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ON IDEALS AND ORTHOGONAL GENERALIZED DERIVATIONS OF SEMIPRIME RINGS

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Abstract. In this paper, some results concerning orthogonal generalized derivations are generalized for a nonzero ideal of a semiprime ring. These results are a generalization of results of M. Brešar and J. Vukman in [3], which are related to a theorem of E. Posner for the product of derivations on a prime ring.

1. Introduction

Throughout R will represent an associative ring. In [2], Brešar defined the following notion. An additive mapping $D : R \to R$ is said to be a *generalized derivation* if there exists a derivation $d : R \to R$ such that

$$D(xy) = D(x)y + xd(y) \qquad \text{for all } x, y \in R.$$

By the above notion it is easily seen that the concept of a generalized derivations covers both the concepts of a derivation and of a left multiplier. This notion is found in P. Ribenboim [9], where some module structures of these higher generalized derivations were treated. Other properties of generalized derivations were given by B. Hvala [4], T-K. Lee [5] and A. Nakajima [6], [7] and [8].

Two additive maps $d, g : R \to R$ are called *orthogonal* if

$$d(x)Rg(y) = 0 = g(y)Rd(x) \qquad \text{for all } x, y \in R.$$

In [3] Brešar and Vukman introduced the notion of orthogonality for two derivations d and g on a semiprime ring, and they presented several necessary and sufficient conditions for d and g to be orthogonal. In [10] the authors replaced R by a non zero ideal I of R, then they showed that some properties in [[3], Theorem] are also valid in this subconstruction. Finally, in [1] the authors introduced orthogonal generalized derivations on a semiprime ring and they presented some results concerning two generalized derivations on a semiprime ring. Their results are a generalization of results of M. Brešar

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and J. Vukman in [3]. In this paper, our aim is to extend their results to orthogonal generalized derivations on a nonzero ideal I of R.

For a semiprime ring R and an ideal I of R, it is well-known that the left and right annihilators of I in R coincide. Note that $I \cap \ell(I) = 0$ (or $I \cap r(I) = 0$) where $\ell(I)$ and $r(I)$ denote the left annihilator and the right annihilator of I, respectively.

Throughout this paper we assume that R is a 2-torsion free semiprime ring and I is a nonzero ideal of R unless stated otherwise.

2. **Preliminaries**

In the following, we give the notation of orthogonal generalized derivations.

Definition. Two generalized derivations (D, d) and (G, g) of R are called *orthogonal* if

$$D(x)RG(y) = 0 = G(y)RD(x) \quad \text{for all } x, y \in R.$$

The following example shows that there are many pairs of generalized derivations which are orthogonal.

Example 1. *Let* $R = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : x, y, z \in \mathbb{Z}, \textit{the set of integers} \right\}$. *Let* m, p *and* s *be fixed two nonzero elements of* $\mathbb{Z}$ *and the additive maps* D, G, d *and* g *define the following;*
$D\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} 0 & mx+mz \\ 0 & 0 \end{bmatrix}$, $d\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} 0 & mx-mz \\ 0 & 0 \end{bmatrix}$,
$G\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} 0 & pz-ys \\ 0 & 0 \end{bmatrix}$ *and* $g\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} 0 & -ys \\ 0 & 0 \end{bmatrix}$. *Then it is easy to see that* d *and* g *are derivations of* R *and that (D,d) and (G,g) are a generalized derivation on* R *such that (D,d) and (G,g) are orthogonal.*

Now, to obtain the main result, we need the following lemmas:

Lemma 1. ([10], Lemma 1). *Let* R *be a 2-torsion free semiprime ring,* I *a nonzero ideal of* R *and* a, b *the elements of* R. *Then the following conditions are equivalent.*

(i) $axb = 0$ *for all* $x \in I$.

(ii) $bxa = 0$ *for all* $x \in I$.

(iii) $axb + bxa = 0$ *for all* $x \in I$.

Moreover, if one of the three conditions is fulfilled and $\ell(I) = 0$, *then* $ab = ba = 0$.

