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CHARACTER VALUES AND DADE'S CONJECTURE

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Introduction

Let G be an arbitrary finite group and fix a prime p. Let S be a Sylow p-subgroup of G. In [19], J. McKay proposed the following conjecture (McKay conjecture) : The numbers of irreducible characters of G and $N_G(S)$ having degrees not divisible by p are equal when $p = 2$. In [1], J. L. Alperin generalized this conjecture for arbitrary primes and p-blocks of G (Alperin-McKay conjecture). Recently, I. M. Isaacs and G. Navarro extended it including the p'-parts of character degrees in [15]. Moreover in [21], Navarro proposed a strong form of the McKay conjecture, which concerns character values.

In [2], Alperin stated the weight conjecture, which described the number of irreducible modular characters by p-local information. In [16], R. Knörr and G. Robinson restated this conjecture using the alternating sum of the numbers of characters. E. C. Dade proposed an extention of this conjecture concerning the defect of characters in [10], and there are sevral forms of Dade's conjecture. Note that Dade's conjecture (the projective form) implies the Alperin-McKay conjecture. (See [11].)

The extended McKay conjecture by Isaacs and Navarro was subdivided into Dade type by K. Uno in [23]. In this paper, moreover, we subdivide the strong form of the McKay conjecture by Navarro into Dade type, and we prove this extended conjecture (Conjecture 1.4) for several sporadic simple groups (Theorem 1.5). For the calculations the `GAP` system is used.

Now we mention some notations used in this paper. We denote by $N : G = N \rtimes G$ and $N.G$ a split and non-split extention of N by G, respectively. We use n to denote a group of order n, and $\mathbb{Z}_n$ to denote a cyclic group of order n. For a prime p, an elementary abelian group of order p^n is denoted by E_{p^n} or simply p^n. We denote by $p_+^{1+2\gamma}$ an extraspecial p-group of order $p^{1+2\gamma}$ and type $+$. We use Q_8 to denote the quaternion group, and D_{2n} to denote the dihedral group of order $2n$. We use $\mathfrak{S}_n$ and $\mathfrak{A}_n$ to denote the symmetric group and the alternating group of degree n, respectively. We denote by F_n^m a Frobenius group with kernel $\mathbb{Z}_n$ and complement $\mathbb{Z}_m$. For a subgroup H of G, we use H' to denote a subgroup of G such that $H \cong H'$ but H' is not G-conjugate to H. Other notations including those for sporadic simple groups are taken from Atlas [9].

Throughout the paper, a character means an irreducible complex character. Let G be a finite group and H a subgroup of G. When a prime p is fixed, for a positive integer n, we denote by n_p the p-part of n, and by $n_{p'}$ the p'-part of n. We denote by $\mathrm{Bl}(H)$ the set of all p-blocks of H. The principal block of H is denoted by

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$B_0 = B_0(H)$. The set of characters χ of H is denoted by $\mathrm{Irr}(H)$. If the p-part of $|H|/\chi(1)$ is p^d, then we say that χ has defect $d = d(\chi)$. For an integer d, we denote by $\mathrm{Irr}(H, d)$ the set of characters χ in $\mathrm{Irr}(H)$ with $\mathrm{d}(\chi) = d$. Also for an integer κ, we denote by $\mathrm{Irr}(H, [\kappa])$ the set of characters χ in $\mathrm{Irr}(H)$ such that

$$(|H|/\chi(1))_{p'} \equiv \pm\kappa \pmod{p} .$$

Moreover, for a p-block B of G, we denote by $\mathrm{Irr}(H, B)$ the set of characters of H belonging to some p-block b of H inducing B. The set $\mathrm{Irr}(H, B, d, [\kappa])$ denotes the intersection of $\mathrm{Irr}(H, B)$, $\mathrm{Irr}(H, d)$ and $\mathrm{Irr}(H, [\kappa])$. The cardinality of $\mathrm{Irr}(*)$ is in general denoted by $\mathrm{k}(*)$. For the other notations and terminology used in this paper, see [22].

1. Dade's conjecture and its extentions

Let G be a finite group and p a prime. A radical p-chain of G is a chain

$$C : P_0 < P_1 < \cdots < P_n$$

of p-subgroups P_i of G such that

$$P_0 = O_p(G) \ \ and \ \ P_i = O_p(\bigcap_{j=0}^{i} N_G(P_j)) \ \ for\ all\ i = 1, 2, \cdots, n.$$

(Here $O_p(G)$ is the largest normal p-subgroup of G.)

For a chain C, we write by $|C|$ the length n of C, and by $N_G(C)$ the normalizer $\bigcap_{j=0}^{n} N_G(P_j)$ of C in G. Let $\mathcal{R}(G)$ be the set of all radical p-chains of G. The group G acts on $\mathcal{R}(G)$ by conjugation. We denote the set of representatives of G-conjugacy classes of $\mathcal{R}(G)$ by $\mathcal{R}(G)/G$.

Now we state the ordinary form of Dade's conjecture [10].

Conjecture 1.1. *Let G be a finite group with $O_p(G) = 1$, p a prime and B a p-block of G with defect $d(B) > 0$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d) = 0$$

for any non-negative integer d.

Uno proposed the following conjecture (it is an extention of Dade's conjecture) in [23].

Conjecture 1.2. *Let G be a finite group with $O_p(G) = 1$, p a prime and B a p-block of G with defect $d(B) > 0$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, [\kappa]) = 0$$

for any non-negative integers d and κ.

Next, we mention a strong form of the McKay conjecture proposed by Navarro in [21].

Conjecture 1.3. *Let G be a finite group and let p be a prime. Let $\zeta_{|G|}$ be a primitive $|G|$-th root of unity. Let e be a non-negative integer and let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{|G|})/\mathbb{Q})$ be any Galois automorphism sending every p'-root of unity ζ to ζ^{p^e}. Then σ fixes the same number of characters in $\mathrm{Irr}_{p'}(G)$ as in $\mathrm{Irr}_{p'}(N_G(S))$.*

For such a σ, let $\mathrm{Irr}(H, B, d, [\kappa], \sigma)$ be the set of σ-invariant characters in $\mathrm{Irr}(H, B, d, [\kappa])$. Now we can extend the conjecture 1.3 to the one of Dade type. See also p.1139 of [21].

Conjecture 1.4. *Let G be a finite group of order n with $O_p(G) = 1$, p a prime and B a p-block of G with defect $d(B) > 0$. Let e be a non-negative integer and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ be any Galois automorphism sending every p'-root of unity ζ to ζ^{p^e}. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, [\kappa], \sigma) = 0$$

for any non-negative integers d and κ.

The main theorem of this paper is the following.

Theorem 1.5. (i) *Conjecture 1.4 holds for all primes and blocks with positive defect for the following sporadic simple groups.*

$$\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3, \mathrm{HS}, \mathrm{McL}, \mathrm{He}, \mathrm{O'N}, \mathrm{Co}_3.$$

(ii) *Conjecture 1.4 holds for all odd primes and blocks with positive defect for the following sporadic simple groups.*

$$\mathrm{Ru}, \mathrm{Suz}, \mathrm{Co}_2.$$

2. Some reduction theorems

We know the following about the conjecture 1.4.

Theorem 2.1. *Suppose that a block B of G has a cyclic defect group. Then conjecture 1.4 holds for B.*

Proof. We suppose that the defect group D of B is cyclic. In [10, §9], Dade shows that it suffices to consider the two chains, $C_1 : 1$ and $C_2 : 1 < \Omega(D)$, where $\Omega(D)$ is the unique subgroup of D having order p. Let b be the block of $N_G(\Omega(D))$ corresponding to B in Brauer's sense. Let σ be as in Conjecture 1.4. Suppose that B is σ-invariant. Applying the deep theory of blocks with cyclic defect groups by Dade [12], the following was claimed by Isaacs, Navarro, J. An and E. A. O'Brien. In [21, Theorem(3.4)], Navarro proved that there exists a bijection $F : \mathrm{Irr}(B) \to \mathrm{Irr}(b)$

such that $F(\chi^\sigma) = F(\chi)^\sigma$ for all $\chi \in \mathrm{Irr}(B)$, and that F sends exceptional (resp. non-exceptional) characters of B onto exceptional (resp. non-exceptional) characters of b. By [15, Theorem(2.1)] of Isaacs and Navarro and [8, Proposition 2.1] of An and O'Brien, a bijection which sends exceptional (resp. non-exceptional) characters of B onto exceptional (resp. non-exceptional) characters of b preserves the defects and $\pm(|G|/\chi(1))_{p'}$ modulo p. Thus Conjecture 1.4 holds for B. □

If $|G|_p = p$, there is no block with the defect group D such that $|D| > p$. Thus it suffices to consider only primes p such that $|G|_p \geq p^2$.

If $\mathrm{Irr}(N_G(C), B, d, [\kappa], \sigma) = \mathrm{Irr}(N_G(C), B, d, [\kappa])$ for all C, d, κ and σ, and Conjecture 1.2 holds for B, then it is clear that Conjecture 1.4 holds for B. The next lemma indicates the cases where it really holds. Note also that for $p = 2$ and 3, it is meaningless to consider κ.

