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CUT LOCI AND DISTANCE FUNCTIONS

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1. Introduction

Let (M, g) be a compact Riemannian manifold and $d(p, q)$ the distance between $p, q \in M$ induced from the metric g. Then the distance function $f := d_p, d_p(x) := d(p, x)$, to a point $p \in M$ plays a fundamental role in Riemannian geometry. Recall that d_p is directionally differentiable at any $q \neq p$, namely for any unit tangent vector $\xi \in U_qM$ we have the first variation formula

$$f'_q(\xi) = -\cos\alpha, \tag{1.1}$$

where α denotes the infimum of angles between ξ and the initial directions of minimal geodesics from q to p.

The behavior of the distance function d_p is closely related to the structure of the cut locus of p. Recall that the cut locus $C(p)$ of $p \in M$ is given as follows: for any unit speed geodesic γ_u emanating from p with the initial direction $u = \dot\gamma_u(0) \in U_pM$, there exists the last parameter value $i_p(u)$ up to which γ_u is a minimal geodesic segment, namely $\gamma_u|[0, t]$ realizes the distance $d(p, \gamma_u(t))$ for $0 < t \leq i_p(u)$. We call $\gamma_u(i_p(u))$ the cut point of p and $i_p(u)$ the cut distance to p along γ_u. Then the cut locus $C(p)$ of p is defined as

$$C(p) := \{\gamma_u(i_p(u)) \mid u \in U_pM\}. \tag{1.2}$$

Recall that q is a cut point of p along γ_u if and only if either there exists another minimal geodesic $\gamma_v, v \in U_pM,\ v \neq u$, emanating from p with $q = \gamma_u(i_p(u)) = \gamma_v(i_p(u))$, or q is a (first) conjugate point to p along γ_u, which means that there exists a nontrivial Jacobi field $Y(t)$ along γ_u with $Y(0) = Y(i_p(u)) = 0$ (see e.g., [16]). The set $\tilde C(p) := \{i_p(u)u \mid u \in U_pM\}$ is called the tangent cut locus of p.

Then if $q(\neq p)$ does not belong to the cut locus $C(p)$, d_p is differentiable at q and its gradient vector $\nabla d_p(q)$ is given by $\dot\gamma(l)$, where γ is a unique minimal

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geodesic parameterized by arclength joining p to q and we set $l = d(p,q)$. Note that $\|\nabla d_p(q)\| = 1$. On the other hand, if $q \in C(p)$ then d_p is in general not differentiable at q.

Now, $q(\neq p)$ is said to be noncritical for $f = d_p$, if there exists a $\xi \in U_qM$ such that $f'_q(\xi) > 0$. Namely, $q(\neq p)$ is a critical point of d_p if for any $\xi \in U_qM$ there exists a minimal geodesic γ from q to p whose initial direction makes an angle $\alpha \leq \pi/2$ with ξ. Note that a critical point $q(\neq p)$ is a cut point of p, where d_p is not differentiable. We consider p itself a critical point of f, since it is a unique minimum point. If f assumes a local maximum at q, then q is a critical point in the above sense. The notion of critical points was first considered by K. Grove and K. Shiohama, and then by M. Gromov ([9], [6]). If q is noncritical, then constructing a gradient-like vector field for $-f$ we may put a neighborhood of q nearer to p by an isotopy of M(isotopy lemma). Therefore, we have an analogy of Morse theory for the case without critical points, and this idea has played an essential roll in problems on curvature and topology of Riemannian manifolds ([9], [6], [7]).

Now since distance function is the most fundamental function on Riemannian manifold, we ask the behavior of the levels of d_p when it passes a critical value. Namely, we ask how to define the notion of index of d_p at a critical point q and how we can get a normal form of d_p around q under some nondegeneracy condition. As mentioned above, d_p is not differentiable at its critical points, and the structure of the cut locus $C(p)$ of p is related to the behavior of d_p around critical points.

In this note we are concerned with the above problem under the assumption that $C(p)$ has rather nice structure, and we discuss an analogy of Morse theory for distance function. We give an application (Corollary 3.12), and hope to give further applications with the present approach ([12] is our motivation for the present work). Indeed, V. Gerschkovich and H. Rubinstein have studied Morse theory for generic distance functions from the view point of min-type functions ([3],[4],[5]), and got results closely related to what we will discuss in the following. Especially, they studied the surface case in detail. Here we try to take a more geometric and direct approach.

Assume that there is a point $p \in M$ such that the tangent cut locus $\tilde{C}(p)$ of p is disjoint from the first tangent conjugate locus, namely any minimal geodesic segment emanating from p is conjugate point-free. Then we say that (M, g) satisfies the condition (C) at $p \in M$. We are interested in the structure of the cut locus $C(p)$ of p under the above condition (C). Especially, if (M, g) admits no conjugate points along all geodesics emanating from p, then the structure of $C(p)$ may be expressed in terms of the Dirichlet domain of the universal covering space $\tilde{M}$ of M with the induced Riemannian metric $\tilde{g}$ (see [15] for these assertions). If (M, g) is nonpositively curved, then for any point $p \in M$ there appear no conjugate points to p along any geodesic emanating from p. On the other hand, A. Weinstein showed that for any compact manifold M with $\dim M \geq 2$ except for S^2 there exists a Riemannian metric such that there is a point $p \in M$ with the cut locus $C(p)$ disjoint from the first conjugate locus ([18]). In this case M satisfies the condition (C) at p.

In §2 we introduce the notion of nondegenerate cut points under the condition (C), and show that the cut locus $C(p)$ admits a nice Whitney stratification if all cut points are nondegenerate. As an application, using this peculiar stratification we give a description of the structure of the cut locus $C(p)$ in a neighborhood of any cut point $q \in C(p)$ in terms of the cone over the cut locus of finitely many unit vectors in general position in the unit sphere S^{n-1} in T_qM (Theorem 2.5). This is also useful to give a normal form of d_p around a critical point under some nondegeneracy condition in §3. We also show that critical points of d_p in the angle sense are critical points of the smooth function, that is the restriction of the distance function d_p to strata containing the critical points, in the usual sense.

In §3, under the above assumption, first we define the notion of index for a critical point of d_p in the angle sense, and we give a normal form of nondegenerate distance function d_p around a critical point by geometric consideration. Then we show that usual procedure of Morse theory works. (Theorem 3.7. Compare [4].) Next, we also consider the condition (F) for $p \in M$, which states that for any unit speed geodesic γ_u emanating from p, and for any Jacobi field Y along γ_u with $Y(0) = 0, \nabla_{\dot{\gamma}} Y(0) \neq 0$, we have $\langle Y(t), \nabla_{\dot{\gamma}_u} Y(t) \rangle > 0$ for the parameter value $t > 0$ up to the cut distance $i_p(u)$ to p. Note that this implies that the condition (C) holds at p, and the condition (F) is satisfied for any $p \in M$ when (M, g) is of nonpositive sectional curvature. Then under the assumption of condition (F), on any stratum of $C(p)$ of codimension less than n, we see that the restriction f of the distance function d_p to the stratum satisfies the following: any critical point r of d_p in the angle sense belonging to the stratum is a strict local minimum of f, namely, r is a critical point of f in the usual sense and its Hessian is positive definite. Therefore, in this case we have simpler procedure of Morse theory (see Theorem 3.11).

The structure of cut loci $C(p)$ for generic Riemannian metrics was studied by applying singularity theory to smooth energy integral on the (finite-dimensional approximation of) the space of piecewise smooth paths emanating from p ([1],[2],[17],[19]). However, it is not clear for us whether such structure theorems directly give information on the Riemannian distance function d_p, and we take here more geometrical approach under somewhat stronger assumption on the cut locus. We are greateful to M. van Manen for his criticism and pointing out several references including [19]. We are greatful to H. Rubinstein for telling us [3],[4],[5]. We would like to also express our sincere appreciation to the referee for his kind suggestion to make the paper more readable.

2. Structure of the cut locus disjoint from the first conjugate locus

Suppose a compact n-dimensional Riemannian manifold (M, g) satisfies the condition (C) at p, namely the tangent cut locus $\tilde{C}(p)$ is disjoint from the first tangent conjugate locus of p. Let $q \in C(p)$. Then from the assumption there are only finitely many minimal geodesics emanating from p to q. We denote by $\{\gamma_0, \cdots, \gamma_k\} (k \geq 1)$ the set of the minimal geodesics parametrized by arclength from p to q, where $k+1$ is called the order of the cut point q. Note that we may find open neighborhoods $U \ni q$ in M and $V_i \ni l\dot{\gamma}_i(0)$ $(i = 0, \ldots, k)$ in T_pM such that $\exp_p : V_i \longrightarrow U$ are diffeomorphisms, where $\exp_p : T_pM \longrightarrow M$ denotes the exponential map at p. We may also assume that for any minimal geodesic γ parameterized by arclength from p to a point $r \in U$ the tangent vector $d(p, r)\dot{\gamma}(0) \in T_pM$ belongs to one of the corresponding V_i's. Then we set

$$F_i := (\exp_p |V_i)^{-1} \tag{2.1}$$

that is a diffeomorphism from U onto V_i for each $i = 0, \dots, k$.

We set $X_i := -\dot{\gamma}_i(l), l = d(p, q), i = 0, \dots, k$ which are pairwise different unit vectors of T_qM. Namely, $\{X_i\}_{0 \le i \le k}$ is the set of initial directions of geodesics from q to p parametrized by arclength. Now we define the notion of nondegenerate cut points as follows:

Definition 2.1. *A cut point $q \in C(p)$ of order $k+1$ is said to be nondegenerate, if $X_0, \dots, X_k$ are in general position in T_qM.*

This means that the dimension of the affine subspace of the tangent space T_qM spanned by $\{X_0, \cdots, X_k\}$ is equal to k. This is also equivalent to the condition that $X_0 - X_1, X_0 - X_2, \dots, X_0 - X_k$ (or equivalently, for fixed i, $X_i - X_j$ $(j \neq i)$) are linearly independent. Then either $X_0, \dots, X_k$ are linearly independent and spans a k-dimensional affine subspace that does not contain the origin, or they are linearly dependent and spans a k-dimensional vector subspace. For instance, cut points of order 2 or of order 3 are always nondegenerate. Note that if q is a nondegenerate cut point, then its order $k+1$ is at most $n+1$.

