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EVALUATION OF THE CONVOLUTION SUMS $\sum_{l+24m=n} \sigma(l)\sigma(m)$ AND $\sum_{3l+8m=n} \sigma(l)\sigma(m)$

Ayşe ALACA, Şaban ALACA and Kenneth S. WILLIAMS

Abstract. The convolution sums $\sum_{l+24m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+8m=n} \sigma(l)\sigma(m)$ are evaluated for all $n \in \mathbb{N}$, and their evaluations used to determine the number of representations of a positive integer n by the form $x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+8(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)$.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers, complex numbers respectively. For $k, n \in \mathbb{N}$ we set

$$\sigma_k(n) = \sum_{d|n} d^k, \tag{1.1}$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. For $a, b \in \mathbb{N}$ with $a \leq b$ we define the convolution sum $W_{a,b}(n)$ by

$$W_{a,b}(n) := \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ al+bm=n}} \sigma(l)\sigma(m). \tag{1.2}$$

Set $g = \gcd(a, b)$. Clearly

$$W_{a,b}(n) = \begin{cases} W_{a/g,b/g}(n/g), & \text{if } g \mid n, \\ 0, & \text{if } g \nmid n. \end{cases} \tag{1.3}$$

Hence we may suppose that $\gcd(a, b) = 1$. When $a = 1$ and $b = k \in \mathbb{N}$ we have

$$W_{1,k}(n) = \sum_{\substack{m \in \mathbb{N} \\ m < n/k}} \sigma(m)\sigma(n-km) \tag{1.4}$$

and we write $W_k(n)$ for $W_{1,k}(n)$. The sum $W_k(n)$ has been evaluated for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 12$ and 16, see [1] for references. The sum $W_{2,3}(n)$ was evaluated in [4] and the sum $W_{3,4}(n)$ in [1]. In this paper we determine

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$W_{24}(n)$ and $W_{3,8}(n)$. These determinations are given in Theorem 2.1 in Section 2. The proof of Theorem 2.1 is given in Section 3. Some related convolution sums are evaluated in [5], [6].

For $k, n \in \mathbb{N}$ we let

$$(1.5)\quad N_k(n) := \mathrm{card}\{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) \in \mathbb{Z}^8 | n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + k(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)\}.$$

The values of $N_1(n)$, $N_2(n)$, $N_3(n)$, $N_4(n)$ and $N_6(n)$ are known, see [10], [4], [13], [1] and [3], respectively. An elementary evaluation of $N_1(n)$ is given in [9]. In Section 4 we use the evaluations of $W_8(n)$ (see [14]), $W_{24}(n)$ and $W_{3,8}(n)$ to determine $N_8(n)$, see Theorem 2.2 in Section 2.

2. Statements of Theorems 2.1 and 2.2

We begin by defining the quantities $c_{1,24}(n)$ and $c_{3,8}(n)$ ($n \in \mathbb{N}$), which will be central to everything that we do.

Definition 2.1. For $n \in \mathbb{N}$ we define $c_{1,24}(n)$ by

$$\begin{aligned}
&61 \sum_{n=1}^{\infty} c_{1,24}(n) q^n \\
&= 34q \prod_{n=1}^{\infty} (1+q^n)(1-q^{2n})(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})(1-q^{12n-6}) \\
(2.1)\quad &+30q \prod_{n=1}^{\infty} (1+q^n)^3(1-q^{2n})^2(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^3(1-q^{12n-6})^2 \\
&-3q \prod_{n=1}^{\infty} (1-q^{2n-1})^2(1+q^{3n})^6(1-q^{4n})^2(1-q^{6n})^6(1-q^{12n-6})^6 \\
&+4q^2 \prod_{n=1}^{\infty} (1+q^n)(1+q^{2n})^2(1-q^{3n})^3(1-q^{4n})^4(1+q^{6n})(1-q^{12n}) \\
&-2q^2 \prod_{n=1}^{\infty} (1+q^n)^2(1-q^{2n})^3(1+q^{3n})^2(1-q^{4n})(1-q^{6n})^3(1-q^{12n})
\end{aligned}$$

and $c_{3,8}(n)$ by

$$\sum_{n=1}^{\infty} c_{3,8}(n) q^n$$

$$
\begin{aligned}
&= q \prod_{n=1}^{\infty} (1-q^{2n-1})^2 (1+q^{3n})^6 (1-q^{4n})^2 (1-q^{6n})^6 (1-q^{12n-6})^6 \\
(2.2)\quad &+2q^2 \prod_{n=1}^{\infty} (1-q^n)^2 (1+q^{2n})^5 (1+q^{3n})^6 (1-q^{6n})^6 (1-q^{12n-6})^3 \\
&+42q^2 \prod_{n=1}^{\infty} (1-q^n)(1-q^{2n})(1+q^{3n})^3 (1-q^{6n})^6 \\
&-30q^2 \prod_{n=1}^{\infty} (1-q^n)(1-q^{2n})^3 (1+q^{3n})^3 (1-q^{4n-2})^2 (1-q^{6n})^2 (1-q^{12n})^2 \\
&+4q^3 \prod_{n=1}^{\infty} (1+q^n)(1-q^{4n})^2 (1-q^{6n-3})^3 (1-q^{12n})^6 \\
&-52q^3 \prod_{n=1}^{\infty} (1-q^{2n})^2 (1-q^{4n-2})^2 (1-q^{12n})^6 .
\end{aligned}
$$