Lemma 2.2. *In the following cases we have* $\mathrm{Irr}(N_G(C), B, d, [\kappa], \sigma) = \mathrm{Irr}(N_G(C), B, d, [\kappa])$ *for any C, B, d, κ and σ.*

$\mathrm{M}_{11}, p = 3$. $\mathrm{M}_{22}, p = 3$. $\mathrm{M}_{23}, p = 2$ *or* 3. $\mathrm{M}_{24}, p = 2$. HS, $p = 3$. Suz, $p = 5$.

Proof. We have only to make it sure that any irrational values of $\chi \in \mathrm{Irr}(N_G(C), B, d, [\kappa])$ is σ-invariant for any such a σ. This can be done by looking at the relevant character tables. □

Next, let H and K be subgroups of G. If $H = K$, then

$$\mathrm{k}(H, B, d, [\kappa], \sigma) = \mathrm{k}(K, B, d, [\kappa], \sigma)$$

for all $B \in \mathrm{Bl}(G)$ with positive defect and for all $d \geq 0$, κ, σ. So we have the following.

Lemma 2.3. *Let $\mathcal{R}(G)/G = \{ C_1, C_2, \cdots, C_n \}$. Let $1 \leq i < j \leq n$ be such that $N_G(C_i) = N_G(C_j)$ and the lengthes of C_i and C_j have different paritis. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ be such that $\mathcal{R}_0(G)/G = \{C_l \mid 1 \leq l \leq n,\ l \neq i,\ j\}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, [\kappa], \sigma) = \sum_{C \in \mathcal{R}_0(G)/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, [\kappa], \sigma)$$

for all $B \in \mathrm{Bl}(G)$ with positive defect and for all $d \geq 0$, κ, σ.

Here, we write order of the sporadic simple groups considered in this paper. $|\mathrm{M}_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. $|\mathrm{M}_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$. $|\mathrm{J}_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. $|\mathrm{M}_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. $|\mathrm{J}_2| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$. $|\mathrm{M}_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. $|\mathrm{HS}| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. $|\mathrm{J}_3| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$. $|\mathrm{M}_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. $|\mathrm{McL}| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$. $|\mathrm{He}| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$. $|\mathrm{O'N}| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$. $|\mathrm{Co}_3| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. $|\mathrm{Ru}| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$. $|\mathrm{Suz}| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. $|\mathrm{Co}_2| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. Then we have the following remark.

Remark. From Theorem 2.1 and Lemma 2.2, in order to verify Theorem 1.5, it suffices to consider the following cases.

group	M_{11}	M_{12}	J_1	M_{22}	J_2	HS	J_3	M_{24}	McL
prime	2	2, 3	2	2	2, 3, 5	2, 5	2, 3	3	2, 3, 5

group	He	O′N	Co_3	Ru	Suz	Co_2
prime	2, 3, 5, 7	2, 3, 7	2, 3, 5	3, 5	3, 5	3, 5

After § 4, we treat only the cases mentioned in the remark above.

3. The consideration to σ

In this chapter, we consider $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{|G|})/\mathbb{Q})$ in Conjecture 1.4.

Let n be a positive integer, and ζ_n be a primitive n-th root of unity. For an integer t relatively prime to n, define $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ by

$$\sigma_t : \zeta_n \longmapsto \zeta_n^t .$$

Then

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \{\sigma_t \mid (t, n) = 1, 1 \le t \le n\} .$$

For a prime p which divides n and a non-negative integer e, we have

$$\begin{aligned} \sigma_t(\zeta_{n_{p'}}) = (\zeta_{n_{p'}})^{p^e} &\Leftrightarrow \sigma_t((\zeta_n)^{n_p}) = (\zeta_n)^{p^e \cdot n_p} \\ &\Leftrightarrow t \cdot n_p \equiv p^e \cdot n_p \pmod{n} \\ &\Leftrightarrow t \equiv p^e \pmod{n_{p'}} . \end{aligned}$$

Thus we may write

$$t = p^e + r \cdot n_{p'} \ \ (r \in \mathbb{Z}) ,$$

and we have the following conditions on t.

$$\left\{ \begin{array}{ll} t = 1 + r \cdot n_{p'} & \Big(1 \le t \le n \ , \ r \cdot n_{p'} \not\equiv -1 \pmod{p} \Big) \quad if\ e = 0, \\ t = p^e + r \cdot n_{p'} & (1 \le t \le n \ , \ p \nmid r) \quad if\ e > 0. \end{array} \right. \tag{3.1}$$

For each G of order n, we henceforth assume that σ_t in $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ satisfies the condition (3.1) above.

Next, we mantion some notations. (See [18].) Let q be an odd prime and a a non zero integer. We denote the quadratic symbol by $\left(\dfrac{a}{q}\right)$. Furthermore, let $m = q_1^{e_1} \cdots q_r^{e_r}$ be an odd positive integer written as a product of primes. We define

$$\left(\frac{a}{m}\right) = \left(\frac{a}{q_1}\right)^{e_1} \cdots \left(\frac{a}{q_r}\right)^{e_r} .$$

We call this also the quadratic symbol, and define the Gauss sum to be

$$G_q = \sum_{a=1}^{q-1} \left(\frac{a}{q}\right) \zeta_q^a$$

for $\zeta_q = e^{\frac{2\pi i}{q}}$. Then we know that the following holds.

Proposition 3.1.

$$G_q = \begin{cases} \sqrt{q} & if \ \ q \equiv 1 \pmod 4. \\ i\sqrt{q} & if \ \ q \equiv 3 \pmod 4. \end{cases}$$

Proposition 3.2. *For an integer t,*

$$\sum_{a=1}^{q-1} \left(\frac{a}{q}\right) \zeta_q^{ta} = \left(\frac{t}{q}\right) G_q \ .$$

Here we mention the Atlas notation for algebraic numbers ([9]), which describes the irrational numbers that appear in character tables. For an odd positive integer m,

$$b_m = \begin{cases} \frac{-1+\sqrt{m}}{2} & \text{if } \ m \equiv 1 \pmod 4. \\ \frac{-1+i\sqrt{m}}{2} & \text{if } \ m \equiv 3 \pmod 4. \end{cases}$$

Let $N = q_0^{e_0} \cdot q_1^{e_1} \cdots q_r^{e_r}$ be a positive integer written as a product of primes. Since $\sigma\left(\sqrt{N}\right) = \sigma(\sqrt{q_0})^{e_0} \cdot \sigma(\sqrt{q_1})^{e_1} \cdots \sigma(\sqrt{q_r})^{e_r}$, it suffices to consider b_q (q : an odd prime), $\sqrt{2}$ and i. For b_q, we use Proposition 3.1 and 3.2. And for $\sqrt{2}$ and i,

$$\sigma_t\left(\sqrt{2}\right) = \begin{cases} -\sqrt{2} & \text{if } \ e : \text{even} \ \text{ and } \ \left(\frac{-1}{t}\right) = -1 \ (p = 2), \ \text{ or } \ e : \text{odd} \ (p = 3). \\ \sqrt{2} & \text{otherwise.} \end{cases}$$

$$\sigma_t(i) = \begin{cases} -i & \text{if } \ e : \text{odd} \ \text{ and } \ \left(\frac{-1}{t}\right) = -1 \ (p = 2), \ \text{ or } \ e : \text{odd} \ (p = 3, 7). \\ i & \text{otherwise.} \end{cases}$$

(In the above equations we consider only those primes which are needed in this paper.)

In the sections thereafter, we prove Theorem 1.4. In the rest of the paper, S always denotes a Sylow p-subgroup of G.

4. M_{11}

Let G be the simple Mathieu group M_{11}. We assume that $p = 2$. By using GAP, the radical 2-chains of G (up to conjugacy) are given in Table 1. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4\}$. Then by Lemma 2.3, we have the following.

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, [\kappa], \sigma_t) = \sum_{C \in \mathcal{R}_0(G)/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, [\kappa], \sigma_t) \ .$$

In the rest of the paper, we define $\mathcal{R}_0(G)$ for several G's, if it is the case where Lemma 2.3 applies similarly. In such cases, we only give a definition of $\mathcal{R}_0(G)$

TABLE 1. The radical 2-chains of M_{11}

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < Q_8$	$Q_8.\mathfrak{S}_3$
$C_2 : 1 < 2^2$	$2^2.\mathfrak{S}_3$	$C_5 : 1 < Q_8 < S$	S
$C_3 : 1 < 2^2 < Q_8$	Q_8	$C_6 : 1 < S$	S

TABLE 2. Conjecture 1.4 for M_{11}, $p = 2$

(d, t_i)	$(4, t_1)$	$(3, t_1)$	$(2, t_1)$	$(4, t_2)$	$(3, t_2)$	$(2, t_2)$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	4	3	1	4	1	1	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_i})$	0	4	1	0	4	1	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_i})$	0	4	1	0	4	1	+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	4	3	1	4	1	1	−

and do not mention further detail. The principal block B_0 of defect 4 is the unique 2-block of G with positive defect.

