Math. J. Okayama Univ. **49** (2007), 59–64

SOME RESULTS ON (σ, τ)-LIE IDEALS

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Abstract. In this note we give some basic results on one sided $(\sigma, \tau)-$Lie ideals of prime rings with characteristic not 2.

1. Introduction

Let R be a ring and σ, τ be two mappings from R into itself. We write $[x, y] = xy - yx$, and $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ for $x, y \in R$. For subsets $A, B \subset R$, let $[A, B]$ be the additive subgroup generated by all $[a, b]$, and $[A, B]_{\sigma,\tau}$ be the additive subgroup generated by all $[a, b]_{\sigma,\tau}$ for $a \in A$ and $b \in B$. We recall that a Lie ideal L is an additive subgroup of R such that $[R, L] \subset L$. We first introduce the generalized Lie ideal in [3] as follows. Let U be an additive subgroup of R. (i) U is called a $(\sigma, \tau)-$right Lie ideal of R if $[U, R]_{\sigma,\tau} \subset U$, (ii) U is called a $(\sigma, \tau)-$left Lie ideal if $[R, U]_{\sigma,\tau} \subset U$. (iii) U is called a $(\sigma, \tau)-$Lie ideal if U is both a $(\sigma, \tau)-$right and a $(\sigma, \tau)-$left Lie ideal. An additive mapping $d : R \longrightarrow R$ is called a $(\sigma, \tau)-$derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$. We write $C_{\sigma,\tau} = \{c \in R \mid c\sigma(r) = \tau(r)c \text{ for } r \in R\}$, and will make extensive use of the following basic commutator identities:

$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$

$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)$

Throughout the present paper, R will represent a prime ring (of $char R \neq 2$, exclude Lemmas 1 and 2) and $\sigma, \tau, \alpha, \beta, \lambda$ and μ will be automorphisms of R. In this note, we give the following proporties on prime rings and some results on one sided $(\sigma, \tau)-$Lie ideals. Let I be a nonzero ideal of R. (1) If $[[I, a]_{\sigma,\tau}, b]_{\alpha,\beta} = 0$ for $a, b \in R$, then $[\tau(a), \beta(b)] = 0$. (2) If $[[a, I]_{\sigma,\tau}, b]_{\alpha,\beta} = 0$ for $a, b \in R$, then $b \in Z$ or $[a, \tau^{-1}\beta(b)]_{\sigma,\tau} = 0$. (3) If $[b, [a, R]_{\sigma,\tau}]_{\alpha,\beta} = 0$ for $a, b \in R$, then $b \in C_{\alpha,\beta}$, $a \in C_{\sigma,\tau}$ or $a + \tau\sigma^{-1}(a) \in C_{\sigma,\tau}$. On the other hand, in [4] Park and Jung proved that if $d : R \longrightarrow R$ is a nonzero $(\sigma, \tau)-$derivation and $a \in R$ such that $d[R, a]_{\sigma,\tau} = 0$, then $\sigma(a) + \tau(a) \in Z$. We prove that if $d : R \longrightarrow R$ is a nonzero $(\sigma, \tau)-$derivation and $a \in R$ such that $d[a, R]_{\alpha,\beta} = 0$, then $a \in C_{\alpha,\beta}$ or $a + \beta\alpha^{-1}(a) \in C_{\alpha,\beta}$.

2. Results

The following Lemmas 1 and 2 are generalizations of [1, Lemma 1.5].

Mathematics Subject Classification. 16N10, 16W25, 16U80.
Key words and phrases. Prime ring, (σ,τ)-Lie ideal, (σ,τ)-derivation.

Lemma 1. *Let I be a nonzero ideal of R and $a, b \in R$. If $[[I, a]_{\sigma,\tau}, b]_{\alpha,\beta} = 0$, then $[\tau(a), \beta(b)] = 0$.*

Proof. Let $[[I, a]_{\sigma,\tau}, b]_{\alpha,\beta} = 0$. Then we have , $0 = [[\tau(a)y, a]_{\sigma,\tau}, b]_{\alpha,\beta} = [\tau(a)[y, a]_{\sigma,\tau} + [\tau(a), \tau(a)]y, b]_{\alpha,\beta} = \tau(a)[[y, a]_{\sigma,\tau}, b]_{\alpha,\beta} + [\tau(a), \beta(b)][y, a]_{\sigma,\tau}$ for all $y \in I$.This gives that

$$[\tau(a), \beta(b)][y, a]_{\sigma,\tau} = 0 \text{ for all } y \in I. \tag{2.1}$$