Clearly the coefficients of q^n $(n \in \mathbb{N})$ on the right hand sides in Definition 2.1 are integers. Hence

$$(2.3) \qquad 61c_{1,24}(n) \in \mathbb{Z}, \ c_{3,8}(n) \in \mathbb{Z} \ (n \in \mathbb{N}).$$

The first thirty values of $61c_{1,24}(n)$ and $c_{3,8}(n)$ are given in the following table.

n	$61c_{1,24}(n)$	$c_{3,8}(n)$	n	$61c_{1,24}(n)$	$c_{3,8}(n)$
1	61	1	16	448	448
2	132	12	17	234	−2766
3	117	−63	18	1188	108
4	112	112	19	860	1100
5	6	126	20	672	672
6	−36	−396	21	1848	3288
7	−136	344	22	3024	−1296
8	−224	−224	23	3048	648
9	−291	−831	24	672	672
10	−648	1512	25	811	−7289
11	−348	−588	26	2136	3336
12	−336	−336	27	1173	−447
13	−322	2198	28	−1792	−1792
14	−672	−1632	29	−2130	9030
15	−618	−258	30	−4536	1944

We note that $c_{1,24}(1) = c_{3,8}(1) = 1$. The table suggests that $61c_{1,24}(4n) = c_{3,8}(4n)$ for all $n \in \mathbb{N}$, and we prove this at the end of Section 4, see (4.11).

In Section 3 we prove the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+24m=n}} \sigma(l)\sigma(m)
&= \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{640}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{160}\sigma_3\left(\frac{n}{4}\right)\\
&\quad + \frac{9}{640}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{8}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{12}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{24}\right)\\
&\quad + \left(\frac{1}{24} - \frac{n}{96}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{24}\right) - \frac{61}{1920}c_{1,24}(n)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{(l,m)\in\mathbb{N}^2\\ 3l+8m=n}} \sigma(l)\sigma(m)
&= \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{640}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{160}\sigma_3\left(\frac{n}{4}\right)\\
&\quad + \frac{9}{640}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{8}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{12}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{24}\right)\\
&\quad + \left(\frac{1}{24} - \frac{n}{32}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{8}\right) - \frac{1}{1920}c_{3,8}(n),
\end{aligned}$$

where $c_{1,24}(n)$ and $c_{3,8}(n)$ are defined in (2.1) *and* (2.2) *respectively.*

Making use of Theorem 2.1, we prove the following result in Section 4.

Theorem 2.2. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
N_8(n) &= \frac{3}{10}\sigma_3(n) + \frac{9}{10}\sigma_3\left(\frac{n}{2}\right) + \frac{27}{10}\sigma_3\left(\frac{n}{3}\right) + \frac{18}{5}\sigma_3\left(\frac{n}{4}\right)\\
&\quad + \frac{81}{10}\sigma_3\left(\frac{n}{6}\right) + \frac{96}{5}\sigma_3\left(\frac{n}{8}\right) + \frac{162}{5}\sigma_3\left(\frac{n}{12}\right) + \frac{864}{5}\sigma_3\left(\frac{n}{24}\right)\\
&\quad - \frac{9}{4}c_{1,8}(n) - \frac{81}{4}c_{1,8}\left(\frac{n}{3}\right) + \frac{549}{40}c_{1,24}(n) + \frac{9}{40}c_{3,8}(n),
\end{aligned}$$

where $c_{1,8}(n)$ *is given by*

$$\sum_{n=1}^{\infty} c_{1,8}(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4. \tag{2.4}$$

3. Proof of Theorem 2.1

The Eisenstein series $L(q)$, $M(q)$ and $N(q)$ are defined by

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \tag{3.1}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \tag{3.2}$$

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \tag{3.3}$$

see for example [11, eqn. (25)], [12, p. 140]. Ramanujan's discriminant function $\Delta(q)$ is given by

$$\Delta(q) := q\prod_{n=1}^{\infty}(1-q^n)^{24} = \frac{1}{1728}\left(M(q)^3 - N(q)^2\right), \tag{3.4}$$

see for example [11, eqn. (44)], [12, p. 144]. The Jacobi theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \ q \in \mathbb{C}, \ |q| < 1. \tag{3.5}$$

Set

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{3.6}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{3.7}$$

It is shown in [4, eqn. (3.84), p. 501] that

$$L(q) - 2L(q^2) = -(1 + 14p + 24p^2 + 14p^3 + p^4)k^2 \tag{3.8}$$

and in [1, eqn. (3.12), p. 34] that

$$L(q) - 12L(q^{12}) = -(11 + 34p + 36p^2 + 16p^3 + 2p^4)k^2. \tag{3.9}$$

The following result is proved in [2, Theorem 9, p. 179].