Let t_1 be such that $\left(\frac{-1}{t_1}\right) = \left(\frac{2}{t_1}\right)$ and t_2 such that $\left(\frac{-1}{t_2}\right) = -\left(\frac{2}{t_2}\right)$. Then we have Table 2.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

5. M_{12}

Let G be the simple Mathieu group M_{12}. We assume that $p = 2$. The radical 2-chains of G (up to conjugacy) are given in Table 3. Here, $N_G(C_6) = N_G(C_7) = \mathbb{Z}_2{\times}D_8$, and $N_G(C_{13}) = N_G(C_{14}) = S$. Then let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4, C_5, C_8, C_9, C_{10}, C_{11}, C_{12}\}$. The principal block B_0 of defect 6 and the block B_1 of defect 2 are 2-blocks of G with positive defect.

Let t_1 and t_2 be such that e is even and odd, respectively, in the condition (3.1) in §3. Then we have Table 4.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

We assume that $p = 3$. The radical 3-chains of G (up to conjugacy) are given in Table 5. (In Table 5, let T be a Sylow 3-group of $N_G(\mathbb{Z}_3)$.) Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_6\}$. The principal block B_0 of defect 3 and the block B_1 of defect 1 are 3-blocks of G with positive defect. By Theorem 2.1, it suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{3}\right) = 1$ and t_2 such that $\left(\frac{t_2}{3}\right) = -1$. Then we have Table 6.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

TABLE 3. The radical 2-chains of M_{12}

$C_1 : 1$	$C_6 : 1 < 2 < 2^3 < 2^2.2^2$	$C_{11} :< 2^2.2^3 < S$
$C_2 : 1 < 2$	$C_7 : 1 < 2 < 2^2.2^2$	$C_{12} :< 2_+^{1+4}$
$C_3 : 1 < 2 < 2^2$	$C_8 : 1 < 2^2$	$C_{13} :< 2_+^{1+4} < S$
$C_4 : 1 < 2 < 2^2 < 2^3$	$C_9 : 1 < 2^2 < 2^3$	$C_{14} :< S$
$C_5 : 1 < 2 < 2^3$	$C_{10} : 1 < 2^2.2^3$	

TABLE 4. Conjecture 1.4 for M_{12}, $p = 2$

(d)	(6)	(5)	(4)	(3)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_1})$	8	2		1		+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_1})$			8	2		−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_1})$				8		+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_1})$				8		−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_1})$			8	2		+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_1})$				8		−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_1})$				8		+
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_1})$	8	6				−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_1})$	8	6	2			+
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_1})$	8	2	2	1		−
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_1})$					4	+
$\mathrm{k}(N_G(C_2), B_1, d, [1], \sigma_{t_1})$					4	−
$\mathrm{k}(N_G(C_3), B_1, d, [1], \sigma_{t_1})$					4	+
$\mathrm{k}(N_G(C_8), B_1, d, [1], \sigma_{t_1})$					4	−
(d)	(6)	(5)	(4)	(3)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	8	2		1		+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$			8	2		−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$				8		+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$				8		−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$			8	2		+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_2})$				4		−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_2})$				4		+
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_2})$	8	6				−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_2})$	8	6	2			+
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_2})$	8	2	2	1		−
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_2})$					2	+
$\mathrm{k}(N_G(C_2), B_1, d, [1], \sigma_{t_2})$					4	−
$\mathrm{k}(N_G(C_3), B_1, d, [1], \sigma_{t_2})$					4	+
$\mathrm{k}(N_G(C_8), B_1, d, [1], \sigma_{t_2})$					2	−

TABLE 5. The radical 3-chains of M_{12}

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_5 : 1 < 3^2 < S$	$S.2^2$
$C_2 : 1 < 3$	$3.\mathfrak{A}_4.2$	$C_6 : 1 < 3^{2\prime}$	$3^2.GL_2(3)$
$C_3 : 1 < 3 < T$	$3.\mathfrak{S}_3$	$C_7 : 1 < 3^{2\prime} < S$	$S.2^2$
$C_4 : 1 < 3^2$	$3^2.GL_2(3)$	$C_8 : 1 < S$	$S.2^2$

TABLE 6. Conjecture 1.4 for M_{12}, $p = 3$

(d, t_i)	$(3, t_1)$	$(2, t_1)$	$(3, t_2)$	$(2, t_2)$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	9	2	9	2	+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$		9		3	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$		9		3	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	9	2	9	2	−
$k(N_G(C_5), B_0, d, [1], \sigma_{t_i})$	9	2	9	2	+
$k(N_G(C_6), B_0, d, [1], \sigma_{t_i})$	9	2	9	2	−

6. J_1

Let G be the simple Janko group J_1. We assume that $p = 2$. The radical 2-chains of G (up to conjugacy) are given in Table 7. The principal block B_0 of defect 3 and the block of defect 1 are 2-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 and t_2 be such that e is even and odd, respectively, in the condition (3.1) in §3. Then the value of $k(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d. Thus the result follows from [10, Theorem 10.1]. For σ_{t_2}, we have Table 8.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

TABLE 7. The radical 2-chains of J_1

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 2 < S$	$2 \times \mathfrak{A}_4$
$C_2 : 1 < 2$	$2 \times \mathfrak{A}_5$	$C_4 : 1 < S$	$S : 3 : 7$

TABLE 8. Conjecture 1.4 for J_1, $p = 2$

(d)	(3)	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	4	+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	4	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	4	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	4	−

Table 9. The radical 2-chains of M_{22}

$C_1 : 1$
$C_2 : 1 < 2^3$
$C_3 : 1 < 2^3 < 2^2.2^3$
$C_4 : 1 < 2^3 < 2^2.2^3 < 2.2^2.2^3$
$C_5 : 1 < 2^3 < 2.2^4$
$C_6 : 1 < 2^3 < 2.2^4 < 2.2^2.2^3$
$C_7 : 1 < 2^3 < 2.2^2.2^3$
$C_8 : 1 < 2^4$
$C_9 : 1 < 2^4 < 2^2.2^4$
$C_{10} : 1 < 2^4 < 2^2.2^4 < S$
$C_{11} : 1 < 2^4 < 2.2^2.2^3$
$C_{12} : 1 < 2^4 < 2.2^2.2^3 < S$
$C_{13} : 1 < 2^4 < S$
$C_{14} : 1 < 2^{4\prime}$
$C_{15} : 1 < 2^{4\prime} < 2^2.2^3$
$C_{16} : 1 < 2^{4\prime} < 2^2.2^3 < 2.2^2.2^3$
$C_{17} : 1 < 2^{4\prime} < 2^2.2^4$
$C_{18} : 1 < 2^{4\prime} < 2^2.2^4 < S$
$C_{19} : 1 < 2^{4\prime} < S$
$C_{20} : 1 < 2^2.2^3$
$C_{21} : 1 < 2^2.2^3 < 2.2^2.2^3$
$C_{22} : 1 < 2.2^2.2^3$
$C_{23} : 1 < 2.2^2.2^3 < S$
$C_{24} : 1 < 2^2.2^4$
$C_{25} : 1 < 2^2.2^4 < S$
$C_{26} : 1 < S$

Table 10. Conjecture 1.4 for M_{22}, $p = 2$

(d)	(7)	(6)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	8	2				+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$		8	2		1	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$		8	6			+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$		8	6	2		−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$		8	2	2	1	+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_2})$	8	2				−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_2})$	8	2	2			+
$\mathrm{k}(N_G(C_{14}), B_0, d, [1], \sigma_{t_2})$	8	2	2			−

7. M_{22}

Let G be the simple Mathieu group M_{22}. We assume that $p = 2$. The radical 2-chains of G (up to conjugacy) are given in Table 9. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4, C_5, C_8, C_9, C_{14}\}$. The principal block B_0 is the unique 2-block of G.

Let t_1 and t_2 be such that e is even and odd, respectively, in the condition (3.1) in §3. Then the value of $\mathrm{k}(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d, and thus the result follows from [14, Theorem 2]. For σ_{t_2}, we have Table 10.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

Table 11. The radical 2-chains of J_2

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_5 : 1 < 2_-^{1+4} < S$	$S.3$
$C_2 : 1 < 2^2$	$\mathfrak{A}_4 \times \mathfrak{A}_5$	$C_6 : 1 < 2^{2+4}$	$2^{2+4} : (3 \times \mathfrak{S}_3)$
$C_3 : 1 < 2^2 < 2^4$	$\mathfrak{A}_4 \times \mathfrak{A}_4$	$C_7 : 1 < 2^{2+4} < S$	$S.3$
$C_4 : 1 < 2_-^{1+4}$	$2_-^{1+4} : \mathfrak{A}_5$	$C_8 : 1 < S$	$S.3$

Table 12. Conjecture 1.4 for J_2, $p = 2$

(d)	(7)	(6)	(5)	(4)	(3)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_1})$	8	6	2		1		$+$
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_1})$				16			$-$
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_1})$				16			$+$
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_1})$	8	2	2	3	1		$-$
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_1})$	8	6	6				$-$
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_1})$	8	2	6	3			$+$
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_1})$						4	$+$
$\mathrm{k}(N_G(C_2), B_1, d, [1], \sigma_{t_1})$						4	$-$
(d)	(7)	(6)	(5)	(4)	(3)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	4	2	2		1		$+$
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$				4			$-$
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$				4			$+$
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	4	2	2	1	1		$-$
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_2})$	4	2	2				$-$
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_2})$	4	2	2	1			$+$
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_2})$						2	$+$
$\mathrm{k}(N_G(C_2), B_1, d, [1], \sigma_{t_2})$						2	$-$

8. J_2

Let G be the simple Janko group J_2. We assume $p = 2$. The radical 2-chains of G (up to conjugacy) are given in Table 11. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4, C_6, C_7\}$. The principal block B_0 of defect 7 and the block B_1 of defect 2 are 2-blocks of G.