Lemma 2. ([10], Lemma 3). *Let R be a semiprime ring and I be a nonzero ideal of R. Suppose that additive mappings F and H of R into itself satisfy $F(x)IH(x) = 0$ for all $x \in I$. Then $F(x)IH(y) = 0$ for all $x, y \in I$.*

3. The Results

The main goal of this section is to prove the following theorem, which corresponds to [[3], Theorem 1].

Theorem 1. *Let (D, d) and (G, g) be generalized derivations of R and I be a nonzero ideal of R such that $\ell(I) = 0$. Then the following conditions are equivalent.*
(i) (D, d) and (G, g) are orthogonal.
(ii) For all $x, y \in I$, the following relations hold.
(a) $D(x)G(y) + G(x)D(y) = 0$.
(b) $d(x)G(y) + g(x)D(y) = 0$.
(iii) $D(x)G(y) = d(x)G(y) = 0$ for all $x, y \in I$.
(iv) $D(x)G(y) = 0$ for all $x, y \in I$ and $dG(x) = dg(x) = 0$ for all $x, y \in I$.
(v) (DG, dg) is a generalized derivation on I and $D(x)G(y) = 0$ for all $x, y \in I$.

For the proof of the Theorem 1 we need the following lemmas. In all that follows $x, y, z \in I$ and $r, s, t \in R$.

Lemma 3. *Let (D, d) and (G, g) be generalized derivations of R and $\ell(I) = 0$. If $D(I)IG(I) = 0$, then $D(R)RG(R) = 0$.*

Proof. By $0 = D(x)zG(y) = G(y)zD(x)$ for all $x, y, z \in I$ and Lemma 1, we have $0 = D(x)g(r) = g(r)D(x)$ and by $g(r)D(x) = 0$, we get $0 = g(r)d(s) = d(s)g(r)$. Using these relations, we have $D(s)xg(r) = 0$ and so by $0 = D(xz)G(y)$, we obtain $d(z)G(y) = 0$. Therefore $0 = D(rx)G(sy) = D(r)xG(s)y$, which shows $D(r)xG(s) = 0$. Replace x by $r'G(s)xD(r)r'$ for some $r' \in R$, we have $D(r)r'G(s) = 0$, as desired.
Moreover, we have the following:

Lemma 4. *Let (D, d) and (G, g) be generalized derivations of R and I an ideal of R such that $\ell(I) = 0$. Then the following conditions are equivalent.*
(i) For any $x, y \in I$, the following relations hold.
(a) $D(x)G(y) + G(x)D(y) = 0$.
(b) $d(x)G(y) + g(x)D(y) = 0$.
(ii) $D(x)G(y) = d(x)G(y) = 0$ for all $x, y \in I$.
(iii) $D(x)G(y) = 0$ for all $x, y \in I$ and $dG = dg = 0$ for all $x, y \in I$.
(iv) (DG, dg) is a generalized derivation from I to R and $D(x)G(y) = 0$ for all $x, y \in I$.

Proof. $(i) \Leftrightarrow (ii)$. By (a), (b) Lemmas 1 and 2, we have $0 = D(x)zG(y) = D(x)G(y)$ and using this $d(z)G(y) = 0$. This shows (ii). And the converse is easily obtained by the relations $D(x)G(y) = G(y)D(x) = 0$ and Lemma 1.

$(ii) \Rightarrow (iii)$. By assumption, $D(x)zG(y) = d(x)zg(y) = 0$. Then by Lemma 3, d and g are orthogonal, which shows $dg = 0$. Moreover, by $0 = d(x)G(y)$ and Lemma 1, we have $0 = d(d(r)sG(y)) = d(r)sdG(y)$ for $r, s \in R$. Take $r = G(y)$, we have $dG(y) = 0$. Since $D(x)G(y) = G(y)d(x) = d(x)g(y) = 0$, using Lemma 3 we obtain $dG(r) = 0$, this gives (iii).

$(iii) \Rightarrow (iv)$. By $dG = dg = 0$, we have

$$G(x)d(y) + d(x)g(y) = 0 = g(x)d(y) + d(x)g(y).$$

Then by the proof of $(i) \Rightarrow (ii)$, we see that $d(x)g(y) = 0$ and so $G(x)d(y) = 0$. Moreover, by $0 = D(x)G(y)$, we have $D(x)g(z) = 0$. Therefore $DG(xy) = DG(x)y$ and thus $(DG, dg = 0)$ is a generalized derivation from I to R.