Duplication principle.

$$p(q^2) = \frac{1+p-p^2-((1-p)(1+p)(1+2p))^{1/2}}{p^2},$$

$$k(q^2) = \frac{(1+p-p^2+((1-p)(1+p)(1+2p))^{1/2})k}{2}.$$

Applying the duplication principle to (3.9), we obtain

$$L(q^2) - 12L(q^{24}) = -\frac{1}{2}\Big(13+26p+12p^2-p^3-\frac{1}{2}p^4\Big)k^2$$

$$-\frac{9}{2}(1+p)((1-p)(1+p)(1+2p))^{1/2}k^2. \tag{3.10}$$

Appealing to (3.8), (3.10) and the trivial identity

$$L(q) - 24L(q^{24}) = (L(q)-2L(q^2)) + 2(L(q^2)-12L(q^{24})),$$

we obtain

$$L(q) - 24L(q^{24}) = -\Big(14+40p+36p^2+13p^3+\frac{1}{2}p^4\Big)k^2$$

$$+9(1+p)((1-p)(1+p)(1+2p))^{1/2}k^2. \tag{3.11}$$

Next applying the duplication principle to (3.8) we obtain

$$L(q^2) - 2L(q^4) = -\Big(1+2p+6p^2+5p^3-\frac{1}{2}p^4\Big)k^2, \tag{3.12}$$

see [1, eqn. (3.40), p. 39]. Then, from (3.8) and (3.12), we obtain

$$L(q) - 4L(q^4) = -(3+18p+36p^2+24p^3)k^2. \tag{3.13}$$

Applying duplication to (3.13), we have

$$L(q^2) - 4L(q^8) = -\Big(\frac{3}{2}+3p+9p^2+\frac{15}{2}p^3-\frac{3}{4}p^4\Big)k^2$$

$$-\Big(\frac{3}{2}+\frac{3}{2}p-3p^2\Big)((1-p)(1+p)(1+2p))^{1/2}k^2. \tag{3.14}$$

Recall from [4, eqn. (3.87), p. 502], [1, eqn. (3.9), p. 33] that

$$L(q) - 3L(q^3) = -(2+16p+36p^2+16p^3+2p^4)k^2. \tag{3.15}$$

Then, using the simple identity

$$3L(q^3) - 8L(q^8) = (L(q)-2L(q^2)) + 2(L(q^2)-4L(q^8)) - (L(q)-3L(q^3)),$$

we obtain

$$3L(q^3) - 8L(q^8) = -\Big(2 + 4p + 6p^2 + 13p^3 - \frac{5}{2}p^4\Big)k^2$$
$$-(3 + 3p - 6p^2)((1-p)(1+p)(1+2p))^{1/2}k^2. \tag{3.16}$$

Squaring (3.11) and (3.16), we deduce the following result.

Lemma 3.1.

(**a**)
$$\begin{aligned}(L(q) - 24L(q^{24}))^2 = \Big(&277 + 1444p + 2932p^2 + 3082p^3 + 1945p^4 \\ &+814p^5 + 205p^6 + 13p^7 + \frac{1}{4}p^8\Big)k^4 \\ &+(252 + 972p + 1368p^2 + 882p^3 \\ &+243p^4 + 9p^5)((1-p)(1+p)(1+2p))^{1/2}k^4.\end{aligned}$$

(**b**)
$$\begin{aligned}(3L(q^3) - 8L(q^8))^2 = \Big(&13 + 52p + 40p^2 - 26p^3 + 85p^4 \\ &+298p^5 + 175p^6 - 137p^7 + \frac{25}{4}p^8\Big)k^4 \\ &+(12 + 36p + 36p^2 + 66p^3 - 9p^4 \\ &-171p^5 + 30p^6)((1-p)(1+p)(1+2p))^{1/2}k^4.\end{aligned}$$

From [1, eqns. (3.14)-(3.19), p. 34] we have

$$\begin{aligned}M(q) &= (1 + 124p + 964p^2 + 2788p^3 + 3910p^4 \\ &\quad +2788p^5 + 964p^6 + 124p^7 + p^8)k^4,\end{aligned} \tag{3.17}$$

$$\begin{aligned}M(q^2) &= (1 + 4p + 64p^2 + 178p^3 + 235p^4 \\ &\quad +178p^5 + 64p^6 + 4p^7 + p^8)k^4,\end{aligned} \tag{3.18}$$

$$\begin{aligned}M(q^3) &= (1 + 4p + 4p^2 + 28p^3 + 70p^4 \\ &\quad +28p^5 + 4p^6 + 4p^7 + p^8)k^4,\end{aligned} \tag{3.19}$$

$$\begin{aligned}M(q^4) &= \Big(1 + 4p + 4p^2 - 2p^3 + 10p^4 + 28p^5 \\ &\quad +\frac{31}{4}p^6 - \frac{29}{4}p^7 + \frac{1}{16}p^8\Big)k^4,\end{aligned} \tag{3.20}$$

$$M(q^6) = (1 + 4p + 4p^2 - 2p^3 - 5p^4 \tag{3.21}$$

$$-2p^5+4p^6+4p^7+p^8)k^4,$$

$$M(q^{12}) = \Big(1+4p+4p^2-2p^3-5p^4-2p^5 + \frac{1}{4}p^6+\frac{1}{4}p^7+\frac{1}{16}p^8\Big)k^4. \tag{3.22}$$

