code of codimension 1 in $C_2(\Pi)$ and $C_2(\Pi) = \langle \text{Hull}_2(\Pi), \mathbf{1}\rangle$. For $n \equiv 2 \pmod 4$ the weights in $C_2(\Pi)$ are 0 or $3 \pmod 4$ and for $n \equiv 0 \pmod 4$ the weights in $C_2(\Pi)$ are all 0 or $1 \pmod 4$.

If $v, w \in C_2(\Pi) - \text{Hull}_2(\Pi)$, then $v + w \in \text{Hull}_2(\Pi)$. Then we have $|\text{Supp}\langle v, w\rangle| = \text{wt}(v) + \text{wt}(w) - |w \wedge v|$ and $\text{wt}(v) + \text{wt}(w) - 2|w \wedge v| \equiv 0 \pmod 4$. For $n \equiv 2 \pmod 4$ we have $3 + 3 - 2|v \wedge w| \equiv 0 \pmod 4$ which implies $|v \wedge w| \equiv 1 \pmod 2$. For $n \equiv 0 \pmod 4$ we have $1 + 1 - 2|v \wedge w| \equiv 0 \pmod 4$ which implies $|v \wedge w| \equiv 1 \pmod 2$. This gives that $|\text{Supp}\langle v, w\rangle| \equiv 1 \pmod 2$.

If $v \in C_2(\Pi)$ and $w \in \text{Hull}_2(\Pi)$ then $v + w \in C_2(\Pi) - \text{Hull}_2(\Pi)$ and a similar argument gives that the size of the support is odd. Hence the only way to have even support is if the 2 dimensional subspace is completely contained in $\text{Hull}_2(\Pi)$. □

If Π is a plane of even order the code $\text{Hull}_2(\Pi)$ is a doubly-even self-orthogonal code. Then the results follows from Theorem 2.2 in [2] give that $d_2(\text{Hull}_2(\Pi)) \geq \frac{3}{2} d_2(\text{Hull}_2(\Pi))$.

Lemma 4.4. *Let* C *be a self-orthogonal binary code. If* $W^2_C(y) = \sum A_i y^i$ *then* $A_i = 0$ *when* i *is odd.*

Proof. We know C is self-orthogonal so $|\text{Supp}\langle v, w\rangle| = \text{wt}(v) + \text{wt}(w) - |w \wedge v| \equiv 0 \pmod 2$. □

Theorem 4.5. *Let* Π *be a plane. If* $W^2_{\text{Hull}_2(\Pi)}(y) = \sum A_i y^i$ *then* $A_i = 0$ *if* i *is odd.*

Proof. Follows from Lemma 4.4. □

Using the previous theorems we can derive the higher weight enumerators of the projective plane of order 2. We give it in Table 2.

TABLE 2. Higher Weight Enumerators of the Projective Plane of Order 2

Weight	3	4	5	6	7
$W^1_{C_2(\Pi)}$	7	7	0	0	1
$W^1_{H_2(\Pi)}$	0	7	0	0	0
$W^2_{C_2(\Pi)}$	0	0	21	7	7
$W^2_{H_2(\Pi)}$	0	0	0	7	0
$W^3_{C_2(\Pi)}$	0	0	0	7	8
$W^3_{H_2(\Pi)}$	0	0	0	0	1
$W^4_{C_2(\Pi)}$	0	0	0	0	1

Surprisingly, the previous results are also enough to give all weight enumerators of the projective plane of order 4, even though the Hull is not the orthogonal of the code in this case. Instead we use the weight enumerator of Hull together with the theorems to get the weight enumerators of the Hull and its orthogonal. Then we can determine the weight enumerators of the code using Theorem 4.3. The weight enumerators are given in Table 3. We list only the code since the weight enumerators of the Hull can be read from these weight enumerators.

5. Biplanes of even order

As a preliminary, we note that if L and M are two lines in Π, a biplane of order n then $[v_L, v_M] = 2$ if L and M are distinct and $[v_L, v_M] = n+2$ if $L = M$.

Theorem 5.1. *Let Π be a biplane of even order n. Then $C_2(\Pi)$ is a self-orthogonal code and* $\mathrm{Hull}_2(\Pi) = C_2(\Pi)$.