Remark 2.2. (1) Tangent cut loci of a point in 2-dimensional flat tori are in general hexagons, in which case all cut points are nondegenerate. If the tangent cut locus of p is given by a rectangle, then the cut point q which is furthest to p and is given by vertices of the rectangle of the tangent cut locus is degenerate. Indeed, we have four minimal geodesics from p to q. However, after slightly deforming the lattice such degenerate cut points disappear in this case. All cut points of p in an n-dimensional flat torus are nondegenerate if and only if the cut points, that are local maximum points of d_p and given by the vertices of the tangent cut locus, are of order $n+1$.

(2) The distance function d_p is a (germ of) min-type function in the sense of [4], namely, in a neighborhood of q we have $d_p(r) := \min\{\|F_i(r)\| \mid 0 \le i \le k\}$.

Now suppose that all cut points of p are nondegenerate. We denote by $C_{k+1} \subset C(p)$ the set of cut points of p of order $k+1$. We assume that C_{k+1} is nonempty. Now for $q \in C_{k+1}$ we denote by $\gamma_0, \dots, \gamma_k$ the set of minimal geodesics parametrized by arclength joining p to q. Recall that for any $r \in U \cap C_{k+1}$ and any minimal geodesic γ joining p to r, $d(p, r)\dot{\gamma}(0)$

belongs to V_i for some $0 \le i \le k$, and that $U \cap C(p)$ consists of cut points of p of order not greater than $k+1$ (see e.g., [15]).

Now, we consider a smooth map $G : U \to \boldsymbol{R}^k$ defined by

$$G(r) := (\|F_0(r)\| - \|F_1(r)\|, \dots, \|F_0(r)\| - \|F_k(r)\|), \tag{2.2}$$

where $F_i = (\exp_p |V_i)^{-1} : U \to V_i \subset T_r M, i = 0, \dots, k$, are given by (2.1). Then we easily see that

$$G^{-1}(0) = C_{k+1} \cap U, \tag{2.3}$$

and that for every $r \in U$ the gradient vector $\nabla G_j(r)$ of the j-th coordinate function $G_j(r) := \|F_0(r)\| - \|F_j(r)\|$ of G is given by

$$\nabla G_j(r) = X_j - X_0 \quad (j = 1, \dots, k) \tag{2.4}$$

by the first variation formula.

Since $\nabla G_j(r)$ are linearly independent for $r \in G^{-1}(0)$ by the assumption, the differential $DG(r) : T_r U \longrightarrow \boldsymbol{R}^k$ of G at any $r \in G^{-1}(0)$ is of rank k. It follows by the implicit function theorem that C_{k+1} is a submanifold of M of codimension k. Equivalently, the hypersurfaces $G_j^{-1}(0)$ $(j = 1, \dots, k)$ intersect transversally at $r \in G^{-1}(0)$. However, note that C_{k+1} is not necessarily connected, and we denote by $C_{k+1,q}$ the connected component of C_{k+1} containing $q \in C_{k+1}$.

Since we have $d_p(r) = \|F_0(r)\| (= \|F_j(r)\|, j = 1, \dots, k)$ for $r \in C_{k+1}$ and F_0 is a diffeomorphism, we see that $f := d_p | C_{k+1,q}$ is a smooth function for $q \in C_{k+1}$. Summing up we get

Lemma 2.3. *Suppose that $C_{k+1} (\neq \emptyset)$ consists of nondegenerate cut points. Then C_{k+1} is a submanifold of codimension k of M, and d_p is a smooth function when restricted to each connected component $C_{k+1,q}$ of C_{k+1}.*

It may happen that some C_{k+1} is empty. For instance, the cut locus of the n-dimensional real projective space with canonical Riemannian metric of constant curvature 1 consists of nondegenerate cut points of order 2, and C_{k+1} $(k \ge 2)$ is empty. In this case the cut locus is an $(n-1)$-dimensional projective subspace and indeed a smooth submanifold. From the condition (C) we see that C_{k+1} is nonempty for some $k \ge 1$ and then so is C_{l+1} for

$1 \le l \le k$ from the nondegeneracy condition. For $q \in C_{k+1}$ we have

$$U \cap C(p) = \bigcup_{1 \le l \le k} C_{l+1} \cap U \quad \text{and} \quad U \setminus C(p) = \bigcup_{i=0}^{k} D_i,$$

where $D_i := \{r \in U \mid \|F_i(r)\| < \|F_j(r)\|; j \neq i\}$. Note that the closure $\bar{C}_{k+1,q}$ of $C_{k+1,q}$ is given by $\bigcup_{l \ge k} C_{l+1,r}$, where $r \in \bar{C}_{k+1,q}$ is of order $l+1$. If $C_{k+2} = \emptyset$ then we see that $\bar{C}_{k+1,q}$ is a smooth submanifold. It follows that we have a stratification of the cut locus by submanifolds $C_{k+1,q}$, and it is easy to verify the Whitney's condition (B) in our case ([8]). Hence we get

Proposition 2.4. *Suppose a compact Riemannian manifold (M, g) satisfies the condition (C) at $p \in M$ and cut points $q \in C(p)$ are nondegenerate. Then the cut locus $C(p)$ of p has a Whitney stratification given as above.*

Next we give a description of the tangent cone of a cut point $q \in C(p)$. Suppose C_{k+1} consists of nondegenerate cut points. Then recall that C_{k+1} is an $(n-k)$-dimensional submanifold of M, and d_p is a smooth function when restricted to each connected component $C_{k+1,q}$.

Theorem 2.5. *Suppose the condition (C) is satisfied at p and all cut points of p are nondegenerate. Let $q \in C_{k+1}$ and let $\gamma_0, \ldots, \gamma_k$ be the minimal geodesics from p to q. Set $X_i = -\dot{\gamma}_i(l) \in U_qM$, $i = 0, \cdots, k$, with $l = d(p,q)$. We denote by $S(q)$ the cut locus of a finite subset $\{X_0, \ldots, X_k\}$ of U_qM, which is considered as the unit $(n-1)$-dimensional sphere with the canonical Riemannian metric. Then $C(p) \cap U$ is homeomorphic to the cone over $S(q)$ in T_qM with origin as the vertex, if we take a sufficiently small open neighborhood U of q.*

First we recall the structure of the cut locus $S(q)$ of the finite set $\{X_0, \ldots, X_k\}$ in the unit sphere U_qM with respect to the canonical metric. Indeed, we have $X \in S(q)$ if and only if there exists at least two minimizing geodesics of U_qM from the set $\{X_0, \ldots, X_k\}$ to X. Namely, $S(q)$ consists of the parts of the bisectors of X_i and X_j $(i < j)$ in the sphere U_qM that are closer or of equidistance to other X_l's, $l \neq i, j$. In our case, $X_0, \cdots, X_k$ spans a k-dimensional affine subspace V_1 and they are contained in a $(k-1)$-dimensional great or small sphere in U_qM. Note that they are contained in a great sphere S^{k-1} if and only if the affine subspace spanned by them contains the origin and is a vector subspace. If they are contained in a small hypersphere $\hat{S}^{k-1}$ in the k-dimensional great sphere $S^k := U_qM \cap \langle X_0, \cdots, X_k\rangle_{\boldsymbol{R}}$, we consider the parallel great sphere S^{k-1} in S^k and the corresponding unit vectors $\tilde{X}_0, \cdots, \tilde{X}_k$ in S^{k-1} that are projections of $X_0, \cdots, X_k$ from the north pole respectively. Let $\tilde{V}_1 = \langle \tilde{X}_0, \cdots, \tilde{X}_k\rangle_{\boldsymbol{R}}$ denote the k-dimensional vector subspace determined by S^{k-1}. If $\{X_0, \cdots, X_k\}$ are contained in a $(k-1)$-dimensional great sphere S^{k-1} of $V_1 = \langle X_0, \cdots, X_k\rangle_{\boldsymbol{R}}$, we set $\tilde{X}_i = X_i (i = 0, \ldots, k)$ and $\tilde{V}_1 = V_1$.

Now we give a description of the structure of the cut locus $S(q)$.

Lemma 2.6. *(i) The cut locus of $\{\tilde{X}_0, \cdots, \tilde{X}_k\}$ in the unit sphere U_qM coincides with $S(q)$, the cut locus of $\{X_0, \ldots, X_k\}$ in U_qM .*

(ii) The cut locus $\tilde{S}_{k-2}(q)$ of $\{\tilde{X}_0, \cdots, \tilde{X}_k\}$ in S^{k-1} is given by the union of the boundaries of the Dirichlet domains (or Voronoi diagrams) $\{u \in S^{k-1} \mid \angle(u, \tilde{X}_i) < \angle(u, \tilde{X}_j)$ for all $j \neq i, 0 \leq j \leq k\}$ determined by $\tilde{X}_i$'s $(0 \leq i \leq k)$. These Dirichlet domains are spherical $(k-1)$-dimensional simplices with totally geodesic boundaries in S^{k-1} and gives a triangulation of S^{k-1}. Then $\tilde{S}_{k-2}(q)$ is the $(k-2)$-skeleton of the triangulation consisting of $k(k+1)/2$ facets, and $(k-l-1)$-dimensional faces are given by $\{u \in S^{k-1} \mid \angle(u, \tilde{X}_{i_0}) = \cdots = \angle(u, \tilde{X}_{i_l}) < \angle(u, \tilde{X}_j); 0 \leq j \leq k, j \neq i_0, \ldots, i_l\}$, which are totally geodesic submanifolds of S^{k-1}.

(iii) The whole cut locus $S(q)$ is given by the spherical join of $\tilde{S}_{k-2}(q)$ and S^{n-k-1}, where S^{n-k-1} consists of points in U_qM of spherical distance $\pi/2$ (or orthogonal) to the given S^{k-1}.

Proof of Lemma. If $k = 1$, $S(q)$ is given by a great sphere S^{n-2} of S^{n-1} obtained as the bisector of X_0 and X_1, and (ii), (iii) hold setting $\tilde{S}_{k-2}(q) = \emptyset$. So we assume $k \geq 2$ in the proof of (ii) and (iii).