Applying the duplication principle to (3.20), we obtain

$$\begin{aligned} M(q^8) = \Big(&\frac{17}{32}+\frac{17}{8}p+\frac{1}{4}p^2-\frac{107}{16}p^3-\frac{55}{32}p^4 \\ &+\frac{163}{16}p^5+\frac{151}{64}p^6-\frac{269}{64}p^7+\frac{1}{256}p^8\Big)k^4 \\ &+\Big(\frac{15}{32}+\frac{45}{32}p+\frac{45}{16}p^2+\frac{105}{32}p^3-\frac{225}{64}p^4 \\ &-\frac{315}{64}p^5+\frac{15}{32}p^6\Big)((1-p)(1+p)(1+2p))^{1/2}k^4. \end{aligned} \tag{3.23}$$

Applying the duplication principle to (3.22), we obtain

$$\begin{aligned} M(q^{24}) = \Big(&\frac{17}{32}+\frac{17}{8}p+\frac{17}{8}p^2-\frac{17}{16}p^3-\frac{85}{32}p^4 \\ &-\frac{17}{16}p^5+\frac{1}{64}p^6+\frac{1}{64}p^7+\frac{1}{256}p^8\Big)k^4 \\ &+\Big(\frac{15}{32}+\frac{45}{32}p+\frac{15}{16}p^2-\frac{15}{32}p^3-\frac{45}{64}p^4 \\ &-\frac{15}{64}p^5\Big)((1-p)(1+p)(1+2p))^{1/2}k^4. \end{aligned} \tag{3.24}$$

From (3.17)-(3.24) we deduce the following result.

Lemma 3.2.

(a) $$\begin{aligned} &\frac{47}{50}M(q)-\frac{9}{50}M(q^2)-\frac{27}{50}M(q^3)-\frac{18}{25}M(q^4) \\ &-\frac{81}{50}M(q^6)-\frac{96}{25}M(q^8)-\frac{162}{25}M(q^{12})+\frac{13536}{25}M(q^{24}) \\ &= \Big(277+\frac{6104}{5}p+\frac{10034}{5}p^2+\frac{10208}{5}p^3+2197p^4 \\ &+\frac{9776}{5}p^5+\frac{4391}{5}p^6+\frac{677}{5}p^7+\frac{1}{4}p^8\Big)k^4 \end{aligned}$$

$$+\Big(252+756p+\frac{2484}{5}p^2-\frac{1332}{5}p^3-\frac{1836}{5}p^4$$
$$-108p^5-\frac{9}{5}p^6\Big)((1-p)(1+p)(1+2p))^{1/2}k^4.$$

(b)
$$-\frac{3}{50}M(q)-\frac{9}{50}M(q^2)+\frac{423}{50}M(q^3)-\frac{18}{25}M(q^4)$$
$$-\frac{81}{50}M(q^6)+\frac{1504}{25}M(q^8)-\frac{162}{25}M(q^{12})-\frac{864}{25}M(q^{24})$$
$$=\Big(13+\frac{224}{5}p-\frac{646}{5}p^2-\frac{1552}{5}p^3+337p^4$$
$$+\frac{3416}{5}p^5+\frac{461}{5}p^6-\frac{1153}{5}p^7+\frac{25}{4}p^8\Big)k^4$$
$$+\Big(12+36p+\frac{684}{5}p^2+\frac{1068}{5}p^3-\frac{936}{5}p^4$$
$$-288p^5+\frac{141}{5}p^6\Big)((1-p)(1+p)(1+2p))^{1/2}k^4.$$

From Lemmas 3.1 and 3.2 we obtain

Lemma 3.3.

(a)
$$(L(q)-24L(q^{24}))^2$$
$$-\Big(\frac{47}{50}M(q)-\frac{9}{50}M(q^2)-\frac{27}{50}M(q^3)-\frac{18}{25}M(q^4)$$
$$-\frac{81}{50}M(q^6)-\frac{96}{25}M(q^8)-\frac{162}{25}M(q^{12})+\frac{13536}{25}M(q^{24})\Big)$$
$$=\frac{18}{5}p(1-p)(1+p)(1+2p)(2+p)(31+51p+17p^2)k^4$$
$$+\frac{9}{5}p(2+p)^2(30+91p+61p^2+p^3)((1-p)(1+p)(1+2p))^{1/2}k^4.$$