Proof. We notice that $[v_L, v_M] = 0$ for any two lines L and M in Π and hence the code is generated by self-orthogonal vectors. Then since $C_2(\Pi) \subseteq C_2(\Pi)^\perp$ we have $\mathrm{Hull}_2(\Pi) = C_2(\Pi) \cap C_2(\Pi)^\perp = C_2(\Pi)$. □

Lemma 5.2. *If Π is a biplane of order $n \equiv 2 \pmod 4$ then all weights in $C_2(\Pi)$ are congruent to $0 \pmod 4$.*

Proof. Since $n \equiv 2 \pmod 4$ the characteristic function of lines have weight $n+2 \equiv 0 \pmod 4$. Moreover, any two of these vectors are orthogonal

TABLE 3. Higher Weight Enumerators of the Projective Plane of Order 4

Weight	$W^1_{C_2(\Pi)}$	$W^2_{C_2(\Pi)}$	$W^3_{C_2(\Pi)}$	$W^4_{C_2(\Pi)}$	$W^5_{C_2(\Pi)}$
5	21	0	0	0	0
6	0	0	0	0	0
7	0	0	0	0	0
8	210	0	0	0	0
9	280	210	0	0	0
10	0	0	0	0	0
11	0	3360	0	0	0
12	280	3850	1120	0	0
13	210	20790	7770	0	0
14	0	10080	49080	2520	0
15	0	43680	182280	41664	168
16	21	17955	453495	327915	19341
17	0	48510	944580	1543500	277830
18	0	10080	1502760	5334000	2239020
19	0	13440	1669080	12945240	11822580
20	0	1470	1147020	19531008	37933434
21	1	826	390530	14018140	56929278
Weight	$W^6_{C_2(\Pi)}$	$W^7_{C_2(\Pi)}$	$W^8_{C_2(\Pi)}$	$W^9_{C_2(\Pi)}$	$W^{10}_{C_2(\Pi)}$
17	5985	0	0	0	0
18	144970	1330	0	0	0
19	1796130	49560	210	0	0
20	12497625	809025	10311	21	0
21	39299277	5487800	163730	1002	1

so the code is generated by orthogonal doubly-even vectors and hence the code is doubly-even. □

Since the code is equal to the Hull we shall introduce a code that will act in many ways like the Hull. Let Π be a biplane of even order n. If $n \equiv 2 \pmod 4$ then $C_2(\Pi)$ is a doubly-even self-orthogonal code. If $n \equiv 0 \pmod 4$ then $C_2(\Pi)$ is a singly-even code. Let $D_2(\Pi)$ be the doubly-even subcode of $C_2(\Pi)$. We have that $C_2(\Pi) = \langle D_2(\Pi), v_L \rangle$ where L is a line of Π.

Theorem 5.3. *Let Π be a biplane of even order n then $d_k(C_2(\Pi)) \leq d_k(D_2(\Pi))$ and $d_2(C_2(\Pi)) > \frac{3}{2}d_1(C_2(\Pi))$ where the inequality is strict for $n \equiv 2 \pmod 4$. If $n \equiv 0 \pmod 4$ then $d_2(D_2(\Pi)) > \frac{3}{2}d_1(D_2(\Pi))$.*

Proof. The proof of the first statement is similar to the proof of Theorem 3.2. The remainder of the statements follow from Theorem 2.2 in [2]. □

Theorem 5.4. *Let* Π *be a biplane of even order. If* $W^2_{C_2(\Pi)}(y) = \sum A_i y^i$ *then* $A_i = 0$ *if* i *is odd.*

Proof. Follows from Lemma 4.4. □

There are 4 known biplanes of even order, namely the biplane of order 2 and the three biplanes of order 4. We can use the previous results to obtain all their higher weight enumerators.

For the biplane of order 2 the code $C_2(\Pi)$ is a $[7, 3, 4]$ code.The code $C_2(\Pi)^\perp$ is the $[7, 4, 3]$ Hamming code. Its weight enumerators are given in Table 4.