(i) Denoting by $\boldsymbol{n}$ the unit vector in $V_1 := \langle X_0, \cdots, X_k\rangle_{\boldsymbol{R}} \cong \boldsymbol{R}^{k+1}$ representing the north pole of S^k, we have

$$X_i = \cos\theta\, \tilde{X}_i + \sin\theta\, \boldsymbol{n}, \tag{2.5}$$

where $\theta = \angle(X_i, \tilde{X}_i)$ is the spherical distance between S^{k-1} and $\hat{S}^{k-1}$. It follows that for $u \in S^{n-1}$ we have $\angle(u, X_i) = \angle(u, X_j)$ (resp. $\angle(u, X_i) < \angle(u, X_j)$) if and only if $\angle(u, \tilde{X}_i) = \angle(u, \tilde{X}_j)$(resp. $\angle(u, \tilde{X}_i) < \angle(u, \tilde{X}_j)$) holds, and the cut locus of $\{\tilde{X}_0, \cdots, \tilde{X}_k\}$ coincides with $S(q)$.

(ii) By the nondegeneracy condition $\tilde{X}_0, \cdots, \tilde{X}_k \in S^{k-1}$ form vertices of a k-simplex in $\tilde{V}_1 \cong \boldsymbol{R}^k$. Then (ii) follows from

$$\tilde{S}_{k-2}(q) = \bigcup_{i<j} \{u \in S^{k-1} \mid \angle(u, \tilde{X}_i) = \angle(u, \tilde{X}_j) \leq \angle(u, \tilde{X}_l)\}$$

where $0 \leq l \leq k, l \neq i, j$.

(iii) Note that the sphere $S^{n-1} = U_qM$ with the canonical Riemannian metric is isometric to the spherical join $S^{k-1} * S^{n-k-1}$ of S^{k-1} and S^{n-k-1}. Namely any $y \in S^{n-1}$ may be written as $\gamma(t), 0 \leq t \leq \pi/2$, where γ is a unit-speed geodesic emanating from $x \in S^{k-1}$ perpendicularly to S^{k-1}. Then we have

$$\cos\angle(\gamma(t), \tilde{X}_i) = \cos t \cos\angle(x, \tilde{X}_i) \tag{2.6}$$

for $0 \leq t \leq \pi/2$. It follows that if $x = \gamma(0) \in \tilde{S}_{k-2}(q)$ then we have $\gamma(t) \in S(q)$, $0 \leq t < \pi/2$ and vice versa. On the other hand, S^{n-k-1} is contained in $S(q)$, since S^{n-k-1} is the set of points of equidistance $\pi/2$ to $\tilde{X}_i$ $(i = 0, \ldots, k)$, namely the set of furthest points to $\{\tilde{X}_i\}$. Note that the $(n-k)$-dimensional vector subspace V_0 of T_qM containing S^{n-k-1} is characterized as the set of points in T_qM which are of equidistance from $\tilde{X}_0, \cdots, \tilde{X}_k$ with respect to the Euclidean metric. □

In the case where $X_0, \cdots, X_k$ lie on a small sphere in S^k, the cut locus $\tilde{S}_{k-1}(q)$ of these unit vectors in S^k is the spherical suspension of the cut locus $\tilde{S}_{k-2}(q)$ of $\tilde{X}_0, \cdots, \tilde{X}_k$ in S^{k-1}, and the cut locus $S_{k-2}(q)$ of these unit vectors in the original small $(k-1)$-sphere $\hat{S}^{k-1}$ is the intersection of $\tilde{S}_{k-1}(q)$ and the k-dimensional affine subspace V_1 determined by these $X_i, 0 \leq i \leq k$. Then $S_{k-2}(q)$ is indeed homeomorphic to $\tilde{S}_{k-2}(q)$. Further note that the cone over $S(q)$ in T_qM is homeomorphic to the product of the subspace V_0 and the cone $\tilde{T}$ over $\tilde{S}_{k-2}(q)$ in $\tilde{V}_1$.

Now we turn to the proof of Theorem 2.5, namely study the local structure of the cut locus $C(p)$ around $q \in C_{k+1}$. We set for $r \in U$

$$x_i(r) := \|F_i(r)\|, \; X_i(r) := -\nabla x_i(r), \; i = 0, \cdots, k, \tag{2.7}$$

where F_i is given in (2.1) and recall that we have $X_i = X_i(q) = -\nabla x_i(q)$.

Then from Lemma 2.3 we may regard that we have "local coordinates" $(x_0, \cdots, x_k, x_{k+1}, \cdots, x_n)$, where $(x_{k+1}, \cdots, x_n)$ denotes local coordinates for C_{k+1} around q guaranteed by Lemma 2.3 on an open neighborhood U of q. In the above, we think $(x_0, \cdots, x_k)$ as "local coordinates" around q for a submanifold N complementary to C_{k+1} with tangent space spanned by $\{X_i - X_0,\ i = 1, \cdots k\}$. More precisely, setting $y_i := x_0 - x_i$, $i = 1, \ldots, k$, we have local coordinates $(y_1, \cdots, y_k, x_{k+1}, \cdots, x_n)$, where $(y_1, \cdots, y_k)$ gives a local coordinates system for N taking $U(\ni q)$ smaller if necessary. However, we also use the above notation. Then around q the cut locus $C(p)$ consists of $k(k+1)/2$ pieces of hypersurfaces $\bar{C}_{i,j}$ $(0 \le i < j \le k)$ of M given by

$$\begin{aligned} \bar{C}_{i,j} := \{ r \in C(p) \cap U \ | G_{ij}(r) := x_i(r) - x_j(r) = 0; \\ x_l(r) \ge x_i(r) = x_j(r) \text{ for } l \ne i, j \} \end{aligned}$$

corresponding to minimal geodesics γ_i and γ_j from p to r. Note that these hypersurfaces are also characterized in terms of $y_i, 1 \le i \le k$, namely for $1 \le i < j \le k$ we obtain

$$\begin{aligned} \bar{C}_{0,i} &:= \{ r \in U \mid 0 = y_i(r) \ge y_l(r),\ l \ne 0, i \} \\ \bar{C}_{i,j} &:= \{ r \in U \mid 0 \le y_i(r) = y_j(r) \ge y_l(r),\ l \ne 0, i, j \}. \end{aligned}$$

The intersection of these hypersurfaces is nothing but $C_{k+1} \cap U$. Now for $I := \{0 \le i_0 < i_1 < \cdots < i_a \le k\}$ we set $\bar{C}_I := \bar{C}_{i_0,i_1} \cap \bar{C}_{i_0,i_2} \cap \cdots \cap \bar{C}_{i_0,i_a}$. Then $C_I := \bar{C}_I \setminus \bigcup \{ \bar{C}_J \mid J \text{ contains } I \text{ with } \sharp J = a+1 \}$ are submanifolds of codimension l in M that give the stratification of the cut locus $C(p)$ in Proposition 2.4. In terms of the coordinates y_i we have

$$\begin{aligned} \bar{C}_I &= \{ r \in U \mid 0 = y_{i_1} = \ldots = y_{i_a}(r) \ge y_l(r), l \notin I \} \ (0 = i_0 \in I), \\ C_I &= \{ r \in U \mid 0 = y_{i_1} = \ldots = y_{i_a}(r) > y_l(r), l \notin I \} \ (0 = i_0 \in I), \\ \bar{C}_I &= \{ r \in U \mid 0 \le y_{i_0} = \ldots = y_{i_a}(r) \ge y_l(r), l \notin I \} \ (0 \notin I), \\ C_I &= \{ r \in U \mid 0 < y_{i_0} = \ldots = y_{i_a}(r) > y_l(r), l \notin I \} \ (0 \notin I). \end{aligned} \tag{2.8}$$

Now, for any tangent vector u to $\bar{C}_{i,j}$ at q we see that u is at the same spherical distance to X_i and X_j. Indeed, taking a curve $s \mapsto x(s)$ in $\bar{C}_{i,j}$ tangent to u, we have $x_i(x(s)) = \|F_i(x(s))\| = \|F_j(x(s))\| = x_j(x(s))$. Then by the first variation formula, it follows that $\langle u, X_i \rangle = \langle u, X_j \rangle$. By the same argument we have $\langle u, X_l \rangle \le \langle u, X_i \rangle = \langle u, X_j \rangle$ for $l \ne i, j$, namely, we have for the spherical distance

$$\angle(u, X_i) = \angle(u, X_j) \le \angle(u, X_l) \ \text{ for } \ l \ne i, j.$$

It follows that $T_qC_{I_a} = \{u \in T_qM \mid \langle u, X_{i_0}\rangle = \langle u, X_{i_1}\rangle = \cdots = \langle u, X_{i_a}\rangle\}$ and $V_0 = T_qC_{k+1} = T_qC_K$ with $K = \{0, 1, \ldots, k\}$.

Now take a section $N(\subset U)$ through $q \in C_{k+1}$ defined by $x_\alpha = const.$ $(\alpha = k+1, \ldots, n)$ in U, that is tangent to $\tilde{V}_1 = \langle X_1(q) - X_0(q), \ldots, X_k(q) - X_0(q)\rangle_{\boldsymbol{R}} = \langle \tilde{X}_1(q) - \tilde{X}_0(q), \ldots, \tilde{X}_k(q) - \tilde{X}_0(q)\rangle_{\boldsymbol{R}}$. Note that T_qN is the orthogonal complement of $V_0 = T_qC_{k+1}$ in T_qM. Then we have

$$C_{0j} \cap N = \{r \in N \mid 0 = y_j(r) > y_l(r), l \neq j\} \quad \text{for } 1 \leq j \leq k,$$
$$C_{ij} \cap N = \{r \in N \mid 0 < y_i(r) = y_j(r) > y_l(r)\} \quad \text{for } 1 \leq i < j \leq k.$$

and also have the similar expressions for $\bar{C}_I \cap N$ and $C_I \cap N$ as in (2.8). Then in terms of the local coordinates $(y_1, \ldots, y_k)$, $C(p) \cap N$ is a cone, and the tangent cone to $\bar{C}_I \cap N$ (resp. $C_I \cap N$) at q is given by

$$\bigcup_{t \geq 0} t\{u \in U_qN \mid \angle(X_{i_0}, u) = \cdots = \angle(X_{i_a}, u) \leq \angle(X_l, u),\ l \neq i_0, \ldots, i_a\},$$
$$\bigcup_{t \geq 0} t\{u \in U_qN \mid \angle(X_{i_0}, u) = \cdots = \angle(X_{i_a}, u) < \angle(X_l, u),\ l \neq i_0, \ldots, i_a\}$$

respectively. It follows that the tangent cone to $N \cap C(p)$ at q is nothing but the cone over $\tilde{S}_{k-2}(q)$ in $\tilde{V}_1 = T_qN$ as described by the above arguments. Therefore $N \cap C(p)$ is homeomorphic to the cone $\tilde{T}$ over $\tilde{S}_{k-2}(q)$ in $\tilde{V}_1$. Note that the above fact also holds for $r \in U \cap C_{k+1,q}$, and taking U $(\ni q)$ small if necessary, $U \cap C(p)$ is homeomorphic to the product of $\tilde{T}$ and an open $(n-k)$-disk, and the latter is homeomorphic to the cone over $S(q)$ in T_qM. In the case of $k = 1$, $C(p) \cap U$ is a hypersurface of M and is homeomorphic to open $(n-1)$-disk, that is the cone over S^{n-2} in U_qM. □

Remark 2.7. We show that Theorem 2.5 does not hold without assuming the nondegeneracy condition. Let (T, g_0) be the 3-dimensional flat torus obtained by identifying the opposite faces of the cube $A := [-10, 10] \times [-10, 10] \times [-10, 10]$ in $\boldsymbol{R}^3$, and we denote by $\phi : A \to T$ this identifying map. Then note that the tangent cut locus of the origin $o = \phi((0, 0, 0))$ coincides with the boundary $\tilde{C}$ of A, and the cut locus C is given by $C = \phi(\tilde{C})$. Now there are exactly four minimal geodesics from the origin to any point in the segment $E = \{\phi((10, 10, t)) \mid -10 < t < 10\}$, and the cut locus around the point is given by four half planes gathering along the segment E.