(b)
$$(3L(q^3)-8L(q^8))^2$$
$$-\Big(-\frac{3}{50}M(q)-\frac{9}{50}M(q^2)+\frac{423}{50}M(q^3)-\frac{18}{25}M(q^4)$$
$$-\frac{81}{50}M(q^6)+\frac{1504}{25}M(q^8)-\frac{162}{25}M(q^{12})-\frac{864}{25}M(q^{24})\Big)$$

$$= \frac{18}{5}p(1-p)(1+p)(1+2p)(2+p)(1+21p-13p^2)k^4.$$

$$-\frac{9}{5}p^2(2+p)(28+27p-63p^2-p^3)((1-p)(1+p)(1+2p))^{1/2}k^4.$$

Before proving the next lemma, we recall from [1, eqns. (3.28)-(3.33), p. 36]

$$\Delta(q) = \frac{1}{16}p(1-p)^{12}(1+p)^4(1+2p)^3(2+p)^3k^{12}, \tag{3.25}$$

$$\Delta(q^2) = \frac{1}{256}p^2(1-p)^6(1+p)^2(1+2p)^6(2+p)^6k^{12}, \tag{3.26}$$

$$\Delta(q^3) = \frac{1}{16}p^3(1-p)^4(1+p)^{12}(1+2p)(2+p)k^{12}, \tag{3.27}$$

$$\Delta(q^4) = \frac{1}{65536}p^4(1-p)^3(1+p)(1+2p)^3(2+p)^{12}k^{12}, \tag{3.28}$$

$$\Delta(q^6) = \frac{1}{256}p^6(1-p)^2(1+p)^6(1+2p)^2(2+p)^2k^{12}, \tag{3.29}$$

$$\Delta(q^{12}) = \frac{1}{65536}p^{12}(1-p)(1+p)^3(1+2p)(2+p)^4k^{12}. \tag{3.30}$$

Lemma 3.4.

(a) $\displaystyle\sum_{n=1}^{\infty} c_{1,24}(n)q^n$

$$= \frac{1}{244}p(1-p)(1+p)(1+2p)(2+p)(31+51p+17p^2)k^4$$

$$+\frac{1}{488}p(2+p)^2(30+91p+61p^2+p^3)((1-p)(1+p)(1+2p))^{1/2}k^4.$$

(b) $\displaystyle\sum_{n=1}^{\infty} c_{3,8}(n)q^n$

$$= \frac{1}{4}p(1-p)(1+p)(1+2p)(2+p)(1+21p-13p^2)k^4$$

$$-\frac{1}{8}p^2(2+p)(28+27p-63p^2-p^3)((1-p)(1+p)(1+2p))^{1/2}k^4.$$

Proof. We just prove part (a) as part (b) can be treated similarly.

We have

$$q\prod_{n=1}^{\infty}(1+q^n)(1-q^{2n})(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})(1-q^{12n-6})$$
$$=q\prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^{2n})^2(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})^2(1-q^{12n})^{-1}$$
$$=\Delta(q)^{-1/24}\Delta(q^2)^{1/12}\Delta(q^3)^{1/8}\Delta(q^4)^{1/8}\Delta(q^6)^{1/12}\Delta(q^{12})^{-1/24}$$
$$=2^{-3}p(1-p)(1+p)^2(1+2p)(2+p)^2k^4.$$

Similarly

$$q\prod_{n=1}^{\infty}(1+q^n)^3(1-q^{2n})^2(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^3(1-q^{12n-6})^2$$
$$=q\prod_{n=1}^{\infty}(1-q^n)^{-3}(1-q^{2n})^5(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^5(1-q^{12n})^{-2}$$
$$=\Delta(q)^{-1/8}\Delta(q^2)^{5/24}\Delta(q^3)^{1/24}\Delta(q^4)^{1/12}\Delta(q^6)^{5/24}\Delta(q^{12})^{-1/12}$$
$$=2^{-3}p(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}(2+p)^2k^4,$$

$$q\prod_{n=1}^{\infty}(1-q^{2n-1})^2(1+q^{3n})^6(1-q^{4n})^2(1-q^{6n})^6(1-q^{12n-6})^6$$
$$=q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^{-2}(1-q^{3n})^{-6}(1-q^{4n})^2(1-q^{6n})^{18}(1-q^{12n})^{-6}$$
$$=\Delta(q)^{1/12}\Delta(q^2)^{-1/12}\Delta(q^3)^{-1/4}\Delta(q^4)^{1/12}\Delta(q^6)^{3/4}\Delta(q^{12})^{-1/4}$$
$$=2^{-2}p(1-p)(1+p)(1+2p)(2+p)k^4,$$

$$q^2\prod_{n=1}^{\infty}(1+q^n)(1+q^{2n})^2(1-q^{3n})^3(1-q^{4n})^4(1+q^{6n})(1-q^{12n})$$
$$=q^2\prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^{2n})^{-1}(1-q^{3n})^3(1-q^{4n})^6(1-q^{6n})^{-1}(1-q^{12n})^2$$
$$=\Delta(q)^{-1/24}\Delta(q^2)^{-1/24}\Delta(q^3)^{1/8}\Delta(q^4)^{1/4}\Delta(q^6)^{-1/24}\Delta(q^{12})^{1/12}$$
$$=2^{-5}p^2(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{1/2}(2+p)^3k^4,$$

and

$$q^2 \prod_{n=1}^{\infty} (1+q^n)^2 (1-q^{2n})^3 (1+q^{3n})^2 (1-q^{4n})(1-q^{6n})^3 (1-q^{12n})$$
$$= q^2 \prod_{n=1}^{\infty} (1-q^n)^{-2} (1-q^{2n})^5 (1-q^{3n})^{-2} (1-q^{4n})(1-q^{6n})^5 (1-q^{12n})$$
$$= \Delta(q)^{-1/12} \Delta(q^2)^{5/24} \Delta(q^3)^{-1/12} \Delta(q^4)^{1/24} \Delta(q^6)^{5/24} \Delta(q^{12})^{1/24}$$
$$= 2^{-4} p^2 (1-p)^{1/2} (1+p)^{1/2} (1+2p)^{3/2} (2+p)^2 k^4.$$