Let t_1 and t_2 be such that e is even and odd, respectively, in the condition (3.1) in §3. Then we have Table 12.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

We assume that $p = 3$. The radical 3-chains of G (up to conjugacy) are given in Table 13. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4\}$. The principal block B_0 of defect 3 and two blocks of defect 1 are 3-blocks of G with positive defect. It suffices to consider only B_0.

Table 13. The radical 3-chains of J_2

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 3'$	$(3 \times \mathfrak{A}_6) : 2$
$C_2 : 1 < 3$	$\mathfrak{A}_4 \times \mathfrak{S}_3$	$C_5 : 1 < 3' < S$	$3_+^{1+2} : 8$
$C_3 : 1 < 3 < 3^2$	$3 \times \mathfrak{S}_3$	$C_6 : 1 < S$	$3_+^{1+2} : 8$

Table 14. Conjecture 1.4 for J_2, $p = 3$

(d, t_i)	$(3, t_1)$	$(2, t_1)$	$(3, t_2)$	$(2, t_2)$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	9	4	9	4	+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$		9		3	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$		9		3	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	9	4	9	4	−
(d, t_i)	$(3, t_3)$	$(2, t_3)$	$(3, t_4)$	$(2, t_4)$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	3	2	3	2	+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$		9		3	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$		9		3	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	3	2	3	2	−

Table 15. The radical 5-chains of J_2

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 5 < S$	$D_{10} \times D_{10}$
$C_2 : 5$	$\mathfrak{A}_5 \times D_{10}$	$C_4 : 1 < S$	$5^2 : D_{12}$

Let t_1 be such that $\left(\frac{t_1}{3}\right) = 1$ and e is even, t_2 such that $\left(\frac{t_2}{3}\right) = -1$ and e is even, t_3 such that $\left(\frac{t_3}{3}\right) = 1$ and e is odd and t_4 such that $\left(\frac{t}{3}\right) = -1$ and e is odd in the condition (3.1) in §3. Then we have Table 14.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

We assume that $p = 5$. The radical 5-chains of G (up to conjugacy) are given in Table 15. The principal block B_0 of defect 2 and the block of defect 1 are 5-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{5}\right) = 1$ and t_2 such that $\left(\frac{t_2}{5}\right) = -1$. Then we have Table 16.

Hence Conjecture 1.4 holds for G in the case of $p = 5$.

9. HS

Let G be the simple Higman-Sims group HS. We assume that $p = 2$. The radical 2-chains of G (up to conjugacy) are given in Table 17. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_{13}, C_{16}, C_{19}, C_{22}\}$. The principal block B_0 of defect 9 and the block B_1 of defect 2 are 2-blocks of G with positive defect.

TABLE 16. Conjecture 1.4 for J_2, $p = 5$

$(d, [\kappa], t_i)$	$(2, [1], t_1)$	$(2, [2], t_1)$	$(2, [1], t_2)$	$(2, [2], t_2)$	Parity
$k(N_G(C_1), B_0, d, [\kappa], \sigma_{t_i})$	2	12	2	4	$+$
$k(N_G(C_2), B_0, d, [\kappa], \sigma_{t_i})$	8	8	4		$-$
$k(N_G(C_3), B_0, d, [\kappa], \sigma_{t_i})$	8	8	4		$+$
$k(N_G(C_4), B_0, d, [\kappa], \sigma_{t_i})$	2	12	2	4	$-$

TABLE 17. The radical 2-chains of HS

$C_1 : 1$	$C_{18} : 1 < 4^3 < 2^2.2^3.2^3 < S$
$C_2 : 1 < 2$	$C_{19} : 1 < 4^3 < 4.2^2.2^4$
$C_3 : 1 < 2 < 2^2$	$C_{20} : 1 < 4^3 < 4.2^2.2^4 < S$
$C_4 : 1 < 2 < 2^2 < 4.2^2$	$C_{21} : 1 < 4^3 < S$
$C_5 : 1 < 2 < 2^{4'}$	$C_{22} : 1 < 4.2^4$
$C_6 : 1 < 2 < 2^{4'} < 2^2.D_8$	$C_{23} : 1 < 4.2^4 < 2.2^3.2^3$
$C_7 : 1 < 2 < 2^2.2.D_8$	$C_{24} : 1 < 4.2^4 < 2.2^3.2^3 < 2.2^2.2^2.2^3$
$C_8 : 1 < 2^2$	$C_{25} : 1 < 4.2^4 < 4.2^2.2^4$
$C_9 : 1 < 2^2 < 4.2^2$	$C_{26} : 1 < 4.2^4 < 4.2^2.2^4 < S$
$C_{10} : 1 < 2^4$	$C_{27} : 1 < 4.2^4 < S$
$C_{11} : 1 < 2^4 < 2.2^3.2^3$	$C_{28} : 1 < 2.2^3.2^3$
$C_{12} : 1 < 2^4 < 2.2^3.2^3 < 2.2^2.2^2.2^3$	$C_{29} : 1 < 2.2^3.2^3 < 2.2^2.2^2.2^3$
$C_{13} : 1 < 2^4 < 2^2.2^5$	$C_{30} : 1 < 4.2^2.2^4$
$C_{14} : 1 < 2^4 < 2^2.2^5 < 2.2^2.2^2.2^3$	$C_{31} : 1 < 4.2^2.2^4 < S$
$C_{15} : 1 < 2^4 < 2.2^2.2^2.2^3$	$C_{32} : 1 < 2^2.2^3.2^3$
$C_{16} : 1 < 4^3$	$C_{33} : 1 < 2^2.2^3.2^3 < S$
$C_{17} : 1 < 4^3 < 2^2.2^3.2^3$	$C_{34} : 1 < S$

Let t_1 be such that $\left(\frac{-1}{t_1}\right) = 1$ and e is even, t_2 such that $\left(\frac{-1}{t_2}\right) = -1$ and e is even, t_3 such that $\left(\frac{-1}{t_3}\right) = 1$ and e is odd and t_4 such that $\left(\frac{-1}{t_4}\right) = -1$ and e is odd in the condition (3.1) in §3. Then the value of $k(N_G(C), B_j, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C, B_j $(j = 1, 2)$ and d, and thus the result follows from [13, Theorem 4.3]. For σ_{t_i} $(i = 2, 3, 4)$, we have Table 18 and Table 19.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

We assume that $p = 5$. The radical 5-chains of G (up to conjugacy) are given in Table 20. The principal block B_0 of defect 3 and the block of defect 1 are 5-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{5}\right) = 1$ and t_2 such that $\left(\frac{t_2}{5}\right) = -1$. Then we have Table 21.

Hence Conjecture 1.4 holds for G in the case of $p = 5$.

TABLE 18. Conjecture 1.4 for HS, $p = 2$

(d)	(9)	(8)	(7)	(6)	(5)	(4)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	8	8		1		1	$+$
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$				16	4	2	$-$
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$						8	$+$
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$						8	$-$
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$					16	4	$+$
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_2})$					16	4	$-$
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_2})$				16	4	2	$+$
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_2})$						8	$-$
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_2})$						8	$+$
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_2})$		16	4			1	$-$
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_2})$		16	12		2	1	$+$
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_2})$		16	12	4	2		$-$
$\mathrm{k}(N_G(C_{13}), B_0, d, [1], \sigma_{t_2})$		16	4	4			$+$
$\mathrm{k}(N_G(C_{16}), B_0, d, [1], \sigma_{t_2})$	8	8		1			$-$
$\mathrm{k}(N_G(C_{19}), B_0, d, [1], \sigma_{t_2})$	8	8	4	3			$+$
$\mathrm{k}(N_G(C_{22}), B_0, d, [1], \sigma_{t_2})$	8	8	4	3		1	$-$
(d)	(9)	(8)	(7)	(6)	(5)	(4)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_3})$	8	8		1		1	$+$
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_3})$				16	4	2	$-$
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_3})$						16	$+$
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_3})$						16	$-$
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_3})$					16	4	$+$
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_3})$					16	4	$-$
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_3})$				16	4	2	$+$
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_3})$						8	$-$
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_3})$						8	$+$
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_3})$		16	4			1	$-$
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_3})$		16	12		2	1	$+$
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_3})$		16	12	4	2		$-$
$\mathrm{k}(N_G(C_{13}), B_0, d, [1], \sigma_{t_3})$		16	4	4			$+$
$\mathrm{k}(N_G(C_{16}), B_0, d, [1], \sigma_{t_3})$	8	8		1			$-$
$\mathrm{k}(N_G(C_{19}), B_0, d, [1], \sigma_{t_3})$	8	10	4	3			$+$
$\mathrm{k}(N_G(C_{22}), B_0, d, [1], \sigma_{t_3})$	8	10	4	3		1	$-$