TABLE 4. Higher Weight Enumerator of the Biplane of Order 2

Weight	3	4	5	6	7
$W^1_{C_2(\Pi)}$	0	7	0	0	0
$W^2_{C_2(\Pi)}$	0	0	0	7	0
$W^3_{C_2(\Pi)}$	0	0	0	0	1

For the biplane B6A of order 4 the code $C_2(\Pi_4)$ is a $[16, 6, 6]$ code and the code $C_2(\Pi)^\perp$ is a $[16, 10, 4]$ code. Its weight enumerators are given in Table 5.

TABLE 5. Higher Weight Enumerator of the Biplane B6A of Order 4

Weight	6	7	8	9	10	11	12	13	14	15	16
$W^1_{C_2(\Pi)}$	16	0	30	0	16	0	0	0	0	0	1
$W^2_{C_2(\Pi)}$	0	0	0	0	120	0	380	0	120	0	31
$W^3_{C_2(\Pi)}$	0	0	0	0	0	0	60	320	480	320	215
$W^4_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	120	256	275
$W^5_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	0	16	47
$W^6_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	0	0	1

For the biplane B6B of order 4 the code $C_2(\Pi_4)$ is a $[16, 7, 4]$ code and the code $C_2(\Pi)^\perp$ is a $[16, 10, 4]$ code. Its weight enumerators are given in Table 6.

TABLE 6. Higher Weight Enumerator of the Biplane B6B of Order 4

Weight	4	5	6	7	8	9	10	11	12	13	14	15	16
$W^1_{C_2(\Pi)}$	4	0	32	0	54	0	32	0	4	0	0	0	1
$W^2_{C_2(\Pi)}$	0	0	0	0	54	0	560	0	1344	0	624	0	85
$W^3_{C_2(\Pi)}$	0	0	0	0	0	0	24	192	892	2432	3672	3008	1591
$W^4_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	28	448	2208	4544	4583
$W^5_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	0	0	120	768	1779
$W^6_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	0	0	0	16	111
$W^7_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE 7. Higher Weight Enumerator of the Biplane B6C of Order 4

Weight	4	5	6	7	8	9	10	11	12	13	14	15	16
$W^1_{C_2(\Pi)}$	12	0	64	0	102	0	64	0	12	0	0	0	1
$W^2_{C_2(\Pi)}$	0	0	8	0	330	0	2352	0	5080	0	2760	0	265
$W^3_{C_2(\Pi)}$	0	0	0	0	2	64	448	2240	8680	18368	29696	25408	12249
$W^4_{C_2(\Pi)}$	0	0	0	0	0	0	8	96	1580	9920	37800	76768	74615
$W^5_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	12	512	5952	29184	61495
$W^6_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	0	0	120	1792	8883
$W^7_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	0	0	0	16	239
$W^8_{C_2(\Pi)}$	0	0	0	0	0	0	0	0	0	0	0	0	1

For the biplane B6C of order 4 the code $C_2(\Pi_4)$ is a $[16, 8, 4]$ code. This code is a Type I, self-dual code. Its weight enumerators are given in Table 7.

We shall show that some of the interesting aspects of these codes are true in general.

Lemma 5.5. *Let Π be a biplane of order n. The minimum weight of $C_2(\Pi)^\perp$ is at least $\frac{n}{2} + 2$.*

Proof. If $w \in C_2(\Pi)^\perp$ then $[w, v_L] = 0$ for all lines L in Π. Hence no line meets the support of w only once.

Assume $|\mathrm{Supp}(w)| < n$. Let $q_1, q_2, \ldots, q_k$ be the points in $\mathrm{Supp}(w)$. There are $n + 2$ lines through q_1. Each of these lines must hit at least one other q_i otherwise it would meet the support only once. Through q_1 and q_i there are exactly two lines, at most $2(k-1)$ of the lines through q_1 can intersect w evenly many times. This gives that k must be at least $\frac{n}{2} + 2$. $\square$

We say that a set of points in a biplane is a k-biarc if no 3 points are collinear.