Thus, appealing to Definition 2.1, we obtain

$$\begin{aligned}
&61 \sum_{n=1}^{\infty} c_{1,24}(n) q^n \\
&= 34 \cdot 2^{-3} p(1-p)(1+p)^2(1+2p)(2+p)^2 k^4 \\
&\quad +30 \cdot 2^{-3} p(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}(2+p)^2 k^4 \\
&\quad -3 \cdot 2^{-2} p(1-p)(1+p)(1+2p)(2+p) k^4 \\
&\quad +4 \cdot 2^{-5} p^2(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{1/2}(2+p)^3 k^4 \\
&\quad -2 \cdot 2^{-4} p^2(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{3/2}(2+p)^2 k^4 \\
&= \frac{1}{4} p(1-p)(1+p)(1+2p)(2+p)(31+51p+17p^2) k^4 \\
&\quad +\frac{1}{8} p(2+p)^2(30+91p+61p^2+p^3)((1-p)(1+p)(1+2p))^{1/2} k^4.
\end{aligned}$$

Dividing both sides by 61, we obtain the assertion of part (a). □

From Lemmas 3.3 and 3.4 we obtain

Lemma 3.5.

(**a**) $(L(q) - 24L(q^{24}))^2$

$$\begin{aligned}
&= 529 + \frac{1}{5} \sum_{n=1}^{\infty} (1128\sigma_3(n) - 216\sigma_3\left(\frac{n}{2}\right) - 648\sigma_3\left(\frac{n}{3}\right) \\
&\quad -864\sigma_3\left(\frac{n}{4}\right) - 1944\sigma_3\left(\frac{n}{6}\right) - 4608\sigma_3\left(\frac{n}{8}\right) \\
&\quad -7776\sigma_3\left(\frac{n}{12}\right) + 649728\sigma_3\left(\frac{n}{24}\right) + 4392 c_{1,24}(n)) q^n.
\end{aligned}$$

(**b**) $(3L(q^3) - 8L(q^8))^2$

$$= 25 + \frac{1}{5}\sum_{n=1}^{\infty}(-72\sigma_3(n) - 216\sigma_3\left(\frac{n}{2}\right) + 10152\sigma_3\left(\frac{n}{3}\right)$$
$$-864\sigma_3\left(\frac{n}{4}\right) - 1944\sigma_3\left(\frac{n}{6}\right) + 72192\sigma_3\left(\frac{n}{8}\right)$$
$$-7776\sigma_3\left(\frac{n}{12}\right) - 41472\sigma_3\left(\frac{n}{24}\right) + 72c_{3,8}(n))q^n.$$

Proof. We just prove part (b) as part (a) can be proved similarly. By Lemma 3.3(b) and Lemma 3.4(b) we have

$$\left(3L(q^3) - 8L(q^8)\right)^2$$
$$= -\frac{3}{50}M(q) - \frac{9}{50}M(q^2) + \frac{423}{50}M(q^3) - \frac{18}{25}M(q^4)$$
$$-\frac{81}{50}M(q^6) + \frac{1504}{25}M(q^8) - \frac{162}{25}M(q^{12}) - \frac{864}{25}M(q^{24})$$
$$+\frac{72}{5}\sum_{n=1}^{\infty}c_{3,8}(n)q^n.$$

Then, appealing to (3.2), we obtain the asserted formula of part (b). □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We just prove the first identity as the second identity can be treated similarly.

We begin by recalling the classical identity

$$L(q)^2 = 1 + \sum_{n=1}^{\infty}(240\sigma_3(n) - 288n\sigma(n))q^n, \tag{3.31}$$

see for example [7], [8]. Mapping $q \to q^{24}$ in (3.31), we obtain

$$L(q^{24})^2 = 1 + \sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{24}\right) - 12n\sigma\left(\frac{n}{24}\right)\right)q^n. \tag{3.32}$$

Also

$$\begin{aligned} L(q)L(q^{24}) &= \left(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n\right)\left(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^{24n}\right) \\ &= 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n - 24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{24}\right)q^n \\ &\quad +576\sum_{n=1}^{\infty}W_{24}(n)q^n. \end{aligned} \tag{3.33}$$

Thus, from (3.31)–(3.33), we have

$$\begin{aligned}
&\left(L(q)-24L(q^{24})\right)^2 \\
&= 529+\sum_{n=1}^{\infty}\Big(240\sigma_3(n)+138240\sigma_3\left(\frac{n}{24}\right) \\
&\quad +1152\Big(1-\frac{n}{4}\Big)\sigma(n)+1152(1-6n)\sigma\left(\frac{n}{24}\right) \\
&\quad -27648W_{24}(n)\Big)q^n.
\end{aligned} \tag{3.34}$$