10. J_3

Let G be the simple Janko group J_3. We assume that $p = 2$. The radical 2-chains of G (up to conjugacy) are given in Table 22. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that

TABLE 19. Conjecture 1.4 for HS, $p = 2$

(d)	(9)	(8)	(7)	(6)	(5)	(4)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_4})$	8	10		1		1	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_4})$				16	4	2	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_4})$						8	+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_4})$						8	−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_4})$					16	4	+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_4})$					16	4	−
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_4})$				16	4	2	+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_4})$						4	−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_4})$						4	+
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_4})$		16	4			1	−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_4})$		16	12		2	1	+
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_4})$		16	12	4	2		−
$\mathrm{k}(N_G(C_{13}), B_0, d, [1], \sigma_{t_4})$		16	4	4			+
$\mathrm{k}(N_G(C_{16}), B_0, d, [1], \sigma_{t_4})$	8	10		1			−
$\mathrm{k}(N_G(C_{19}), B_0, d, [1], \sigma_{t_4})$	8	8	4	5			+
$\mathrm{k}(N_G(C_{22}), B_0, d, [1], \sigma_{t_4})$	8	8	4	5		1	−

(d, t_i)	$(2, t_2)$	$(2, t_3)$	$(2, t_4)$	Parity
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_i})$	4	2	2	+
$\mathrm{k}(N_G(C_2), B_1, d, [1], \sigma_{t_i})$	4	4	4	−
$\mathrm{k}(N_G(C_3), B_1, d, [1], \sigma_{t_i})$	4	4	4	+
$\mathrm{k}(N_G(C_8), B_1, d, [1], \sigma_{t_i})$	4	2	2	−

TABLE 20. The radical 5-chains of HS

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 5 < 5^2$	$(5:4) \times D_{10}$
$C_2 : 1 < 5$	$(5:4) \times \mathfrak{A}_5$	$C_4 : 1 < S$	$(5^{1+2}_+ : 8) : 2$

$\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_6\}$. The principal block B_0 of defect 7 is the unique 2-block of G with positive defect.

Let t_1 and t_2 be such that e is even and odd, respectively, in the condition (3.1) in §3. Then the value of $\mathrm{k}(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d, and thus the result follows from [17, Theorem 2.10.1]. For σ_{t_2}, we have Table 23.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

We assume that $p = 3$. The radical 3-chains of G (up to conjugacy) are given in Table 24. The principal block B_0 of defect 5 and the block of defect 1 are 3-blocks of G with positive defect. It suffices to consider only B_0.

TABLE 21. Conjecture 1.4 for HS, $p = 5$

$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_1})$	9	4	4		+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_1})$			10	10	−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_1})$			10	10	+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_1})$	9	4	4		−
$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_2})$	9	4	2		+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_2})$				10	−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_2})$				10	+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_2})$	9	4	2		−

TABLE 22. The radical 2-chains of J_3

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_5 : 1 < 2_-^{1+4} < S$	$S : 3$
$C_2 : 1 < 2^4$	$2^4 : GL_2(4)$	$C_6 : 1 < 2^{2+4}$	$2^{2+4} : (3 \times \mathfrak{S}_3)$
$C_3 : 1 < 2^4 < 2^{2+4}$	$2^{2+4} : 3^2$	$C_7 : 1 < 2^{2+4} < S$	$S : 3$
$C_4 : 1 < 2_-^{1+4}$	$2_-^{1+4} : \mathfrak{A}_5$	$C_8 : 1 < S$	$S : 3$

TABLE 23. Conjecture 1.4 for J_3, $p = 2$

(d)	(7)	(6)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	4	2	2		1	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$		4		1		−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$		4		1		+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	4	2	2	1	1	−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$	4	2	2	1		+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_2})$	4	2	2			−

TABLE 24. The radical 3-chains of J_3

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 3 < 3^3$	$3^3 : 8$
$C_2 : 1 < 3$	$(3 \times \mathfrak{A}_6) : 2$	$C_4 : 1 < S$	$3^2.(3 \times 3^2) : 8$

Let t_i $(1 \leq i \leq 12)$ satisfy $t_1 \equiv t_7 \equiv 1$, $t_2 \equiv t_8 \equiv 2$, $t_3 \equiv t_9 \equiv 4$, $t_4 \equiv t_{10} \equiv 5$, $t_5 \equiv t_7 \equiv 7$, $t_6 \equiv t_{12} \equiv 8 \pmod 9$, and define $t_1, \cdots, t_6$ for an even e and $t_7, \cdots, t_{12}$ for an odd e in the condition (3.1) in §3. Then we have Table 25.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

TABLE 25. Conjecture 1.4 for J_3, $p = 3$

(d, t_i)	$(5, t_1)$	$(4, t_1)$	$(3, t_1)$	$(5, t_2)$	$(4, t_2)$	$(3, t_2)$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	9	7		9	4		+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$			15			13	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$			15			13	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	9	7		9	4		−

(d, t_i)	$(5, t_3)$	$(4, t_3)$	$(3, t_3)$	$(5, t_4)$	$(4, t_4)$	$(3, t_4)$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	9	4		9	4		+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$			15			13	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$			15			13	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	9	4		9	4		−

(d, t_i)	$(5, t_5)$	$(4, t_5)$	$(3, t_5)$	$(5, t_6)$	$(4, t_6)$	$(3, t_6)$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	9	4		9	7		+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$			15			13	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$			15			13	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	9	4		9	7		−

(d, t_i)	$(5, t_7)$	$(4, t_7)$	$(3, t_7)$	$(5, t_8)$	$(4, t_8)$	$(3, t_8)$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	3	5		3	2		+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$			7			5	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$			7			5	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	3	5		3	2		−

(d, t_i)	$(5, t_9)$	$(4, t_9)$	$(3, t_9)$	$(5, t_{10})$	$(4, t_{10})$	$(3, t_{10})$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	3	2		3	2		+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$			7			5	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$			7			5	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	3	2		3	2		−

(d, t_i)	$(5, t_{11})$	$(4, t_{11})$	$(3, t_{11})$	$(5, t_{12})$	$(4, t_{12})$	$(3, t_{12})$	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	3	2		3	5		+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_i})$			7			5	−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_i})$			7			5	+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_i})$	3	2		3	5		−

TABLE 26. The radical 3-chains of M_{24}

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 3'$	$3.2 \times L_3(2)$
$C_2 : 1 < 3$	$3.\mathfrak{S}_6$	$C_5 : 1 < 3^2 < 3_+^{1+2}$	$3_+^{1+2}.2^2$
$C_3 : 1 < 3' < 3^{2'}$	$3^2.2^2$	$C_6 : 1 < 3^2$	$3^2.GL_2(3)$

TABLE 27. Conjecture 1.4 for M_{24}, $p = 3$

(d)	(3)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	9	2	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	9	2	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$		9	+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$		9	−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$	9	2	+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_2})$	9	2	−

11. M_{24}

Let G be the simple Mathieu group M_{24}. We assume that $p = 3$. By [5, (5A)], let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_6\}$ in Table 26. The principal block B_0 of defect 3 and four blocks of defect 1 are 3-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{15}\right) = 1$ and t_2 such that $\left(\frac{t_2}{15}\right) = -1$. Then the value of $\mathrm{k}(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d, and thus the result follows from [5, 5C]. For σ_{t_2}, we have Table 27.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

12. McL

Let G be the simple McLaufhlin group McL. We assume that $p = 2$. By [20, (TABLE 4.3)], let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_8\}$ in Table 28. The principal block B_0 of defect 7 and the block of defect 1 are 2-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 and t_2 be such that e is even and odd, respectively, in the condition (3.1) in §3. Then the value of $\mathrm{k}(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d, and thus the result follows from [20, Theorem 4.8.3]. For σ_{t_2}, we have Table 29.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

We assume that $p = 3$. The radical 3-chains of G (up to conjugacy) are given in Table 30. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4\}$. The principal block B_0 of defect 6 is the unique 3-block of G with positive defect.