$(iv) \Rightarrow (ii)$. (DG, dg) is a generalized derivation if and only if

$$G(x)d(y) + D(x)g(y) = 0 = d(x)g(y) + g(x)d(y).$$

So we obtain $dg = 0$. Furthermore by $0 = D(x)G(y)$, we get $D(x)g(y) = 0$ and by the above relation, we see $G(x)d(y) = 0$. Therefore $G(x)zd(y) = 0$ and by Lemma 1, we arrive at $d(y)G(x) = 0$. This shows (ii).

Using Lemma 3 and Lemma 4, the proof of Theorem 1 is easily seen as follows:

Proof of Theorem 1. $(i) \Rightarrow (ii), (iii), (iv)$ and (v) are clear by [[1], Theorem 1]. Since (ii), (iii), (iv) and (v) are equivalent by Lemma 4, we assume (iii). This implies that $0 = (D(x)z + xd(z))G(y) = D(x)zG(y)$. Then we have $D(I)IG(I) = 0$. Thus by Lemma 3, we have Theorem 1 $(iii) \Rightarrow (i)$.

Remark 1. *If (DG, dg) is a generalized derivations on I and $\ell(I) = 0$ then (DG, dg) is a generalized derivations on R.*

Proof. It is easily seen that (DG, dg) is a generalized derivations on I if and only if

$$G(x)d(y) + D(x)g(y) = 0, \quad d(x)g(y) + g(x)d(y) = 0.$$

Then by the second relation, we have d and g are orthogonal. By the first relation $0 = G(x)d(y) + D(x)g(y)$, we get $0 = G(x)zd(y) + D(x)zg(y)$. Hence replacing z by $g(y)z$ in this relation and using the orthogonality of the derivations d and g, we obtain $0 = D(x)g(y)zg(y)$ which implies that $D(x)g(y) = G(x)d(y) = 0$. Moreover by $0 = D(x)g(yr)$, we get $0 = D(x)g(r)$ for all $r \in R$. Using this relation we have $D(s)xg(r) = 0$ and similarly we can see that $D(s)g(r) = G(s)d(r) = 0$. Thus we obtain $DG(rs) = DG(r)s$ for all $r, s \in R$ which completes the proof.

Theorem 2. *Let (D, d) and (G, g) be generalized derivations of R and $\ell(I) = 0$. Then the following conditions are equivalent.*
(i) (DG, dg) is a generalized derivation on I.
(ii) (GD, gd) is a generalized derivation on I.
(iii) D and g are orthogonal, and G and d are orthogonal.

The proof of the Theorem 2 is clear by Remark 1 and [[1], Theorem 2].

Corollary 1. *Let (D, d) be a generalized derivations of R and $\ell(I) = 0$. If (D^2, d^2) is a generalized derivation on I, then $d = 0$.*

Proof. The fact that (D^2, d^2) is a generalized derivation on I is implies that d and d are orthogonal. Therefore we get $d = 0$ by the semiprimeness of R.

Corollary 2. *Let (D, d) be a generalized derivations of R and $\ell(I) = 0$. If $D(x)D(y) = 0$ for all $x, y \in I$, then $D = d = 0$.*

Proof. By the hypothesis we have $0 = D(x)D(yz) = D(x)D(y)z + D(x)yd(z) = D(x)yd(z)$ for all $x, y, z \in I$. In particular, we see that $d(z)D(x) = 0$ for all $x, z \in I$ by Lemma 1. Replacing x by xy in the last relation we get $0 = d(z)D(x)y + d(z)xd(y) = d(z)xd(y)$ for all $x, y, z \in I$. By [[10], Lemma 2, (a) and (b)], we obtain $d(s)Rd(r) = 0$ for all $s, r \in R$. In particular $d(s)Rd(s) = 0$ for all $s \in R$. Thus $d = 0$ by the semiprimeness of R. Then we have $0 = D(xz)D(y) = D(x)zD(y)$ for all $x, y, z \in I$. By Lemma 3, we arrive at $D(r)RD(s) = 0$ for all $r, s \in R$. In particular, $D(r)RD(r) = 0$ for all $r \in R$ which implies $D = 0$, as desired.

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