Now for any positive integer n, let B_n (resp. B'_n) be the $\frac{1}{2^{n+3}}$-ball centered at $(1, 1, \tan \frac{1}{2^n})$ (resp. $(1, -1, \tan \frac{1}{2^n})$). We denote by $\frac{1}{2}B_n$ (resp. $\frac{1}{2}B'_n$) the $\frac{1}{2^{n+4}}$-ball with the same center as B_n (resp. B'_n). Now for an $\epsilon > 0$ small enough take a smooth function χ_n (resp. χ'_n) that is equal to $\epsilon/2^{2n}$ on $\frac{1}{2}B_n$ (resp. $\frac{1}{2}B'_n$), vanishes outside B_n (resp. B'_n), and is nonincreasing along radii. Then setting $g_n := (1 + \chi_n)g_0$ (resp. $g'_n := (1 + \chi'_n)g_0$), we get a new 3-dimensional almost flat torus. Note that $s \to \phi(s, s, s \tan \frac{1}{2^n})$ (resp. $\phi(s, -s, s \tan \frac{1}{2^n})$), $0 \leq s \leq 10$, is not a g_n (resp. g'_n)-minimal geodesic from o to $q_n := \phi((10, 10, 10 \tan \frac{1}{2^n}))(= \phi((10, -10, 10 \tan \frac{1}{2^n})))$. But still there are exactly three g_n (resp. g'_n)-minimal geodesics from o to q_n. Namely, q_n is a cut point of o and the cut locus around q_n with respect to the metric g_n (resp. g'_n) locally consists of three half planes P_1, P_2, P_3 (resp. P'_1, P'_2, P'_3) gathering along the segment E, where we set $P_1 : y = 10, x > 10; P_2 : x = 10, y > 10; P_3 : x = y < 10$ (resp. $P'_1 : y = 10, x < 10; P'_2 : x = 10, y < 10; P'_3 : x = y > 10$).

Note that $\{B_{2n}, B_{2n+2}\}_{n=1}^{\infty}$ are pairwise disjoint, and they are disjoint from the segment $s \mapsto \phi(s, s, 0)$. We take a new 3-dimensional almost flat torus $(T, \tilde{g})$ given by

$$\tilde{g} = (1 + \sum_n (\chi_{2n} + \chi'_{2n+1}))g_0.$$

Then $q := \phi(10, 10, 0)$ is again a cut point of o with respect to the deformed Riemannian metric $\tilde{g}$ and in fact there are exactly four $\tilde{g}$-minimal geodesics from the origin, that are also g_0-minimal geodesics. It follows that the cut locus $S(q)$ in U_qT consists of four half great circles joining two antipodes, and the cone P over $S(q)$ consists of four half planes gathering along a segment. However, the cut locus of the origin with respect to $\tilde{g}$ is not homeomorphic to C, since both of the sequences $\{q_{2n}\}$ and $\{q_{2n+1}\}$) converges to q and the local structure of the cut locus around $\{q_{2n}\}$ is different from the one around $\{q_{2n+1}\}$.

Remark 2.8. Even for real analytic metrics, the assumption of nondegeneracy seems necessary in Theorem 2.5, as is suggested by the following example: Take the functions $f_1 := z + c, f_2 := -z + c, f_3 := y + x^2 + c, f_4 := -y + x^2 + c$ defined on $\boldsymbol{R}^3 = \{(x, y, z) \mid x, y, z \in \boldsymbol{R}\}$, where c is a positive constant. Put $f := \min\{f_1, f_2, f_3, f_4\}$ and consider the set C of points in $\boldsymbol{R}^3$

such that there are at least two $1 \le i < j \le 4$ with $f = f_i = f_j$. Although f is precisely not a distance function, the set C is similar to the cut locus. Note that the origin $(0,0,0)$ belongs to C with $f = f_i = c(1 \le i \le 4)$, and that gradient vectors of f_i at the origin are given by $\pm\partial/\partial z, \pm\partial/\partial y$. In this case C consists of the 4 half parabolic cylinders (given by $y = z - x^2, z \le 0$; $y = z + x^2, z \ge 0$; $y = -z - x^2, z \ge 0$; $y = -z + x^2, z \le 0$) and 2 cusp regions $x \ge \sqrt{|y|}, x \le -\sqrt{|y|}$ in the xy-plane. On the other hand, $S((0,0,0))$ consists of 4 great half circles of S^2 joining the antipodes. Hence the cone over $S((0,0,0))$ is not homeomorphic to C.

Now we set $f := d_p|C_{k+1,q}$ and give the gradient vector ∇f of f at $x \in C_{k+1,q}$. Indeed, let $u \in T_xC_{k+1,q}$ and $s \mapsto x(s)$ be a curve in $C_{k+1,q}$ with $\dot{x}(0) = u$. Then noting that $f(x(s)) = \|F_i(x(s))\| = x_i(x(s))$ for any $0 \le i \le k$, we obtain by the first variation formula (see e.g., [16])

$$\langle \nabla f, u\rangle = \frac{d}{ds}\|F_i(x(s))\|_{s=0} = -\langle X_i, u\rangle, \tag{2.9}$$

where $\langle X_i, u\rangle$ is independent of i by the definition of C_{k+1}. It follows that $\nabla f(x)$ is the orthogonal projection of any $-X_i$ to $T_xC_{k+1,q}$ for $i = 0, \cdots, k$. Therefore, x is a critical points of $f = d_p|C_{k+1,q}$ in usual sense if and only if all of X_i $(i = 0, \dots, k)$ are orthogonal to $C_{k+1,q}$ and are located in a $(k-1)$-dimensional great sphere of U_qM.

Now how about the Hessian $D^2f(x)$ of f at a critical point $x \in C_{k+1,q}$ of f? Let u and $s \mapsto x(s)$ be as before. Then we have $D^2f(u,u) = \frac{d^2}{ds^2}\|F_i(x(s))\|_{s=0}$ for each i $(0 \le i \le k)$. Take a variation of γ_i given by

$$\alpha(t,s) := \exp_p \frac{t}{l}F_i(x(s)), \quad (0 \le t \le l := d(p,x), \ -\epsilon \le s \le \epsilon).$$

Then the variation vector field is a unique Jacobi field $Y_i(t)$ along γ_i with $Y_i(0) = 0$ and $Y_i(l) = u$. Note that Y_i is perpendicular to γ_i. Then we get by the second variation formula (see e.g., [16])

(2.10)

$$\begin{aligned} D^2f(u,u) = \int_0^l &\{\langle \nabla_{\dot{\gamma}_i}Y_i(t), \nabla_{\dot{\gamma}_i}Y_i(t)\rangle - \langle R(Y_i(t), \dot{\gamma}_i(t))\dot{\gamma}_i(t), Y_i(t)\rangle\}dt \\ &- \langle X_i, \nabla_u\dot{x}(s)\rangle = \langle u, \nabla_{\dot{\gamma}_i}Y_i(l)\rangle - \langle X_i, \nabla_u\dot{x}(s)\rangle, \end{aligned}$$

where we set $l = d(p,x)$.

Now, if a point p of a compact Riemannian manifold (M, g) admits no conjugate points along all geodesics emanating from p (e.g., for any point of a nonpositively curved manifold), then the structure of $C(p)$ may be expressed in terms of the Dirichlet domain of the universal covering space $\tilde{M}$ of M with the induced Riemannian metric $\tilde{g}$. We briefly explain this case (see also [15]). Let $\pi : \tilde{M} \to M$ be the covering projection, and set $\pi^{-1}(p) = \{\tilde{p} = \tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_N, \ldots\}$, that may be identified with the deck transformation group $\Gamma = \{g_0 = e, g_1, \ldots, g_N, \ldots\}$ via $\tilde{p}_i = g_i(\tilde{p}_0)$, where e denotes the identity. Note that $\tilde{p}_i\,(i = 0, 1, \ldots)$ are poles, namely $\exp_{\tilde{p}_i} : T_{\tilde{p}_i}\tilde{M} \to \tilde{M}$ are diffeomorphisms, and for any $\tilde{q} \in \tilde{M}$ there exists a unique minimal geodesic joining $\tilde{p}_i$ and $\tilde{q}$. We fix $\tilde{p}$ as a base point. Then the cut locus $C(p)$ may be described as follows: Let $\Delta_{\tilde{p}}$ be the Dirichlet domain of Γ. Namely,

$$\Delta_{\tilde{p}} = \bigcap_{g(\neq e)\in\Gamma} \{H_{\tilde{p},g(\tilde{p})} \mid i = 1, 2, \ldots\}$$

with $H_{\tilde{p},\tilde{q}} = \{\tilde{r} \in \tilde{M} \mid \tilde{d}(\tilde{p}, \tilde{r}) < \tilde{d}(\tilde{q}, \tilde{r})\}$. Then we have $C(p) = \pi(\partial\Delta_{\tilde{p}})$. Since M is compact, it suffices to consider a finite number of $g \in \Gamma$ such that $\tilde{d}(\tilde{p}, g\tilde{p}) \le d(M)$ so that we may write $\Delta_{\tilde{p}} = \bigcap_{j=1,\ldots,N} H_{\tilde{p},g_j(\tilde{p})}$.