TABLE 28. The radical 2-chains of McL

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_5 : 1 < 2$	$2.\mathfrak{A}_8$
$C_2 : 1 < 2^4$	$2^4 : \mathfrak{A}_7$	$C_6 : 1 < 2 < 2^4$	$2^4 : GL_3(2)$
$C_3 : 1 < 2^4 < 2^2.2^4$	$(2^2.2^4) : (3^2 : 2)$	$C_7 : 1 < 2 < 2^4 < 2^2.2^4$	$2^4 : \mathfrak{S}_4$
$C_4 : 1 < 2^{4\prime}$	$2^4 : \mathfrak{A}_7$	$C_8 : 1 < 2 < 2^{4\prime}$	$2^4 : GL_3(2)$

TABLE 29. Conjecture 1.4 for McL, $p = 2$

(d)	(7)	(6)	(5)	(4)	(3)	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	8	6	2	1	1	+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	8	6		1		−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	8	6	2	1		+
$k(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	8	6		1		−
$k(N_G(C_5), B_0, d, [1], \sigma_{t_2})$	8	2	2	6	1	−
$k(N_G(C_6), B_0, d, [1], \sigma_{t_2})$	8	2		4		+
$k(N_G(C_7), B_0, d, [1], \sigma_{t_2})$	8	2	2	2		−
$k(N_G(C_8), B_0, d, [1], \sigma_{t_2})$	8	2		4		+

TABLE 30. The radical 3-chains of McL

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 3_+^{1+4}$	$3_+^{1+4} : (2.\mathfrak{S}_5)$
$C_2 : 1 < 3^4$	$3^4 : M_{10}$	$C_5 : 1 < 3_+^{1+4} < S$	$3_+^{1+4} : 3 : Q_8$
$C_3 : 1 < 3^4 < S$	$3_+^{1+4} : 3 : Q_8$	$C_6 : 1 < S$	$3_+^{1+4} : 3 : Q_8$

Let t_1 be such that $\left(\frac{t_1}{3}\right) = 1$ and e is even, t_2 such that $\left(\frac{t_2}{3}\right) = -1$ and e is even, t_3 such that $\left(\frac{t_3}{3}\right) = 1$ and e is odd and t_4 such that $\left(\frac{t_4}{3}\right) = -1$ and e is odd in the condition (3.1) in §3. Then the value of $k(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d, and thus the result follows from [20, Theorem 3.6.2]. For σ_{t_i} $(i = 2, 3, 4)$, we have Table 31.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

We assume that $p = 5$. The radical 5-chains of G (up to conjugacy) are given in Table 32. The principal block B_0 of defect 3 is the unique 5-block of G with positive defect.

Let t_1 be such that $\left(\frac{t_1}{5}\right) = 1$ and e is even, t_2 such that $\left(\frac{t_2}{5}\right) = -1$ and e is even, t_3 such that $\left(\frac{t_3}{5}\right) = 1$ and e is odd and t_4 such that $\left(\frac{t_4}{5}\right) = -1$ and e is odd in the condition (3.1) in §3. Then we have Table 33.

Hence Conjecture 1.4 holds for G in the case of $p = 5$.

TABLE 31. Conjecture 1.4 for McL, $p = 3$

(d)	(6)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	8	3	3	1	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	8	3	3		−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	8	3	4		+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	8	3	4	1	−

(d)	(6)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_3})$	10	1	3	1	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_3})$	10	3	3		−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_3})$	10	3	4		+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_3})$	10	1	4	1	−

(d)	(6)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_4})$	10	1	3	3	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_4})$	10	3	3		−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_4})$	10	3	6		+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_4})$	10	1	6	3	−

TABLE 32. The radical 5-chains of McL

Chain C	$N_G(C)$
$C_1 : 1$	G
$C_2 : 1 < S$	$5_+^{1+2} : 3 : 8$

TABLE 33. Conjecture 1.4 for McL, $p = 5$

$(d, [\kappa]$	$(3, [1])$	$(3, [2])$	$(2, [1])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_1})$	9	4	6	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_1})$	9	4	6	−

$(d, [\kappa], t_i)$	$(3, [1])$	$(3, [2])$	$(2, [1])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_2})$	9	4	2	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_2})$	9	4	2	−

$(d, [\kappa]$	$(3, [1])$	$(3, [2])$	$(2, [1])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_3})$	5	4	6	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_3})$	5	4	6	−

$(d, [\kappa], t_i)$	$(3, [1])$	$(3, [2])$	$(2, [1])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_4})$	5	4	2	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_4})$	5	4	2	−

13. He

Let G be the simple Held group He. We assume that $p = 2$. By [3, (4D)] , let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_{12}\}$ in Table 34. The principal

TABLE 34. The radical 2-chains of He

$C_1 : 1$	$C_7 : 1 < (2^6)' < (2^{1+6}_+.2^2)'$
$C_2 : 1 < 2^{1+6}_+$	$C_8 : 1 < (2^6)' < 2^4.2^4 < S$
$C_3 : 1 < 2^6 < 2^{1+6}_+.2^2$	$C_9 : 1 < 2^2$
$C_4 : 1 < 2^6$	$C_{10} : 1 < 2^2 < (2^6)'$
$C_5 : 1 < (2^6)' < 2^4.2^4$	$C_{11} : 1 < 2^2 < 2^6$
$C_6 : 1 < (2^6)'$	$C_{12} : 1 < 2^2 < (2^6)' < 2^4.2^4$

TABLE 35. Conjecture 1.4 for He, $p = 2$

(d)	(10)	(9)	(8)	(7)	(6)	(4)	(3)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	16	4	2	2	1	1			$+$
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	16	4	2	6	1	1			$-$
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	16	12	10	6	1				$+$
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	16	12	2	2	1				$-$
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$	16	20	2	2	1				$+$
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_2})$	16	12	2	2	1				$-$
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_2})$	16	12	10	6	1				$+$
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_2})$	16	20	18	6	1				$-$
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_2})$		8	12	4	2				$+$
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_2})$		8	12	4	2				$-$
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_2})$		8	12	4	2				$+$
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_2})$		8	12	4	2				$-$
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_2})$							4	1	$+$
$\mathrm{k}(N_G(C_9), B_1, d, [1], \sigma_{t_2})$							4	1	$-$

block B_0 of defect 10 and the block B_1 of defect 3 are 2-blocks of G with positive defect.

Let t_1 be such that $\left(\frac{-1}{t_1}\right) = 1$ and t_2 such that $\left(\frac{-1}{t_2}\right) = -1$. Then the value of $\mathrm{k}(N_G(C), B_j, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C, B_j ($j = 1, 2$) and d, and thus the result follows from [3, (5B)]. For σ_{t_2}, we have Table 35.

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

We assume that $p = 3$. By [3, (4C)], let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_6\}$ in Table 36. The principal block B_0 of defect 3, the block B_1 of defect 2 and three blocks of defect 1 are 3-blocks of G with positive defect. It suffices to consider B_0 and B_1.

Let t_1 be such that $\left(\frac{t_1}{3}\right) = 1$ and e is even, t_2 such that $\left(\frac{t_2}{3}\right) = -1$ and e is even, t_3 such that $\left(\frac{t_3}{3}\right) = 1$ and e is odd and t_4 such that $\left(\frac{t_4}{3}\right) = -1$ and e is odd in the condition (3.1) in §3. Then the value of $\mathrm{k}(N_G(C), B_j, d, [\kappa], \sigma_{t_1})$ is equal to that

TABLE 36. The radical 3-chains of He

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 3'$	$\mathfrak{S}_3 \times L_2(7)$
$C_2 : 1 < 3$	$3.\mathfrak{S}_7$	$C_5 : 1 < 3' < 3^{2'}$	$\mathfrak{S}_3 \times \mathfrak{S}_3$
$C_3 : 1 < 3 < 3^2$	$(3^2 \times 2^2).(2 \times \mathfrak{S}_3)$	$C_6 : 1 < 3^2$	$(3^2 \times 2^2).GL_2(3).2$

TABLE 37. Conjecture 1.4 for He, $p = 3$

(d, t_i)	$(3, t_2)$	$(2, t_2)$	$(3, t_3)$	$(2, t_3)$	$(3, t_4)$	$(2, t_4)$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_i})$	9	2	9	2	9	4	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_i})$	9	2	9	2	9	4	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_i})$	9	2	9	2	9	2	+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_i})$		9		9		9	−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_i})$		9		9		9	+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_i})$	9	2	9	2	9	2	−
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_i})$		9		3		3	+
$\mathrm{k}(N_G(C_2), B_1, d, [1], \sigma_{t_i})$		9		9		9	−
$\mathrm{k}(N_G(C_3), B_1, d, [1], \sigma_{t_i})$		9		9		9	+
$\mathrm{k}(N_G(C_6), B_1, d, [1], \sigma_{t_i})$		9		3		3	−

TABLE 38. The radical 5-chains of He

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 5 < S$	$5^2.(4 \times 2)$
$C_2 : 1 < 5$	$(5 \times \mathfrak{A}_5).4$	$C_4 : 1 < S$	$5^2 : (4\mathfrak{A}_4)$

without κ and σ_{t_1} for every C, B_j $(j = 1, 2)$ and d, and thus the result follows from [3, (5A)]. For σ_{t_i} $(i = 2, 3, 4)$, we have Table 37.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

We assume that $p = 5$. The radical 5-chains of G (up to conjugacy) are given in Table 38. The principal block B_0 of defect 2 and the block of defect 1 are 5-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 and t_2 be such that e is even and odd, respectively, in the condition (3.1) in §3. Then we have Table 39.