Equating the coefficients of q^n ($n \in \mathbb{N}$) on the right hand sides of Lemma 3.5(a) and (3.34), we obtain

$$\begin{aligned}
&\frac{1}{5}\Big(1128\sigma_3(n)-216\sigma_3\left(\frac{n}{2}\right)-648\sigma_3\left(\frac{n}{3}\right)-864\sigma_3\left(\frac{n}{4}\right) \\
&\quad -1944\sigma_3\left(\frac{n}{6}\right)-4608\sigma_3\left(\frac{n}{8}\right)-7776\sigma_3\left(\frac{n}{12}\right) \\
&\quad +649728\sigma_3\left(\frac{n}{24}\right)+4392c_{1,24}(n)\Big) \\
&= 240\sigma_3(n)+138240\sigma_3\left(\frac{n}{24}\right) \\
&\quad +1152\Big(1-\frac{n}{4}\Big)\sigma(n)+1152(1-6n)\sigma\left(\frac{n}{24}\right) \\
&\quad -27648W_{24}(n).
\end{aligned} \tag{3.35}$$

Solving (3.35) for $W_{24}(n)$ we obtain the first identity.

4. Proof of Theorem 2.2

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $l \in \mathbb{N}_0$ we set

$$r(l) = \mathrm{card}\{(x_1,x_2,x_3,x_4) \in \mathbb{Z}^4 \mid l = x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2\} \tag{4.1}$$

so that $r(0)=1$. It is known that [9, Theorem 13, p. 266], [10, p. 12]

$$r(l) = 12\sigma(l) - 36\sigma\left(\frac{l}{3}\right),\ l \in \mathbb{N}. \tag{4.2}$$

By (1.5) and (4.1) we have

$$\begin{aligned}
N_8(n) &= \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ l+8m=n}} r(l)r(m) \\
&= r(n)r(0)+r(0)r\left(\frac{n}{8}\right)+\sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+8m=n}} r(l)r(m).
\end{aligned}$$

Thus

$$\begin{aligned} N_8(n) &= \left(12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 12\sigma\left(\frac{n}{8}\right) - 36\sigma\left(\frac{n}{24}\right)\right) \\ &\quad +144 \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n}} \sigma(l)\sigma(m) - 432 \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) \qquad (4.3) \\ &\quad -432 \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) + 1296 \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n}} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right). \end{aligned}$$

By a result of Williams [14, Theorem 1] the first sum is

$$\begin{aligned} \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n}} \sigma(l)\sigma(m) &= \sum_{m<n/8} \sigma(m)\sigma(n-8m) \\ &= \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) \qquad (4.4) \\ &\quad + \left(\frac{1}{24} - \frac{n}{32}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{8}\right) - \frac{1}{64}c_{1,8}(n), \end{aligned}$$

where $c_{1,8}(n)$ is defined in (2.4).

By Theorem 2.1 the second sum is

$$\begin{aligned} \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) &= \sum_{\substack{(l,m)\in\mathbb{N}^2\\ 3l+8m=n}} \sigma(l)\sigma(m) \\ &= \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{640}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{160}\sigma_3\left(\frac{n}{4}\right) \qquad (4.5) \\ &\quad + \frac{9}{640}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{8}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{12}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{24}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{32}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{8}\right) - \frac{1}{1920}c_{3,8}(n). \end{aligned}$$

By Theorem 2.1 the third sum is

$$\begin{aligned} \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) &= \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+24m=n}} \sigma(l)\sigma(m) \\ &= \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{640}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{160}\sigma_3\left(\frac{n}{4}\right) \qquad (4.6) \end{aligned}$$

$$+\frac{9}{640}\sigma_3\left(\frac{n}{6}\right)+\frac{1}{30}\sigma_3\left(\frac{n}{8}\right)+\frac{9}{160}\sigma_3\left(\frac{n}{12}\right)+\frac{3}{10}\sigma_3\left(\frac{n}{24}\right)$$
$$+\left(\frac{1}{24}-\frac{n}{96}\right)\sigma(n)+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma\left(\frac{n}{24}\right)-\frac{61}{1920}c_{1,24}(n).$$

From (4.4) with n replaced by $n/3$, the fourth sum is

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n}}\sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right)=\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+8m=n/3}}\sigma(l)\sigma(m)$$
$$(4.7)\qquad =\frac{1}{192}\sigma_3\left(\frac{n}{3}\right)+\frac{1}{64}\sigma_3\left(\frac{n}{6}\right)+\frac{1}{16}\sigma_3\left(\frac{n}{12}\right)+\frac{1}{3}\sigma_3\left(\frac{n}{24}\right)$$
$$+\left(\frac{1}{24}-\frac{n}{96}\right)\sigma\left(\frac{n}{3}\right)+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{24}\right)-\frac{1}{64}c_{1,8}\left(\frac{n}{3}\right).$$

Using the evaluations (4.4), (4.5), (4.6) and (4.7) in the formula (4.3) for $N_8(n)$, we obtain after some simplification the assertion of Theorem 2.2. □