Replacing $yr, r \in R$ by y in (2.1), we get $0 = [\tau(a), \beta(b)]y[r, \sigma(a)] + [\tau(a), \beta(b)][y, a]_{\sigma,\tau}r$ and so

$$[\tau(a), \beta(b)]y[r, \sigma(a)] = 0 \quad \text{for all} \quad y \in I, r \in R. \tag{2.2}$$

Since R is prime, we get

$$[\tau(a), \beta(b)] = 0 \text{ or } a \in Z. \tag{2.3}$$

Thus, $[\tau(a), \beta(b)] = 0$ is obtained for two cases in (2.3) □

Corollary 1. ***(1)*** *If I is a nonzero ideal of R and $a \in R$ such that $[I, a]_{\alpha,\beta} \subset C_{\lambda,\mu}$, then $a \in Z$.*

(2) *Let U be a nonzero $(\sigma, \tau)-$right(left) Lie ideal of R and I a nonzero ideal of R. If $[[I, I]_{\sigma,\tau}, U]_{\alpha,\beta} = 0$ then $U \subset Z$.*

(3) *If $a \in R$ such that $[[I, I]_{\sigma,\tau}, a]_{\alpha,\beta} = 0$ then $a \in Z$.*

Proof. **(1)** $[I, a]_{\alpha,\beta} \subset C_{\lambda,\mu}$ implies that $[[I, a]_{\alpha,\beta}, R]_{\lambda,\mu} = 0$. By Lemma 1 we obtain that $[\beta(a), \mu(R)] = 0$. Since μ is onto, we have $\beta(a) \in Z$ and so $a \in Z$.

(2) By Lemma 1 we have $[\tau(I), \beta(U)] = 0$ and so $U \subset Z$.

(3) $[[I, I]_{\sigma,\tau}, a]_{\alpha,\beta} = 0$ implies that $[\tau(I), \beta(a)] = 0$ by Lemma 1 and so $a \in Z$. □

Lemma 2. *Let I be a nonzero ideal of R. If $a, b \in R$ and $[[a, I]_{\sigma,\tau}, b]_{\alpha,\beta} = 0$, then $b \in Z$ or $[a, \tau^{-1}\beta(b)]_{\sigma,\tau} = 0$.*

Proof. For any $x, y \in I$ we have

$$\begin{aligned} 0 &= [[a, xy]_{\sigma,\tau}, b]_{\alpha,\beta} \\ &= [\tau(x)[a, y]_{\sigma,\tau} + [a, x]_{\sigma,\tau}\sigma(y), b]_{\alpha,\beta} \\ &= \tau(x)[[a, y]_{\sigma,\tau}, b]_{\alpha,\beta} + [\tau(x), \beta(b)][a, y]_{\sigma,\tau} + [a, x]_{\sigma,\tau}[\sigma(y), \alpha(b)] \\ &\quad + [[a, x]_{\sigma,\tau}, b]_{\alpha,\beta}\sigma(y) \end{aligned}$$

and so

$$[\tau(x), \beta(b)][a, y]_{\sigma,\tau} + [a, x]_{\sigma,\tau}[\sigma(y), \alpha(b)] = 0 \text{ for all } x, y \in I. \tag{2.4}$$

Replacing x by $rx, r \in R$ in (2.4) we get

$$\begin{aligned}0 &= [\tau(rx),\beta(b)][a,y]_{\sigma,\tau} + [a,rx]_{\sigma,\tau}[\sigma(y),\alpha(b)]\\ &= \tau(r)[\tau(x),\beta(b)][a,y]_{\sigma,\tau} + [\tau(r),\beta(b)]\tau(x)[a,y]_{\sigma,\tau} + \tau(r)[a,x]_{\sigma,\tau}[\sigma(y),\alpha(b)]\\ &\quad + [a,r]_{\sigma,\tau}\sigma(x)[\sigma(y),\alpha(b)].\end{aligned}$$

That is
(2.5)
$$[\tau(r),\beta(b)]\tau(x)[a,y]_{\sigma,\tau} + [a,r]_{\sigma,\tau}\sigma(x)[\sigma(y),\alpha(b)] = 0 \text{ for all } x,y \in I, r \in R.$$