This also means that the distance function d_p is a min-type function in the sense of [4], namely we have $d_p(q) := \min\{\tilde{d}(\tilde{q}, g_1\tilde{p}), \ldots, \tilde{d}(\tilde{q}, g_N\tilde{p})\}$, where $\tilde{q} \in \pi^{-1}(q)$. Let $q \in C(p)$ be a cut point of order $k + 1$ and $\gamma_0, \cdots \gamma_k$ be the minimal geodesics from p to q with length $l = d(p, q)$. As before we set $X_i = -\dot{\gamma}_i(l) \in U_qM$ $(i = 0, \ldots, k)$. Take the lift of γ_0 emanating from $\tilde{p}$ with respect to the universal covering π, and we denote by $\tilde{q}$ the end point of the lift. Then there exist $g_{i_0} = e, g_{i_1}, \ldots, g_{i_k}$ in Γ such that γ_j is expressed as the projection of a unique minimal geodesic $\tilde{\gamma}_j$ in $\tilde{M}$ joining $g_{i_j}\tilde{p}$ and $\tilde{q}$ $(j = 0, \ldots, k)$ with $\tilde{d}(g_{i_j}\tilde{p}, \tilde{q}) = \tilde{d}(\tilde{p}, \tilde{q}) = l$. Now we set for $I = \{i_1, \ldots, i_k\}$

$$\tilde{C}_I := \{\tilde{r} \in \tilde{M} \mid \tilde{d}(g_i\tilde{p}, \tilde{r}) = \tilde{d}(\tilde{p}, \tilde{r}) < \tilde{d}(g_j\tilde{p}, \tilde{r}) \text{ for any } i \in I,\ j \notin I\}.$$

Then $g_{i_0} = e, g_{i_1}, \ldots, g_{i_k}$ are chosen in common in the connected component containing q of the set of cut points of order $k+1$, and we have $\pi(\tilde{C}_I) \subset C(p)$. If we take the lift of γ_j instead of γ_0 in the above, then we have $g_{i_j}^{-1}\tilde{q}$ and $g_{i_j}^{-1}\tilde{C}_I$ instead of $\tilde{q}$ and $\tilde{C}_I$, respectively. Now suppose that all cut points of p are nondegenerate. Then $\tilde{X}_0, \ldots, \tilde{X}_k$ are in general position in $T_{\tilde{q}}\tilde{M}$, where we set $\tilde{X}_i = -\dot{\tilde{\gamma}}_i(l)$. Then if $\tilde{C}_I \neq \emptyset$, $\tilde{C}_I$ (resp. $\pi(\tilde{C}_I)$) is a submanifold of dimension $n-k$ of $\tilde{M}$ (resp. M), and the cut locus $C(p)$ is stratified by the strata $\pi(\tilde{C}_I)$ by the same arguments as above.

Remark 2.9. (1) We suspect whether Riemannian metrics such that all cut points of p are nondegenerate are open and dense in the set of all Riemannian metrics satisfying the condition (C) at p. For two-dimensional case, V. Gershkovich asserts that the above assertion holds ([3], [5]). This also follows applying a recent result of [13] to our situation. We suspect that their approach is helpful for the above problem.

(2) For 3-dimensional case, approximating a Riemannian metric of A. Weinstein ([18]) in §1 by M. Buchner's cut stable metrics, we get Riemannian metrics satisfying the condition (C) for $p \in M$ such that all cut points of p are nondegenerate ([2], this holds up to dimension 6).

(3) M. van Manen pointed out that Y. Yodomin has considered cones over the $(n-2)$-skelton of simplices for central sets in $\boldsymbol{R}^n$ in [19] that is related to our cut locus case.

3. Morse theory for distance functions

First we recall the notion of a critical point of the distance function d_p in the angle sense (see §1 and [9], [6]). A point $q \in M$ is said to be a critical point of d_p, if for any unit tangent vector $v \in T_qM$ there exists a minimal geodesic γ from p to q such that $\angle(v, -\dot{\gamma}(l)) \leq \pi/2$ holds with $l = d(p,q)$. Note that any critical point q of d_p is a cut point of p. Now we set

(3.1)
$$\Gamma(q) := \{-\dot{\gamma}(l) \in U_qM \mid \gamma : [0,l] \to M;\ \text{minimal geodesic from } p \text{ to } q\}$$

and define the set $\hat{\Gamma}_q \subset T_qM$ as the convex hull of $\Gamma(q)$. Then, the above condition for q to be a critical point of d_p means that $\hat{\Gamma}_q$ contains the origin 0 of T_qM.

Definition 3.1. *For any critical point q of d_p we define its degree as the dimension of the (vector) subspace spanned by $\Gamma(q)$.*

Recall that we set $X_i := -\dot{\gamma}_i(l), l = d(p,q)$ for minimal geodesics γ_i $(i = 0, \cdots, k)$ joining p to q. If $C(p)$ is nondegenerate and $q \in C_{k+1}$, then $X_0, \cdots, X_k$ span a k or $(k+1)$-dimensional (vector) subspace V. If q is a critical point, then V is a subspace of dimension k, since the convex hull $\hat{\Gamma}_q$ contains 0. Therefore in this nondegenerate case, degree of q is equal to k.

Now, suppose there exist no critical points in the annulus $R(r_1, r_2) := d_p^{-1}([r_1, r_2]) = \{x \in M \mid r_1 \le d_p(x) \le r_2\}, 0 < r_1 < r_2$. Then the isotopy lemma asserts the following: All the levels $d_p^{-1}(r)$, $r_1 \le r \le r_2$ are homeomorphic to each other, and $R(r_1, r_2)$ is homeomorphic to the product $d_p^{-1}(r_1) \times [r_1, r_2]$. This may be proved as in usual Morse theory by considering a gradient-like vector field of d_p (see e.g., [9], [6], [10] for more detail).

Next, suppose that the cut locus $C(p)$ of p consists of nondegenerate cut points. Then C_{k+1} is a submanifold of dimension $n-k$ of M, and d_p is a smooth function when restricted to each connected component $C_{k+1,q}$. First we will be concerned with the relation between the two kinds of critical points of the distance function, which was also obtained by V. Gershkovich and H. Rubinstein ([4]).

Lemma 3.2. *Suppose $r \in C_{k+1,q}$ is a critical point of d_p in the angle sense. Then r is a critical point of the smooth function $f := d_p \mid C_{k+1,q}$ in the usual sense. If $r \in C_{2,q}$ is a critical point of the smooth function $f := d_p \mid C_{2,q}$, then r is a critical point of d_p in the angle sense.*

Proof. Suppose $r \in C_{k+1,q}$ is a critical point of d_p in the angle sense and let γ_i $(i = 0, \dots, k)$ be a minimal geodesic parametrized by arclength joining p to r. To see that r is a critical point of $f = d_p \mid C_{k+1,q}$ in the usual sense, by (2.9) it suffices to show that $\alpha = \langle u, X_i \rangle$ is equal to 0 for any $u \in T_r C_{k+1,q}$, where $X_i = -\dot{\gamma}_i(l), l = d(p,q)$. Recall that α is independent of i. Then from the assumption we may choose $a_i \ge 0$ $(i = 0, \dots, k)$ with $\sum a_i = 1$ such that $\sum a_i X_i = 0$, and it follows that

$$\alpha = \sum a_i \alpha = \langle u, \sum a_i X_i \rangle = 0.$$

Next suppose r is a critical point of $d_p \mid C_{2,q}$ in the usual sense. Note that $\dim C_{2,q} = n - 1$. We have unit tangent vectors X_0, X_1 at r of minimal geodesics γ_0, γ_1 joining p to r, respectively. Then X_0, X_1 are different unit vectors perpendicular to the hypersubspace $T_r C_{2,q}$ of $T_r M$ by (2.6), and therefore should satisfy $X_0 + X_1 = 0$. It follows that r is a critical point of d_p in the angle sense. $\square$

In general for $k > 1$, critical points of $d_p \mid C_{k+1,q}$ are not necessarily critical points of d_p in the angle sense. For instance, if $k = n$ then $\dim C_{n+1,q} = 0$ and every $r \in C_{n+1}$ are critical points in the usual sense. However, r may not assume local maximum of d_p (see Figure 1).

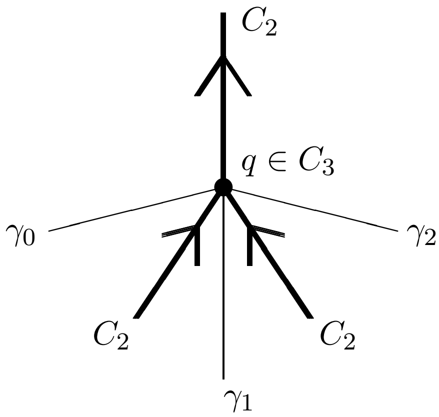

Figure 1.(Arrows denote the direction in which f increases.)

Next we define the notion of nondegeneracy for a distance function as follows:

Definition 3.3. *Suppose the cut locus $C(p)$ consists of nondegenerate cut points of p under the condition (C). Then we call the distance function d_p nondegenerate, if the following hold:*

(1) All critical points of d_p in the angle sense are isolated, and $\hat{\Gamma}_q$ contains 0 in its interior for each critical point q of d_p. Namely, for a critical point $q \in C_{k+1}$ in the angle sense we may find (unique) $a_i > 0$ $(0 \le i \le k)$ with $\sum a_i = 1$ such that $\sum a_i X_i = 0$ holds.

(2) For any critical point $q \in C_{k+1}$ of d_p in the angle sense, q is a nondegenerate critical point in the usual sense of $f := d_p \mid C_{k+1}$, the distance function restricted to the stratum C_{k+1} of $C(p)$.

The last condition in (1) above means that for a critical point $q \in C_{k+1}$, the cone $\bigcup_{t \ge 0} t\hat{\Gamma}_q$ forms a vector subspace of dimension $k = \dim \hat{\Gamma}_q$ that is the orthogonal complement of T_qC_{k+1} in T_qM. Note that this property holds for critical points of d_p which are cut points of order 2. For instance, the distance functions d_p of generic flat tori are nondegenerate for arbitrary points p.