Using the simple identity

$$\sigma(3n)=4\sigma(n)-3\sigma\left(\frac{n}{3}\right),\ n\in\mathbb{N},$$

it is easy to show that

$$W_{24}(3n)+3W_{3,8}(n)=4W_8(n)$$

and

$$3W_{24}(n)+W_{3,8}(3n)=4W_8(n).$$

Appealing to Theorem 2.1 and (4.4), we obtain after some calculation

$$c_{1,8}(n)=\frac{61}{120}c_{1,24}(3n)+\frac{1}{40}c_{3,8}(n)$$
$$(4.8)\qquad =\frac{61}{40}c_{1,24}(n)+\frac{1}{120}c_{3,8}(3n),\ n\in\mathbb{N}.$$

Next using the easily proved identity

$$\sigma(4n)=7\sigma(n)-6\sigma\left(\frac{n}{2}\right),\ n\in\mathbb{N},$$

we can show that

$$W_{24}(4n)=7W_6(n)-6W_3\left(\frac{n}{2}\right)$$

and

$$W_{3,8}(4n)=7W_{2,3}(n)-6W_3\left(\frac{n}{2}\right).$$

Appealing to [4, Theorem 1, p. 493] for the evaluations of $W_6(n)$ and $W_{2,3}(n)$, and to [9, Theorem 3, p. 248] for the evaluation of $W_3(n)$, we obtain after some calculation

$$c_{1,6}(n) = \frac{61}{112} c_{1,24}(4n) = \frac{1}{112} c_{3,8}(4n), \tag{4.9}$$

where

$$\sum_{n=1}^{\infty} c_{1,6}(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{2n})^2 (1-q^{3n})^2 (1-q^{6n})^2. \tag{4.10}$$

This establishes that

$$61 c_{1,24}(n) = c_{3,8}(n), \text{ if } n \equiv 0 \pmod 4, \tag{4.11}$$

which was asserted in Section 2.

It was proved in [4, p. 509] that

$$c_{1,6}(2^\alpha \cdot 3^\beta) = (-1)^{\alpha+\beta} \cdot 2^\alpha \cdot 3^\beta, \ \alpha, \beta \in \mathbb{N}_0. \tag{4.12}$$

From (4.9) and (4.12) we deduce

$$c_{1,24}(2^\alpha \cdot 3^\beta) = (-1)^{\alpha+\beta} \cdot 2^{\alpha+2} \cdot 3^\beta \cdot 7/61, \ \alpha \geq 2, \beta \geq 0 \tag{4.13}$$

and

$$c_{3,8}(2^\alpha \cdot 3^\beta) = (-1)^{\alpha+\beta} \cdot 2^{\alpha+2} \cdot 3^\beta \cdot 7, \ \alpha \geq 2, \beta \geq 0. \tag{4.14}$$

From (2.4) we deduce that

$$c_{1,8}(n) = 0, \ n \equiv 0 \pmod 2). \tag{4.15}$$

Appealing to (4.8) and (4.15) with $n = 2 \cdot 3^\beta$ ($\beta \in \mathbb{N}_0$) we obtain

$$c_{1,24}(2 \cdot 3^{\beta+1}) = -\frac{3}{61} c_{3,8}(2 \cdot 3^\beta) \tag{4.16}$$

and

$$c_{1,24}(2 \cdot 3^\beta) = -\frac{1}{183} c_{3,8}(2 \cdot 3^{\beta+1}). \tag{4.17}$$

As $c_{1,24}(2) = \frac{132}{61}$, $c_{3,8}(2) = 12$, $c_{1,24}(6) = -\frac{36}{61}$ and $c_{3,8}(6) = -396$, we deduce from (4.16) and (4.17) that for $\beta \in \mathbb{N}_0$

$$c_{1,24}(2 \cdot 3^{2\beta}) = 2^2 \cdot 3^{2\beta+1} \cdot 11/61, \tag{4.18}$$

$$c_{1,24}(2 \cdot 3^{2\beta+1}) = -2^2 \cdot 3^{2\beta+2}/61, \tag{4.19}$$

$$c_{3,8}(2 \cdot 3^{2\beta}) = 2^2 \cdot 3^{2\beta+1}, \tag{4.20}$$

$$c_{3,8}(2 \cdot 3^{2\beta+1}) = -2^2 \cdot 3^{2\beta+2} \cdot 11. \tag{4.21}$$

There does not appear to be a simple formula for $c_{1,24}(3^\beta)$ or for $c_{3,8}(3^\beta)$ ($\beta \in \mathbb{N}_0$).

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Ayşe Alaca
Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics, Carleton University
Ottawa, Ontario, Canada K1S 5B6

e-mail address: aalaca@connect.carleton.ca

Şaban Alaca
Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics, Carleton University
Ottawa, Ontario, Canada K1S 5B6
e-mail address: salaca@connect.carleton.ca

Kenneth S. Williams
Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics, Carleton University
Ottawa, Ontario, Canada K1S 5B6
e-mail address: kwilliam@connect.carleton.ca

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