Hence Conjecture 1.4 holds for G in the case of $p = 5$.

We assume that $p = 7$. The radical 7-chains of G (up to conjugacy) are given in Table 40. The principal block B_0 of defect 3 and the block of defect 1 are 7-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{7}\right) = 1$ and t_2 such that $\left(\frac{t_2}{7}\right) = -1$. Then we have Table 41.

Hence Conjecture 1.4 holds for G in the case of $p = 7$.

TABLE 39. Conjecture 1.4 for He, $p = 5$

$(d, [\kappa], t_i)$	$(2, [1], t_1)$	$(2, [2], t_1)$	$(2, [1], t_2)$	$(2, [2], t_2)$	Parity
$k(N_G(C_1), B_0, d, [\kappa], \sigma_{t_i})$	8	8	4	4	+
$k(N_G(C_2), B_0, d, [\kappa], \sigma_{t_i})$	2	12	2	12	−
$k(N_G(C_3), B_0, d, [\kappa], \sigma_{t_i})$	2	12	2	12	+
$k(N_G(C_4), B_0, d, [\kappa], \sigma_{t_i})$	8	8	4	4	−

TABLE 40. The radical 7-chains of He

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 7^2$	$7^2.SL_2(7)$
$C_2 : 1 < 7$	$F_7^3 \times L_2(7)$	$C_5 : 1 < 7^2 < S$	$S.6$
$C_3 : 1 < 7 < 7^{2'}$	$F_7^3 \times F_7^3$	$C_6 : 1 < S$	$S.(\mathfrak{S}_3 \times 3)$

TABLE 41. Conjecture 1.4 for He, $p = 7$

$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(3, [3])$	$(2, [1])$	$(2, [2])$	$(2, [3])$	Parity
$k(N_G(C_1), B_0, d, [\kappa], \sigma_{t_1})$	1	7	12			3	+
$k(N_G(C_2), B_0, d, [\kappa], \sigma_{t_1})$				4	9	12	−
$k(N_G(C_3), B_0, d, [\kappa], \sigma_{t_1})$				4	9	12	+
$k(N_G(C_4), B_0, d, [\kappa], \sigma_{t_1})$	13		4	1			−
$k(N_G(C_5), B_0, d, [\kappa], \sigma_{t_1})$	13		4	1			+
$k(N_G(C_6), B_0, d, [\kappa], \sigma_{t_1})$	1	7	12			3	−
$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(3, [3])$	$(2, [1])$	$(2, [2])$	$(2, [3])$	Parity
$k(N_G(C_1), B_0, d, [\kappa], \sigma_{t_2})$	1	3	6			1	+
$k(N_G(C_2), B_0, d, [\kappa], \sigma_{t_2})$					9		−
$k(N_G(C_3), B_0, d, [\kappa], \sigma_{t_2})$					9		+
$k(N_G(C_4), B_0, d, [\kappa], \sigma_{t_2})$	7			1			−
$k(N_G(C_5), B_0, d, [\kappa], \sigma_{t_2})$	7			1			+
$k(N_G(C_6), B_0, d, [\kappa], \sigma_{t_2})$	1	3	6			1	−

14. O′N

Let G be the simple O'Nan group O′N. We assume that $p = 2$. By [24, Proposition 5.1], let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4\}$ in Table 42. The principal block B_0 of defect 9 and the block B_1 of defect 3 are 2-blocks of G with positive defect.

Let t_1 be such that $\left(\frac{2}{t_1}\right) = 1$ and e is even, t_2 such that $\left(\frac{2}{t_2}\right) = -1$ and e is even, t_3 such that $\left(\frac{2}{t_3}\right) = 1$ and e is odd and t_4 such that $\left(\frac{2}{t_4}\right) = -1$ and e is odd in the condition (3.1) in §3. Then the value of $k(N_G(C), B_j, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C, B_j $(j = 1, 2)$ and d, and thus the result follows from [24, 5.6]. For σ_{t_i} $(i = 2, 3, 4)$, we have Table 43.

TABLE 42. The radical 2-chains of O′N

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 4^3$	$4^3.L_3(2)$
$C_2 : 1 < 4$	$4.L_3(4) : 2$	$C_4 : 1 < 4^3 < (4 \times 2^2).2^4$	$(4 \times 2^2).2^4.\mathfrak{S}_3$

TABLE 43. Conjecture 1.4 for O′N, $p = 2$

(d)	(9)	(8)	(7)	(6)	(4)	(3)	(2)	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	8	2	1	1				+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	8	2	5	3				−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	8	2	1	1				−
$k(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	8	2	5	3				+
$k(N_G(C_1), B_1, d, [1], \sigma_{t_2})$						4	1	+
$k(N_G(C_2), B_1, d, [1], \sigma_{t_2})$						4	1	−
(d)	(9)	(8)	(7)	(6)	(4)	(3)	(2)	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_3})$	8	6	1	1	2			+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_3})$	8	6	5	3	2			−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_3})$	8	6	1	1				−
$k(N_G(C_4), B_0, d, [1], \sigma_{t_3})$	8	6	5	3				+
$k(N_G(C_1), B_1, d, [1], \sigma_{t_3})$						4	1	+
$k(N_G(C_2), B_1, d, [1], \sigma_{t_3})$						4	1	−
(d)	(9)	(8)	(7)	(6)	(4)	(3)	(2)	Parity
$k(N_G(C_1), B_0, d, [1], \sigma_{t_4})$	8	2	3	1				+
$k(N_G(C_2), B_0, d, [1], \sigma_{t_4})$	8	2	5	5				−
$k(N_G(C_3), B_0, d, [1], \sigma_{t_4})$	8	2	3	1				−
$k(N_G(C_4), B_0, d, [1], \sigma_{t_4})$	8	2	5	5				+
$k(N_G(C_1), B_1, d, [1], \sigma_{t_4})$						4	1	+
$k(N_G(C_2), B_1, d, [1], \sigma_{t_4})$						4	1	−

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

We assume that $p = 3$. The radical 3-chains of G (up to conjugacy) are given in Table 44. The principal block B_0 of defect 4 and the block B_1 of defect 2 are 3-blocks of G with positive defect.

Let t_1 and t_2 be such that e is even and odd, respectively, in the condition (3.1) in §3. Then the value of $k(N_G(C), B_j, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C, B_j ($j = 1, 2$) and d, and thus the result follows from [24, 6.6]. For σ_{t_2}, we have Table 45.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

TABLE 44. The radical 3-chains of O′N

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 3^2 < S$	$3^4.2^2.2^3$
$C_2 : 1 < 3^2$	$(3^2 : 4 \times \mathfrak{A}_6).2$	$C_4 : 1 < S$	$3^4 : 2_-^{1+4}.D_{10}$

TABLE 45. Conjecture 1.4 for O′N, $p = 3$

(d)	(4)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	12		$+$
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	20		$-$
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	20		$+$
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	12		$-$
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_2})$		6	$+$
$\mathrm{k}(N_G(C_2), B_1, d, [1], \sigma_{t_2})$		6	$-$

TABLE 46. The radical 7-chains of O′N

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 7^{2'}$	$7^2 : SL_2(7) : 2$
$C_2 : 1 < 7^2$	$7^2 : SL_2(7) : 2$	$C_5 : 1 < 7^{2'} < S$	$7_+^{1+2} : (3 \times 2^2)$
$C_3 : 1 < 7^2 < S$	$7_+^{1+2} : (3 \times 2^2)$	$C_6 : 1 < S$	$7_+^{1+2} : (3 \times D_8)$

We assume that $p = 7$. The radical 7-chains of G (up to conjugacy) are given in Table 46. The principal block B_0 of defect 3 is the unique 7-block of G with positive defect.

Let t_1 be such that $\left(\frac{t_1}{7}\right) = 1$ and e is even, t_2 such that $\left(\frac{t_2}{7}\right) = -1$ and e is even, t_3 such that $\left(\frac{t_3}{7}\right) = 1$ and e is odd and t_4 such that $\left(\frac{t_4}{7}\right) = -1$ and e is odd in the condition (3.1) in §3. For σ_{t_i} $(i = 1, 4)$ and σ_{t_j} $(j = 2, 3)$, we have Table 47.

Hence Conjecture 1.4 holds for G in the case of $p = 7$.

15. Co_3

Let G be the simple Conway's third group Co_3. We assume that $p = 2$. By [4, (3C)], let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_{16}\}$ in Table 48. The principal block B_0 of defect 10, the block B_1 of defect 3 and the block of defect 1 are 2-blocks of G with positive defect. It suffices to consider B_0 and B_1.

Let t_1 be such that $\left(\frac{-1}{t_1}\right) = 1$ and e is even, t_2 such that $\left(\frac{-1}{t_2}\right) = -1$ and e is even, t_3 such that $\left(\frac{-1}{t_3}\right) = 1$ and e is odd and t_4 such that $\left(\frac{-1}{t_4}\right) = -1$ and e is odd in the condition (3.1) in §3. Then the value of $\mathrm{k}(N_G(C), B_j, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C, B_j $(j = 1, 2)$ and d, and thus the result follows from [4, (4B)]. For σ_{t_i} $(i = 2, 3, 4)$, we have Table 49 and Table 50.