In the following we want to give a normal form of nondegenerate distance function d_p in a neighborhood U of a critical point $q \in C(p)$ of d_p in the angle sense by choosing carefully a kind of local coordinates around q adapted to the local structure of $C(p)$. For $q \in C_{k+1}$ recall that $x_i(r) = \|F_i(r)\|,\ X_i(r) = -\nabla x_i(r)(r \in U,\ i = 0, \ldots, k)$ were given in (2.7), and we have $X_i = X_i(q)$. Recall also that a small neighborhood $U \cap C(p)$ of q in the cut locus is homeomorphic to the cone over $S(q)$ in T_qM, where $S(q)$ is the cut locus of $\{X_0, \ldots, X_k\}$ in the unit sphere U_qM (see Theorem 2.5). Then $U \setminus C(p)$ is divided into $k+1$ components $D_0, \ldots, D_k$ where each D_i contains the direction X_i, namely, contains $\gamma_i((l-\epsilon, l))$ with $l = d(p,q)$, where $\{\gamma_i\}_{0 \leq i \leq k}$ are minimal geodesics from p to q and $\epsilon > 0$ is sufficiently small. D_i $(i = 0, \ldots, k)$ is indeed given by

$$D_i = \{r \in U \mid x_i(r) < x_l(r) \text{ for any } l \neq i, 0 \leq l \leq k\},$$

and note that we have $d_p(r) = x_i(r)$ for $r \in D_i$. Then it follows that $\bar{D}_i \cap \bar{D}_j = \bar{C}_{i,j} \cap U$ for $i < j$, where $\bar{D}_i$ denotes the closure of D_i and $\bar{C}_{i,j}$ is given just before the formula (2.8).

Now let d_p be nondegenerate distance function in the sense of Definition 3.3 and q a critical point of d_p in the angle sense. To make our approach more understandable we begin with the simplest case where q is a critical point of order 2, namely $q \in C_2$. Then we have $X_0 + X_1 = 0$, and $S(q)$ is a great hypersphere S^{n-2} of U_qM. $U \setminus C_{2,q}$ is divided into domains D_0 and D_1. For every point $r \in D_i$ $(i = 0, 1)$ take a unique minimal geodesic γ_r joining p to r, and denote by $r_1 = \gamma_x(l_r) \in C_{2,q}$ the cut point of p along γ_r. Note that $l_r = d(p, r_1)$ is the cut distance along the geodesic γ_r and depends smoothly on $r \in D_i$, since γ_r intersects $C_{2,q}$ transversely. Now we set

$$z := l_r - d(p,r) = i_p(\dot{\gamma}_r(0)) - d_p(r) > 0.$$

For $r \in C_{2,q}$ we set $r_1 = r$ and $z = 0$. Then any $r \in \bar{D}_i, i = 0, 1$, may be uniquely expressed as $r = (r_1, z) \in C_{2,q} \times \boldsymbol{R}^+$. Since $q = (q, 0)$ is a critical point of d_p, q is a critical point of $f := d_p \mid C_{2,q}$ by Lemma 3.2. Since d_p is nondegenerate, q is a nondegenerate critical point of f and taking local coordinates $(x_2,\ldots,x_n)$ around $q \in C_{2,q}$ we may write

$$d_p(r) = d_p(q) \pm x_2^2 \ldots \pm x_n^2 - z. \tag{3.2}$$

Therefore in this case, we may define the index of d_p at q as the sum of the index of $f := d_p|C_{2,q}$ at q and 1, where recall that 1 is the degree of the critical point q.

Now we turn to a critical point $q \in C_{k+1}$ of d_p for general $k \geq 2$. Then $\{X_0, X_1, \ldots, X_k\}$ span a k-dimensional subspace of T_qM orthogonal to the subspace $T_qC_{k+1,q}$ of dimension $n-k$, and the convex hull of $\{X_0, X_1, \ldots, X_k\}$ contains the origin 0 of T_qM in its interior. Now for any proper subset I of $K := \{0, 1, \ldots, k\}$ with $\sharp I \geq 2$ we set

$$C_I = \{r \in C(p) \mid x_i(r) = x_j(r) < x_l(r) \text{ for any } i, j \in I \text{ and } l \notin I\},$$

and $\bar{C}_I$ denotes the closure of C_I (see also (2.8)). For $r \in \bar{C}_I$ we denote by $T_rC_I := \{u \in T_r(M) \mid \langle u, X_i(r)\rangle = \langle u, X_j(r)\rangle \text{ for any } i, j \in I\}(= \lim_{r_i \in C_I \to r} T_{r_i}C_I)$ the tangent space to C_I at r. Then $f_I := d_p|C_I$ is a smooth function and ∇f_I denotes the gradient vector of f_I. For $r \in \bar{C}_I$ we also use the notation $\nabla f_I(r) (\in T_rC_I)$ that is given as $\lim_{r_i \in C_I \to r} \nabla f_I(r_i)$. Then we have the following lemmas.

Lemma 3.4. *Let $q \in C_{k+1}$ be a critical point of nondegenerate distance function d_p. Let $I \subset \{0, 1, \ldots, k\}$ with $\sharp I \geq 2$ be a proper subset. Now for any $i \in I$ and $r \in \bar{C}_I$ we denote by $X_i^\top(r)$ the orthogonal projection of $X_i(r)$ to T_rC_I. Then we have*

$$X_i^\top(r) = -\nabla f_I(r) \ \textit{ for any } \ i \in I, r \in \bar{C}_I$$

and we denote the above vector also by $X_I^\top(r)$. Furthermore, there exist an open neighborhood U around q and $\delta > 0$ such that

$$\|\nabla f_I(r)\| \geq \delta \ \textit{ on } \ U \cap C_I.$$

Proof. For any curve $t \to x(t)$ in $\bar{C}_I$ emanating from r we have $d_p(x(t)) = f_I(x(t)) = x_i(x(t))$ for any $i \in I$. Differentiating this equation with respect to t at $t = 0$, it follows that

$$\langle \nabla f_I(r), \dot{x}(0)\rangle = \langle \nabla x_i(r)), \dot{x}(0)\rangle = -\langle X_i(r), \dot{x}(0)\rangle = -\langle X_i^\top(r), \dot{x}(0)\rangle$$

from which the first assertion follows. For the second assertion it suffices to show that $X_I^\top(q) \neq 0$ at the critical point q of d_p. Indeed, otherwise for a fixed $i_0 \in I = \{i_0, \ldots, i_a\}$ we see that $\{X_i - X_{i_0}\}_{i \in I \setminus \{i_0\}}$ forms a basis of a subspace $(T_q C_I)^\top$ of dimension a. Since q is a critical point of d_p, $\{X_i\}_{i \in I}$ span an a-dimensional subspace of $T_q M$. However this contradicts the nondegeneracy condition that the faces of the convex hull of $\{X_0, X_1, \ldots, X_k\}$ cannot contain the origin 0 of $T_q M$. □

Note that the orthogonal projection of of X_i to $T_q C_{k+1}$ is equal to zero since q is a critical point of d_p, but the lemma asserts that the orthogonal projection $X_i^\top$ of $X_i (i \in I)$ to $T_q C_I$ never vanishes for any proper subset I of $\{0, 1, \ldots, k\}$.

Lemma 3.5. *Under the assumption of the previous lemma there exist positive constants a_i $(i = 0, \ldots, k)$ with $\sum a_i = 1$ such that*

$$-(\sum_{i \in I} a_i) \langle X_I^\top(q), X_I^\top(q) \rangle = \sum_{j \notin I} a_j \langle \nabla f_I(q), -X_j(q) \rangle. \tag{3.3}$$

Indeed, we have $\sum_{0 \le l \le k} a_l X_l(q) = 0$ at q for some $a_l > 0$ $(l = 0, \ldots, k)$ with $\sum a_l = 1$ by Definition 3.3. Then considering the orthogonal projection of $\sum a_l X_l(q)$ to $T_q C_I$, we see that

$$\sum a_l \langle X_l(q), \nabla f_I(q) \rangle = -\sum_{i \in I} a_i \langle X_i^\top(q), X_i^\top(q) \rangle + \sum_{j \notin I} a_j \langle \nabla f_I(q), X_j(q) \rangle$$

vanishes by the previous lemma, and (3.3) follows.

We also note that there exist no critical points of d_p (and also of f_I) except q in U by the nondegeneracy condition.

Now suppose $r \in D_{i_0} (0 \le i_0 \le k)$. First take a unique minimal geodesic γ_r joining p to r, and denote by $r_1 = \gamma_r(l_r) \in C(p)$ the cut point of p along γ_r, where $l_r = d(p, r_1) = i_p(\dot\gamma_r(0))$ is the cut distance to p along γ_r. Then r_1 lies in some $\bar C_{i_0, i_1}$ that is a subset of the boundary of D_{i_0}. Note that here we do not assume that $i_0 < i_1$. For generic $r \in D_{i_0}$ we have $r_1 \in C_{i_0, i_1} := \{r \in U \mid x_{i_0}(r) = x_{i_1}(r) < x_l(r) \text{ for any } l \ne i_0, i_1\}$ for some i_1, and γ_r intersects C_{i_0, i_1} transversely at r_1. Then we denote the above r_1 by r_{i_1}, and set $z_{i_0} := l_r - d(p, r) (> 0)$. By Lemma 3.4 the gradient vector ∇f of $f (= f_{i_0 i_1}) := d_p | C_{i_0, i_1}$ at r_{i_1} is given by $-X_{i_0}^\top = -X_{i_1}^\top$ which is the orthogonal projection of $-X_{i_0}$ (or $-X_{i_1}$) to $T_{r_1} C_{i_0, i_1}$, and does not vanish. Since r_{i_1} is not a critical point of f, we may move r_{i_1} along the trajectory of ∇f to a point $r_2 \in \bar C_{i_0, i_1, i_2}$ in general in the following manner. If $k = 2$, then i_2 is uniquely determined, and we have $a_{i_0}X_{i_0} + a_{i_1}X_{i_1} + a_{i_2}X_{i_2} = 0$ at q for some $a_{i_j} > 0$. Since we have $f = x_{i_0}(= \|F_{i_0}\|) = x_{i_1}(= \|F_{i_1}\|)$ and $\langle \nabla f(r), -X_{i_0}(r)\rangle = \langle \nabla f(r), -X_{i_1}\rangle = \|\nabla f(r)\|^2 \geq \delta^2 > 0$ by Lemma 3.4, $x_{i_0} = x_{i_1}$ increases along the trajectory of ∇f. On the other hand, x_{i_2} decreases along the trajectory, because from (3.3) we have at q

$$\langle -X_{i_2}(q), \nabla f(q)\rangle = -(a_{i_0} + a_{i_1})\|X^\top\|^2(q)/a_{i_2} < 0,$$

and $\langle -X_{i_2}(r), \nabla f(r)\rangle < 0$, $r \in U$ for small U.