Proposition 5.6. *In a biplane of order n, if a k-biarc exists then $k \leq \frac{n}{2}+2$, with equality only possible when n is even.*

Proof. Allowing $q_1, q_2, \ldots, q_k$ to be the points in the k-biarc and applying the same reasoning as in the proof of Lemma 5.5, we see that $k \leq \frac{n}{2}+2$. For equality to occur we need n to be divisible by 2. □

In this case with equality, we shall call the $(\frac{n}{2}+2)$ points a bihyperoval. It is clear that the weight $(\frac{n}{2}+2)$ vectors in $C_2(\Pi)^{\perp}$ are bihyperovals. These results give the following.

Theorem 5.7. *Let Π be a biplane of even order n then the minimum weight of $C_2(\Pi)^{\perp}$ is $\frac{n}{2}+2$ if and only if there exist bihyperovals.*

Notice that the biplane of order 2 and all the biplanes of order 4 have bihyperovals.

6. Ternary Codes

In this section we shall investigate the codes of planes and biplanes where 3 divides their order. If Π is a plane or a biplane with 3 dividing n then all weights in $\mathrm{Hull}_3(\Pi)$ are congruent to $0 \pmod 3$. This follows from the fact that the code $\mathrm{Hull}_3(\Pi)$ is a self-orthogonal code and all self-orthogonal vectors over $\mathbb{F}_3$ have weight congruent to $0 \pmod 3$.

We know for odd prime p, $\langle \mathrm{Hull}_3(\Pi), \mathbf{1}\rangle = C_p(\Pi)$ with $\mathbf{1} \notin \mathrm{Hull}_p(\Pi)$ for biplanes, and $\langle \mathrm{Hull}_3(\Pi), \mathbf{1}\rangle = C_p(\Pi)$ with $\mathbf{1} \notin \mathrm{Hull}_p(\Pi)$ for all projective planes, see[1]. Then for $p = 3$, for $v \in \mathrm{Hull}_p(\Pi)$, $\alpha \neq 0$, $[v+\alpha\mathbf{1}, v+\alpha\mathbf{1}] = \alpha^2[\mathbf{1},\mathbf{1}] = [\mathbf{1},\mathbf{1}]$.

Hence for a projective plane the weights in the code are either 1 or 0 $\pmod 3$ and for a biplane the weights in the code are either 2 or 0 $\pmod 3$. In both cases the vectors that have weight 0 $\pmod 3$ are precisely those vectors that are in $\mathrm{Hull}_3(\Pi)$.

This gives the following.

Theorem 6.1. *For a projective plane the weights in the code are either* 1 *or* 0 $\pmod 3$ *and for a biplane the weights in the code are either* 2 *or* 0 $\pmod 3$*. Set* $W^1(C_3(\Pi); y) = \sum A_i y^i$ *and* $W^1(\mathrm{Hull}_3(\Pi); y) = \sum B_i y^i$*. If* $i \equiv 0 \pmod 3$ *then* $A_i = B_i$.

Using the previous results we are able to compute all the higher weight enumerators of the biplane of order 3.

TABLE 8. Higher Weight Enumerator of the Biplane of Order 3

Weight	5	6	7	8	9	10	11
$W^1_{C_3(\Pi)}$	66	66	0	165	55	0	12
$W^1_{H_3(\Pi)}$	0	66	0	0	55	0	0
$W^2_{C_3(\Pi)}$	0	0	330	825	2695	4125	3036
$W^2_{H_3(\Pi)}$	0	0	0	165	220	495	330
$W^3_{C_3(\Pi)}$	0	0	0	165	1705	9405	22605
$W^3_{H_3(\Pi)}$	0	0	0	0	55	330	825
$W^4_{C_3(\Pi)}$	0	0	0	0	55	1221	9735
$W^4_{H_3(\Pi)}$	0	0	0	0	0	11	110
$W^5_{C_3(\Pi)}$	0	0	0	0	0	11	353
$W^5_{H_3(\Pi)}$	0	0	0	0	0	0	1
$W^6_{C_3(\Pi)}$	0	0	0	0	0	0	1

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SCRANTON
SCRANTON, PA 18510, USA

e-mail address: doughertys1@scranton.edu

Department of Mathematics
University of Illinois at Chicago
Chicago, IL 60607, USA
e-mail address: rramad1@uic.edu

(*Received May 18, 2006*)
(*Revised March 24, 2007*)