If we take $\tau^{-1}\beta(b)$ instead of r in (2.5) then we have

$$[a,\tau^{-1}\beta(b)]_{\sigma,\tau}\sigma(I)[\sigma(I),\alpha(b)] = 0. \tag{2.6}$$

Since $\sigma(I) \neq 0$ an ideal of R and R is prime we get

$$[a,\tau^{-1}\beta(b)]_{\sigma,\tau} = 0 \text{ or } [\sigma(I),\alpha(b)] = 0. \tag{2.7}$$

Since R is prime, $[\sigma(I),\alpha(b)] = 0$ implies that $b \in Z$. Thus $[a,\tau^{-1}\beta(b)]_{\sigma,\tau} = 0$ or $b \in Z$ is obtained. □

Lemma 3. *Let U be a nonzero $(\sigma,\tau)-$right Lie ideal of R and $a \in R$. If $[U,a]_{\alpha,\beta} = 0$, then $a \in Z$ or $U \subset C_{\sigma,\tau}$.*

Proof. Since $[[U,R]_{\sigma,\tau},a]_{\alpha,\beta} \subset [U,a]_{\alpha,\beta} = 0$ then we have

$$a \in Z \text{ or } [U,\tau^{-1}\beta(a)]_{\sigma,\tau} = 0$$

by Lemma 2. If $[U,\tau^{-1}\beta(a)]_{\sigma,\tau} = 0$ then $a \in Z$ or $U \subset C_{\sigma,\tau}$ by [6, Lemma 2]. □

Theorem 1. *Let U be a nonzero $(\sigma,\tau)-$right Lie ideal of R and $I \neq 0$ an ideal of R.*

(1) *If $a \in R$ and $[[U,I]_{\alpha,\beta},a]_{\lambda,\mu} = 0$, then $a \in Z$ or $U \subset C_{\sigma,\tau}$.*

(2) *If $[U,I]_{\alpha,\beta} \subset C_{\lambda,\mu}$, then $U \subset C_{\sigma,\tau}$ or R is commutative.*

Proof. **(1)** $[[U,I]_{\alpha,\beta},a]_{\lambda,\mu} = 0$ implies that $a \in Z$ or $[U,\beta^{-1}\mu(a)]_{\alpha,\beta} = 0$, by Lemma 2. If $[U,\beta^{-1}\mu(a)]_{\alpha,\beta} = 0$ then $a \in Z$ or $U \subset C_{\sigma,\tau}$ by Lemma 3.

(2) Let $[U,I]_{\alpha,\beta} \subset C_{\lambda,\mu}$ then we have $[[U,I]_{\alpha,\beta},R]_{\lambda,\mu} = 0$. If we use (1) we get $R \subset Z$ or $U \subset C_{\sigma,\tau}$ and so $U \subset C_{\sigma,\tau}$ or R is commutative. □

Theorem 2. *Let d be a nonzero $(\sigma,\tau)-$derivation on R and $a \in R$. If $d[a,R]_{\alpha,\beta} = 0$, then $a \in C_{\alpha,\beta}$ or $a + \beta\alpha^{-1}(a) \in C_{\alpha,\beta}$.*

Proof. For any $x,y \in R$ we have

$$\begin{aligned}0 = d[a,xy]_{\alpha,\beta} &= d(\beta(x)[a,y]_{\alpha,\beta} + [a,x]_{\alpha,\beta}\alpha(y))\\ &= d\beta(x)\sigma[a,y]_{\alpha,\beta} + \tau[a,x]_{\alpha,\beta}d\alpha(y)\end{aligned}$$

Replacing x by $\beta^{-1}[a,z]_{\alpha,\beta}$ in the last relation we get

$$[a,\beta^{-1}[a,z]_{\alpha,\beta}]_{\alpha,\beta}d\alpha(y)=0 \text{ for all } y,z\in R$$

and so

$$[a,\beta^{-1}[a,z]_{\alpha,\beta}]_{\alpha,\beta}=0 \text{ for all } z\in R \tag{2.8}$$