If $k > 2$, then from Lemma 3.5 there exists at least one index i_2 $(0 \leq i_2 \leq k)$ different from i_0, i_1 such that $\langle X_{i_2}, \nabla f\rangle > 0$. It follows again that $x_{i_0} = x_{i_1}$ increases while x_{i_2} decreases for such an i_2 along the trajectory of ∇f. Therefore, along the trajectory we reach the point r_2 such that the values of $x_{i_0}, x_{i_1}, x_{i_2}$ are equal, while the value of other $x_j(= \|F_j\|)$ $(j \neq i_0, i_1, i_2)$ is not less than this value, namely a point of $\bar{C}_{i_0,i_1,i_2}$ for some i_2. Note that for a starting point $r \in D_{i_0}$ the stratum C_I, $I \supset \{i_0, i_1, i_2\}$ of $C(p)$ containing the above r_2 is uniquely determined and the trajectory is transversal to C_I at r_2 unless $I = K$. We have $l = 2$ for generic r, and then we denote the above r_2 also by r_{i_2}. Let z_{i_1} the parameter value of r_2 of the trajectory, namely $z_{i_1} = d(p, r_{i_2}) - d(p, r_{i_1})(> 0)$, which is also uniquely determined from r and depends smoothly on r. Now for generic $r \in D_{i_0}$, repeating this procedure k times, we may have $r_{i_1}, \ldots, r_{i_k}$ and $z_{i_0}, \ldots, z_{i_{k-1}}$, where $r_{i_k} \in C_{k+1,q}$ and $z_{i_{k-1}} = d(p, r_{i_k}) - d(p, r_{i_{k-1}})(> 0)$. Recall that we have local coordinates $(x_{k+1}, \ldots, x_n)$ of C_{k+1} around q adapted to the smooth Morse function $f := d_p \mid C_{k+1,q}$.

On the other hand, if $r \in D_{i_0}$ moves along γ_r to $r_1 \in C_I, I = \{i_0, \ldots, i_a\}$ $(a < k)$, then γ_r intersects C_I transversely. In this case, we set $r_{i_1} = \cdots = r_{i_a} := r_1$, and $z_{i_0} = d(p, r_1) - d(p, r) = l_r - d(p, r) > 0$, $z_{i_1} = \cdots = z_{i_{a-1}} = 0$. We make a similar arrangement for the case where the starting point r (resp. r_j) belongs to C_I $(a < k)$ (resp. C_I $(j < a)$), and we get $r_{i_1}, \cdots, r_{i_k}$ and $z_{i_0}, \cdots, z_{i_{k-1}}(\geq 0)$ for every $r \in \bar{D}_{i_0}$, the closure of D_{i_0}. For instance, for $r \in C_I$ we set $r_{i_1} = \cdots = r_{i_a} := r$ and $z_{i_0} = \cdots = z_{i_{a-1}} = 0, z_{i_a} = d(p, r_{i_{a+1}}) - d(p, r)$, etc.

Then setting $D_{i_0,I}$ $(I = \{i_0, i_1, \ldots, i_a\} \subset K = \{0, 1, \ldots, k\})$ as the set of points $r \in D_{i_0}$ such that r reaches the point r_1 of C_I along the geodesic γ_r in the above manner, D_{i_0} is stratified by $D_{i_0,I}$'s. Indeed, $D_{i_0,I}$ is a submanifold of codimension $a-1$ and is independent of the order of $\{i_1,\dots,i_a\}$, in fact we have $D_{i_0,I}=(\bar{D}_{i_0,i_1}\cap\cdots\cap\bar{D}_{i_0,i_l})\setminus C(p)$. It follows that

$$U=\{C(p)\cap U\}\bigcup\{\cup_{0\le i_0\le k}(\cup_{I\ni i_0}D_{i_0,I})\},$$

namely, U is stratified into submanifolds given by $\{D_{i_0,I}\}$ and the stratification $\{C_I\}$ of $C(p)$ given in §2. The boundary $\partial D_{i_0,I}$ of $D_{i_0,I}$ consists of $C_J\subset C(p)$ with $J\supset I$ and $D_{i_0,J}$ with $J\supsetneq I$.

Note that on D_{i_0} we have the smooth unit vector field $-X_{i_0}=\nabla d_p$ which is transversal to the boundary, and on a stratum $C_I\subset C(p)$ where I is a proper subset of $\{0,1,\dots,k\}$ with $\sharp I\ge 2$ we have the nonvanishing smooth vector field $-X_I^{\top}=\nabla f_I$ transversal to the boundary. For I with $\sharp I\ge 2$ we also write $D_I=C_I$ in the above notation. Then $r_{i_1},\dots,r_{i_k},z_{i_0},\dots,z_{i_{k-1}}$ mentioned above are obtained by the successive trajetories of these vector fields on D_I. Indeed, for $\emptyset\ne I_0\subsetneq I_1\subsetneq\cdots\subsetneq I_l\subsetneq K:=\{0,\dots,k\}$ we denote by $D_{I_0,I_1,\dots,I_l}$ the set of points $r\in D_{I_0}$ that reach the points $\pi(r)=r_{i_k}$ of $C_{k+1}\cap U$ along the successive trajectories of $-X_{i_0}$(or $-X_{I_0}^{\top}$) and $-X_{I_1}^{\top},\dots,-X_{I_l}^{\top}$. We have an adapted chart $(x_{k+1},\dots,x_n)$ on C_{k+1} around q to the Morse function $f:=d_p\mid C_{k+1}$, and we set $x_{k+j}(r):=x_{k+j}(\pi(r))$. Then $D_{I_0,I_1,\dots,I_l}$ is a submanifold of codimension $k-l-1$ and its closure consists of $D_{\bar{I}_0,\dots,\bar{I}_{l'}}$ $(l'\le l)$ with $I_0\subset\bar{I}_0,\dots I_{l'}\subset\bar{I}_{l'}$. Once $D_{I_0,I_1,\dots,I_l}$ is fixed, $(x_{k+1},\dots,x_n,z_{i_0},\dots,z_{i_{k-1}})$ are uniquely determined for $r\in D_{I_0,I_1,\dots,I_l}$ and smooth where precisely $(l+1)$ of z_i's are positive. Note that

$$D_{I_0,I_1,\dots,I_l}\ni r\mapsto(x_{k+1}(r),\dots,x_n(r),z_{i_0}(r),\dots,z_{i_{k-1}}(r))$$

is an embedding into $\boldsymbol{R}^n$, since $r\mapsto(r_{i_1},z_{i_0})\mapsto(r_{i_2},z_{i_0},z_{i_1})\mapsto\cdots\mapsto(r_{i_k},z_{i_0},\dots,z_{i_{k-1}})$ define an embedding at each stage. U is stratifed into submanifolds $D_{I_0,I_1,\dots,I_l}$ and $D_K:=C_{k+1}\cap U$ (corresponding to $l=-1$), and above embeddings are piecewise smoothly extended to the whole U. Thus we have "local coordinates" $(x_{k+1},\dots,x_n,z_1,\dots,z_k)$ around q, where $(z_1,\dots,z_k)$ is given by $(z_{i_0},\dots,z_{i_{k-1}})$ when restricted to $D_{I_0,I_1,\dots,I_l}$ and Lipschitz on U.

Now recalling the local structure around q of M given by the cone structure of the cut locus mentioned above, we see that for a fixed point $r\in C_{k+1}\cap U$ with $z_1=\ldots=z_k=0$, the set defined by the equation $z_1+\ldots+z_k\le\delta$

(resp.$= \delta$) with $x_{k+1} = x_{k+1}(r), \dots, x_n = x_n(r)$ is homeomorphic to k-disk (resp.$(k-1)$-dimensional sphere). Indeed, we reverse the above procedure of defining $(z_1, z_2, \dots, z_k)$. First for a fixed $r \in C_{k+1} \cap U$, note that the equation $z_1 = \cdots = z_{k-1} = 0, z_k = \delta$ represents k points (i.e., 0-disks) $r_{i_k} \in C_{K\setminus\{i_k\}}$ with $d(r, r_{i_k}) = \delta$ such that r_{i_k} moves to r along the trajectory of $\nabla f_{K\setminus\{i_k\}}$ in $C_{K\setminus\{i_k\}}$. Next the intersection of the set $z_1 = \cdots = z_{k-2} = 0, z_{k-1} + z_k = \delta$ with $C_{K\setminus\{i_{k-1}, i_k\}}$ is a curve (i.e., 1-disk) joining $r_{i_{k-1}}$ and r_{i_k}. Then $z_1 = \cdots = z_{k-2} = 0, z_{k-1} + z_k = \delta$ defines a family of curves corresponding to the 1-skelton of the triangulation of $\tilde{S}_{k-2}(q)$ given in Lemma 2.6. Repeating the procedure we see that the intersection of the set $z_1 + \cdots + z_k = \delta$ with each $\bar{D}_{i_0}$ is a $(k-1)$-disk, and that the set given by $z_1 + \cdots + z_k = \delta$ is Lipschitz homeomorphic to a $(k-1)$-sphere.

Now suppose $k < n$. Since q is a nondegenerate critical point of $f := d_p | C_{k+1,q}$ by Lemma 3.2, we may take local coordinates $\{x_{k+1}, \dots, x_n\}$ around q in $C_{k+1,q}$ so that we may write

$$d(\pi(r), p) = d(r_{i_k}, p) = d(p, q) \pm x_{k+1}^2 \pm \cdots \pm x_n^2.$$

It follows that we have for r in an open neighborhood U of q

$$d_p(r) = d(p, q) \pm x_{k+1}^2(r) \pm \cdots \pm x_n^2(r) - z_1(r) - \cdots - z_k(r), \tag{3.4}$$

where x_{k+j} are smooth and z_j are nonnegative Lipschitz functions. Therefore, we may consider that the index of d_p at the critical point $q \in C_{k+1}$ is given by the sum of the index of $f := d_p | C_{k+1,q}$ at q and k, where k is also equal to the degree of q (see Definition 3.1).

If $k = n$, then $\dim C_{n+1} = 0$ and C_{n+1} consists of vertices, i.e., strata of dimension 0 of $C(p)$. If such a vertex q is a critical point of d_p in the angle sense, then above procedure implies that we may write around q

$$d_p(r) = d(p, q) - z_1(r) - \cdots - z_n(r)$$

with $z_j \geq 0$ and we may consider that q is a critical point of index n.

Summing up we have

Lemma 3.6. *Let $q \in C_{k+1}$ be a critical point in the angle sense of nondegenerate distance function d_p. Then we have a stratification of a neighborhood U of q by submanifolds $D_{I_0, I_1, \dots, I_l}$'s and embeddings*

$$\phi : D_{I_0, I_1, \dots, I_l} \ni r \mapsto (z_1(r), \dots, z_k(r), x_{k+i}(r), \dots, x_n(r)) \in \boldsymbol{R}^n$$

with $z_i(r) \geq 0$ such that

$$\begin{aligned} d_p(r) =& d(p,q) - z_1(r) - \cdots - z_k(r) - x_{k+1}(r)^2 - \cdots - x_{k+j}(r)^2 \\ &+ x_{k+j+1}(r)^2 + \cdots + x_n(r)^2 \end{aligned}$$

where j is the index of $f := d_p|C_{k+1}$ at q in the usual sense. $i := k+j$ is called the index of d_p at the critical point q. ϕ's are piecewise smoothly extended to the whole U and x_{k+i}'s are smooth and z_i's are Lipschitz functions on U. Moreover for a fixed point $r \in C_{k+1} \cap U$ with $z_1 = \ldots = z_k = 0$, the set defined by the equation $z_1 + \ldots + z_k \leq \delta$ with $x_{k+1} = x_{k+1}(r), \ldots, x_n = x_n(r)$ is homeomorphic to k-disk.

Now if q is not a critical point of d_p, we may make use of the isotopy lemma. Next suppose $q \in C(p)$ is a critical point of index i of d_p in the angle sense that is isolated by the nondegeneracy assumption. Then we may apply the usual procedure of Morse theory ([14]). Indeed, around q we have

$$d_p(r) = d(p,q) - z_1 - \cdots - z_k - x_{k+1}^2 - \cdots - x_{k+j}^2 + x_{k+j+1}^2 + \cdots + x_n^2,$$

where $i = k+j$ is equal to the index of q. Then the subset defined by

$$\begin{gathered} -z_1 - \cdots - z_k - x_{k+1}^2 - \cdots - x_{k+j}^2 + x_{k+j+1}^2 + \cdots + x_n^2 = c \\ (\text{resp.} - z_1 - \cdots - z_k - x_{k+1}^2 - \cdots - x_{k+j}^2 + x_{k+j+1}^2 + \cdots + x_n^2 = -c) \end{gathered}$$

for sufficiently small $c > 0$ is homeomorphic to $S^{n-k-j-1} \times I^{k+j}$ (resp. $(S^{j-1} * S^{k-1}) \times I^{n-k-j}$, where $S^{j-1} * S^{k-1}$ denotes the spherical join and homeomorphic to S^{k+j-1}).

Then for sufficiently small $\epsilon > 0$, levels $d_p^{-1}(t) \cap U$ are homeomorphic to $S^{n-i-1} \times I^i$ (resp. $S^{i-1} \times I^{n-i}$) for t with $0 < t - l \leq \epsilon$ (resp. $-\epsilon \leq t - l < 0$), and $\{r \in U \mid d(p,r) \leq l + \epsilon\}$ has the homotopy type of $\{r \in U \mid d(p,r) \leq l - \epsilon\}$ with an i-cell attached. In the above, we set $S^{n-i-1} \times I^i = \emptyset$ for $i = n$. Since we may construct a nowhere vanishing gradient-like vector field of d_p outside of a small neighborhood of the set of critical points of d_p, we apply the usual procedure of Morse theory to get the following main result in this section.

Theorem 3.7. *Let (M,g) satisfy the condition (C) at p. Suppose all cut points of p are nondegenerate cut points and d_p is a nondegenerate distance function. Let q be a critical point of d_p in the angle sense. Note that q is* *a cut point of p and we denote by k the degree of q, namely $q \in C_{k+1}(p)$. Then q is a critical point of a smooth function $f := d_p|C_{k+1}$ on $C_{k+1,q}$ and we define the index of d_p at q as the sum of k and the index of f at q in the usual sense. Then M has the homotopy type of a CW-complex, with one cell of dimension i for each critical point in the angle sense of index i.*

Remark 3.8. Among all Riemannian metrics on M satisfying the condition (C) at p, is the set of Riemannian metrics on M such that all cut points of p are nondegenerate cut points and d_p is a nondegenerate distance function dense in C^∞ topology ? For two-dimensional case, V. Gershkovich asserts that the assertion holds ([3], [5]).

Now we consider the case where the Riemannian metric g satisfies the following condition (F):

Definition 3.9. *A compact Riemannian manifold (M, g) satisfies the condition (F) at $p \in M$, if for any unit speed geodesic γ emanating from p and any Jacobi field J along γ satisfying the initial condition $Y(0) = 0, \nabla_{\dot\gamma} Y(0) \neq 0$, we have $\langle Y(t), \nabla_{\dot\gamma} Y(t)\rangle > 0$ for parameter values $t > 0$ up to the cut distance to p along γ.*

If (M, g) satisfies the condition (F) at p, then minimal geodesics are conjugate points-free, namely the condition (C) holds. However, considering the real projective spaces of positive constant curvature for which the cut locus of any point is disjoint from the conjugate locus, we see that the converse does not necessarily holds. On the other hand, if (M, g) is of nonpositive sectional curvature, then it satisfies the condition (F) at every point $p \in M$. Note that the condition (F) means that cut points are not focal points, and is also essentially given in [5].

Lemma 3.10. *Suppose (M, g) satisfies the condition (F) at p. Then for any critical point $q \in C_{k+1}$ $(k < n)$ of d_p in the angle sense, that is also a critical point of the smooth function $f = d_p|C_{k+1,q}$, its Hessian $D^2 f(q)$ is positive definite. Namely, f assumes a local minimum at r.*

Proof. To see the assertion recall the second variation formula (2.10). As before choose $a_i \geq 0$ $(0 \leq i \leq k)$ with $\sum a_i = 1$ so that $\sum a_i X_i = 0$ holds. Recall that $\dim C_{k+1,q} > 0$. Then for any $u \in T_q C_{k+1}(u \neq 0)$ take a unique

Jacobi field Y_i along γ_i with $Y_i(0) = 0, Y_i(l) = u$, $l = d(p,q)$. By (2.10) we obtain

$$\begin{aligned} D^2 f(u,u) &= \sum a_i D^2 f(u,u) \\ &= \sum a_i \langle Y_i(l), \nabla_{\dot\gamma_i} Y_i(l)\rangle - \langle \sum a_i X_i, \nabla_u \dot x(s)\rangle \\ &= \sum a_i \langle Y_i(l), \nabla_{\dot\gamma_i} Y_i(l)\rangle > 0, \end{aligned} \tag{3.5}$$

and this completes the proof of the lemma. Note that here we do not need to assume that the critical point q is nondegenerate in the sense of Definition 3.3. □

Therefore, we may abbreviate the condition (2) in Definition 3.3 of nondegeneracy for a distance function d_p if the condition (F) is satisfied. Now suppose a compact Riemannian manifold (M, g) of dimension n satisfies the condition (F) at p and the cut locus $C(p)$ consists of nondegenerate cut points. Let $q \in C_{k+1}$ be a critical point of a nondegenerate d_p. We want to describe the behavior of the distance function d_p in a neighborhood U of q, and follow the argument as before. For instance, if $k = 1$ then by (3.2) and Lemma 3.10 we have

$$d_p(x) = d_p(q) + y_1^2 + \cdots + y_{n-1}^2 - z, \tag{3.6}$$

with $z = d(p, x_1) - d(p, x)$ where x_1 denotes the cut point of p along the minimal geodesic joining p to x. Therefore, we may consider that the index of d_p at the critical point q is given by 1, which is also equal to the degree of q (see Definition 3.1). Next, suppose $k < n$. Since q is a local minimum point of $f := d_p | C_{k+1,q}$ by Lemma 3.10, we may take local coordinates $\{x_{k+1}, \ldots, x_n\}$ around q in $C_{k+1,q}$ so that we may write for $r \in D_{i_0}$

$$d(r_{i_k}, p) = d(p,q) + x_{k+1}^2 + \cdots + x_n^2,$$

where $r_{i_k} \in C_{k+1,q}$ is uniquely determined from r by the procedure given before the statement of Theorem 3.7. It follows that we have for $r \in U$

$$d_p(r) = d(p,q) + x_{k+1}^2 + \cdots + x_n^2 - z_1 - \cdots - z_k, \tag{3.7}$$

where z_j are nonnegative and given by $(z_{i_0}, z_{i_1}, \ldots, z_{i_k})$ as described before. Therefore, we may regard that the index of d_p at the critical point $q \in C_{k+1}$ is given by k, which is also equal to the degree of q (see Definition 3.1). If $k = n$, then the situation is the same as before.

Note that if (M,g) satisfies the condition (F) at p then $C_{n+1} \neq \emptyset$ for $C(p)$, since points which are furthest from p are of index n and therefore of order $n+1$.

Theorem 3.11. *Suppose (M,g) satisfies the condition (F) at p. If all cut points of p are nondegenerate and d_p is a nondegenerate distance function, then we may perform the k-cell attaching at each critical point in C_{k+1} of d_p in the angle sense.*

Finally we give an easy application of Theorem 3.11.

Corollary 3.12. *Let (M,g) be a Riemannian manifold satisfying the condition (F) at $p \in M$. If all cut points of p are nondegenerate cut points and d_p is a nondegenerate distance function, then the number of connected components of C_{k+1} is greater than or equal to the k-th Betti number of M.*

Proof. The critical points of the distance function are local minimal points on C_{k+1}. Then the index of each critical points on C_k is equal to k and we get the corollary by the Morse inequality. □

Remark 3.13. In [11] some kind of Morse theory was discussed by using distance function. In particular, in section 3 of [11] one of the authors constructed a metric by using handle attaching in Morse theory.

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