by [5,Lemma 3]. Taking zy for z in (2.8) we get

$$\begin{aligned}0&=[a,\beta^{-1}[a,zy]_{\alpha,\beta}]_{\alpha,\beta}=[a,\beta^{-1}(\beta(z)[a,y]_{\alpha,\beta}+[a,z]_{\alpha,\beta}\alpha(y))]_{\alpha,\beta}\\&=[a,z\beta^{-1}[a,y]_{\alpha,\beta}+\beta^{-1}[a,z]_{\alpha,\beta}\beta^{-1}\alpha(y))]_{\alpha,\beta}\\&=[a,z]_{\alpha,\beta}\alpha\beta^{-1}[a,y]_{\alpha,\beta}+[a,z]_{\alpha,\beta}[a,\beta^{-1}\alpha(y)]_{\alpha,\beta}\end{aligned}$$

which leads to

$$[a,z]_{\alpha,\beta}(\alpha\beta^{-1}[a,y]_{\alpha,\beta}+[a,\beta^{-1}\alpha(y)]_{\alpha,\beta})=0 \text{ for all } z,y\in R. \tag{2.9}$$

Replacing z by zt in (2.9), we get

(2.10)

$$[a,z]_{\alpha,\beta}=0,\forall\ z\in R \text{ or } \alpha\beta^{-1}[a,y]_{\alpha,\beta}+[a,\beta^{-1}\alpha(y)]_{\alpha,\beta}=0 \text{ for all } y\in R.$$

Hence $a\in C_{\alpha,\beta}$ or $0=\alpha\beta^{-1}[a,y]_{\alpha,\beta}+a\alpha\beta^{-1}\alpha(y)-\alpha(y)a$ for all $y\in R$. If we apply α^{-1} and β to the last relation we have $a\alpha(y)-\beta(y)a+\beta\alpha^{-1}(a)\alpha(y)-\beta(y)\beta\alpha^{-1}(a)=0$ for all $y\in R$. This implies that $(a+\beta\alpha^{-1}(a))\alpha(y)-\beta(y)(a+\beta\alpha^{-1}(a))=0$ and so $a+\beta\alpha^{-1}(a)\in C_{\alpha,\beta}$ for all $y\in R$. Thus we obtain $a\in C_{\alpha,\beta}$ or $a+\beta\alpha^{-1}(a)\in C_{\alpha,\beta}$ by (2.10). □

Corollary 2. *If $[b,[a,R]_{\sigma,\tau}]_{\alpha,\beta}=0$, then $a\in C_{\sigma,\tau}$ or $b\in C_{\alpha,\beta}$ or $a+\tau\sigma^{-1}(a)\in C_{\sigma,\tau}$.*

Proof. $d(x)=[b,x]_{\alpha,\beta}$ is a $(\alpha,\beta)-$derivation on R. Furthermore $d[a,R]_{\sigma,\tau}=0$. This implies that $a\in C_{\sigma,\tau}$, $b\in C_{\alpha,\beta}$ or $a+\tau\sigma^{-1}(a)\in C_{\sigma,\tau}$ by Theorem 2. □

Theorem 3. *Let U be a nonzero $(\sigma,\tau)-$right Lie ideal of R and $d:R\longrightarrow R$ a nonzero $(\lambda,\mu)-$derivation.*
  (1) *If $d(U)=0$, then $v+\tau\sigma^{-1}(v)\in C_{\sigma,\tau}$ for all $v\in U$.*
  (2) *If $d[U,R]=0$, then $U\subset Z$.*

Proof. **(1)** Suppose that $d(U)=0$. Then $d[U,R]_{\sigma,\tau}=0$. This implies that $U\subset C_{\sigma,\tau}$ or $v+\tau\sigma^{-1}(v)\in C_{\sigma,\tau}$ for all $v\in U$ by Theorem 2.
  (2) Taking $\alpha=\beta=1$ in Theorem 2, we have $U\subset Z$. □

Theorem 4. *Let U be a nonzero $(\sigma,\tau)-$left Lie ideal of R and $d:R\longrightarrow R$ a nonzero $(\alpha,\beta)-$derivation.*
  (1) *If $d(U)=0$, then $\sigma(v)+\tau(v)\in Z$ for all $v\in U$.*
  (2) *If $a\in R$ and $[U,a]=0$, then $a\in Z$ or $\sigma(v)+\tau(v)\in Z$ for all $v\in U$.*

(3) If $a \in R$ and $[U, a]_{\alpha,\beta} = 0$, then $a \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in U$.

(4) If $[[R, U]_{\alpha,\beta}, a]_{\lambda,\mu} = 0$ then $a \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in U$.

Proof. **(1)** Suppose that $d(U) = 0$. Then $d[R, v]_{\sigma,\tau} = 0$ for all $v \in U$. This implies that $\sigma(v) + \tau(v) \in Z$ for all $v \in U$ by [4, Corollary 5] for all $v \in U$.

(2) Let $d(x) = [x, a]$ for all $x \in R$. Then d is a derivation and furthermore $d(U) = 0$. Thus we have $a \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in U$ by (1).

(3) Since $[[R, U]_{\sigma,\tau}, a]_{\alpha,\beta} \subset [U, a]_{\alpha,\beta} = 0$ we have $[\tau(U), \beta(a)] = 0$ by Lemma 1. That is $[U, \tau^{-1}\beta(a)] = 0$. This implies that $a \in Z$ or $\sigma(v)+\tau(v) \in Z$ for all $v \in U$ by (2).

(4) By Lemma 1 and hypothesis, we have $[\beta(U), \mu(a)] = 0$. That is $[U, \beta^{-1}\mu(a)] = 0$. This implies that $a \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in U$ by (2). □

Remark 1. *Let U be a nonzero $(\sigma, \tau)-$left Lie ideal of R such that $[U, U]_{\alpha,\beta} = 0$. Then we have $\sigma(v) + \tau(v) \in Z$ for all $v \in U$.*

Proof. By Theorem 4(3) we have $\sigma(v) + \tau(v) \in Z$ for all $v \in U$. □

Theorem 5. *Let U be a nonzero $(\sigma, \tau)-$left Lie ideal of R and $a \in R$.*

(1) If $[a, U]_{\alpha,\beta} = 0$, then $a \in C_{\alpha,\beta}$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in U$.

(2) If $[a, [R, U]_{\alpha,\beta}]_{\lambda,\mu} = 0$, then $a \in C_{\lambda,\mu}$ or $\alpha(v)+\beta(v) \in Z$ for all $v \in U$.

(3) If $[R, U]_{\alpha,\beta} \subset C_{\lambda,\mu}$, then R is commutative or $\sigma(v) = \tau(v)$ for all $v \in U$.

(4) If $U \subset C_{\lambda,\mu}$, then $\sigma(v) = \tau(v)$ for all $v \in U$ or R is commutative.

Proof. **(1)** Let $d(x) = [a, x]_{\alpha,\beta}$ for all $x \in R$. Then d is $(\alpha, \beta)-$derivation of R. Since $[a, [R, U]_{\sigma,\tau}]_{\alpha,\beta} \subset [a, U]_{\alpha,\beta} = 0$ then we have $d[R, U]_{\sigma,\tau} = 0$. This implies that $a \in C_{\alpha,\beta}$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in U$ by [4,Corollary 5].

(2) Considering as in the proof (1) we obtain the result.

(3) Suppose that $[R, U]_{\alpha,\beta} \subset C_{\lambda,\mu}$. Then we have $[[R, U]_{\alpha,\beta}, R]_{\lambda,\mu} = 0$. This gives $[\beta(U), \mu(R)] = 0$ by Lemma 1 and so $U \subset Z$. Thus $[R, U]_{\sigma,\tau} \subset U \subset Z$ is obtained. For any $r, s \in R, v \in U$ we have $0 = [[r, v]_{\sigma,\tau}, s] = [r\sigma(v) - \tau(v)r, s] = [r(\sigma(v) - \tau(v)), s] = r[\sigma(v) - \tau(v), s] + [r, s]\ (\sigma(v) - \tau(v))$ which leads to

$$[r, s](\sigma(v) - \tau(v)) = 0 \text{ for all } r, s \in R, v \in U. \tag{2.11}$$

Since R is prime and $\sigma(v) - \tau(v) \in Z$ we get

$$[r, s] = 0 \text{ for all } r, s \in R \text{ or } \sigma(v) = \tau(v) \text{ for all } v \in U. \tag{2.12}$$

and so R is commutative or $\sigma(v) = \tau(v)$ for all $v \in U$.

(4) If $U \subset C_{\lambda,\mu}$, then $[R, U]_{\sigma,\tau} \subset C_{\lambda,\mu}$. This implies that R is commutative or $\sigma(v) = \tau(v)$ for all $v \in U$ by (3). □

Acknowledgement

The authors wish to thank the referee for valuable suggestions.

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(*Received November 28, 2005*)
(*Revised September 1, 2006*)