Table 47. Conjecture 1.4 for O′N, $p = 7$

$(d, [\kappa])$	$(3,[1])$	$(3,[2])$	$(3,[3])$	$(2,[2])$	$(2,[3])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_i})$	1	7	12		4	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_i})$	3	16		2		−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_i})$	3	16		2		+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_i})$	3	16		2		−
$\mathrm{k}(N_G(C_5), B_0, d, [\kappa], \sigma_{t_i})$	3	16		2		+
$\mathrm{k}(N_G(C_6), B_0, d, [\kappa], \sigma_{t_i})$	1	7	12		4	−
$(d, [\kappa])$	$(3,[1])$	$(3,[2])$	$(3,[3])$	$(2,[2])$	$(2,[3])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_j})$	1	7	12		2	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_j})$	3	16		2		−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_j})$	3	16		2		+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_j})$	3	16		2		−
$\mathrm{k}(N_G(C_5), B_0, d, [\kappa], \sigma_{t_j})$	3	16		2		+
$\mathrm{k}(N_G(C_6), B_0, d, [\kappa], \sigma_{t_j})$	1	7	12		2	−

Table 48. The radical 2-chains of Co_3

$C_1 : 1$	$C_9 : 1 < 2' < 2^3.2^3$
$C_2 : 1 < 2$	$C_{10} : 1 < 2'$
$C_3 : 1 < 2 < 2^2.2^6$	$C_{11} : 1 < 2' < 2.2^{1+4}_+$
$C_4 : 1 < 2^2.2^6$	$C_{12} : 1 < 2' < 2^3 < (2^4)'$
$C_5 : 1 < 2^4 < 2^{1+6}_+$	$C_{13} : 1 < 2' < 2^3$
$C_6 : 1 < 2^4$	$C_{14} : 1 < 2' < 2^3.2^3 < 2^3.2^3.2$
$C_7 : 1 < 2^4 < 2^2.2^6$	$C_{15} : 1 < 2^3 < (2^4)'$
$C_8 : 1 < 2^4 < 2^2.2^6 < 2.2^3.2^5$	$C_{16} : 1 < 2^3$

Hence Conjecture 1.4 holds for G in the case of $p = 2$.

We assume that $p = 3$. By [4, (3B)], let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_6\}$ in Table 51. The principal block B_0 of defect 7 and the block of defect 1 are 3-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{3}\right) = 1$ and e is even, t_2 such that $\left(\frac{t_2}{3}\right) = -1$ and e is even, t_3 such that $\left(\frac{t_3}{3}\right) = 1$ and e is odd and t_4 such that $\left(\frac{t_4}{3}\right) = -1$ and e is odd in the condition (3.1) in §3. Then the value of $\mathrm{k}(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d, and thus the result follows from [4, (4A)]. For σ_{t_i} $(i = 2, 3, 4)$, we have Table 52.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

We assume that $p = 5$. The radical 5-chains of G (up to conjugacy) are given in Table 53. The principal block B_0 of defect 3 and the block of defect 1 are 5-blocks of G with positive defect. It suffices to consider only B_0.

TABLE 49. Conjecture 1.4 for Co_3, $p = 2$

(d)	(10)	(9)	(8)	(7)	(6)	(5)	(4)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	16	4	2	6	1		1	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	16	4	2	10	6		1	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	16	12	10	10	6			+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	16	12	10	6	1			−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$	16	4	2	6	1		1	+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_2})$	16	4	2	2			1	−
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_2})$	16	12	10	2				+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_2})$	16	12	10	6	1			−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_2})$				16	12			+
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_2})$				16	4		2	−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_2})$				16	4	4	2	+
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_2})$							16	−
$\mathrm{k}(N_G(C_{13}), B_0, d, [1], \sigma_{t_2})$							16	+
$\mathrm{k}(N_G(C_{14}), B_0, d, [1], \sigma_{t_2})$				16	12	4		−
$\mathrm{k}(N_G(C_{15}), B_0, d, [1], \sigma_{t_2})$							16	+
$\mathrm{k}(N_G(C_{16}), B_0, d, [1], \sigma_{t_2})$							16	−

(d)	(10)	(9)	(8)	(7)	(6)	(5)	(4)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_3})$	16	4	2	6	1		1	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_3})$	16	4	2	10	6		1	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_3})$	16	12	10	10	6			+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_3})$	16	12	10	6	1			−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_3})$	16	4	2	6	1		1	+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_3})$	16	4	2	2			1	−
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_3})$	16	12	10	2				+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_3})$	16	12	10	6	1			−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_3})$				16	12			+
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_3})$				16	4		2	−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_3})$				16	4	4	2	+
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_3})$							8	−
$\mathrm{k}(N_G(C_{13}), B_0, d, [1], \sigma_{t_3})$							8	+
$\mathrm{k}(N_G(C_{14}), B_0, d, [1], \sigma_{t_3})$				16	12	4		−
$\mathrm{k}(N_G(C_{15}), B_0, d, [1], \sigma_{t_3})$							8	+
$\mathrm{k}(N_G(C_{16}), B_0, d, [1], \sigma_{t_3})$							8	−

Let t_1 be such that $\left(\frac{t_1}{5}\right) = 1$ and t_2 such that $\left(\frac{t_2}{5}\right) = -1$. Then we have Table 54. Hence Conjecture 1.4 holds for G in the case of $p = 5$.

TABLE 50. Conjecture 1.4 for Co_3, $p = 2$

(d)	(10)	(9)	(8)	(7)	(6)	(5)	(4)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_4})$	16	4	2	6	1		3	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_4})$	16	4	2	10	6		3	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_4})$	16	12	10	10	6			+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_4})$	16	12	10	6	1			−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_4})$	16	4	2	6	1		1	+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_4})$	16	4	2	2			1	−
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_4})$	16	12	10	2				+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_4})$	16	12	10	6	1			−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_4})$				16	12			+
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_4})$				16	4		2	−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_4})$				16	4	4	2	+
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_4})$							8	−
$\mathrm{k}(N_G(C_{13}), B_0, d, [1], \sigma_{t_4})$							8	+
$\mathrm{k}(N_G(C_{14}), B_0, d, [1], \sigma_{t_4})$				16	12	4		−
$\mathrm{k}(N_G(C_{15}), B_0, d, [1], \sigma_{t_4})$							8	+
$\mathrm{k}(N_G(C_{16}), B_0, d, [1], \sigma_{t_4})$							8	−

(d, t_i)	$(3, t_2)$	$(3, t_3)$	$(3, t_4)$	Parity
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_i})$	8	4	4	+
$\mathrm{k}(N_G(C_{10}), B_1, d, [1], \sigma_{t_i})$	8	4	4	−
$\mathrm{k}(N_G(C_{13}), B_1, d, [1], \sigma_{t_i})$	8	4	4	+
$\mathrm{k}(N_G(C_{16}), B_1, d, [1], \sigma_{t_i})$	8	4	4	−

TABLE 51. The radical 3-chains of Co_3

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 3^5$	$3^5 : (2 \times \mathrm{M}_{11})$
$C_2 : 1 < 3$	$\mathfrak{S}_3 \times L_2(8) : 3$	$C_5 : 1 < 3^5 < S$	$S : (2 \times SD_{2^4})$
$C_3 : 1 < 3 < 3 \times (9 : 3)$	$(3 \times (9 : 3)).2^2$	$C_6 : 1 < 3_+^{1+4}$	$3_+^{1+4} : 4\mathfrak{S}_6$

TABLE 54. Conjecture 1.4 for Co_3, $p = 5$

$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_1})$	10	10	2	4	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_1})$			10	10	−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_1})$			10	10	+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_1})$	10	10	2	4	−

$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_2})$	10	10	2	2	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_2})$				10	−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_2})$				10	+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_2})$	10	10	2	2	−

TABLE 52. Conjecture 1.4 for Co_3, $p = 3$

(d)	(7)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	33	4		2	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$			9	3	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$			9	3	+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	33	4			−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$	33	19			+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_2})$	33	19		2	−
(d)	(7)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_3})$	33	4		2	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_3})$			27	3	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_3})$			27	3	+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_3})$	33	4			−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_3})$	33	17			+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_3})$	33	17		2	−
(d)	(7)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_4})$	33	4		2	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_4})$			9	3	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_4})$			9	3	+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_4})$	33	4			−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_4})$	33	17			+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_4})$	33	17		2	−

TABLE 53. The radical 5-chains of Co_3

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 5 < 5^2$	$(5 \times F_5^2).4$
$C_2 : 1 < 5$	$(5 \times \mathfrak{A}_5).4$	$C_4 : 1 < S$	$5_+^{1+2}.24.2$

TABLE 55. The radical 3-chains of Ru

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 3^2$	$3^2 : GL_2(3)$
$C_2 : 1 < 3$	$(3.\mathfrak{A}_6).2^2$	$C_4 : 1 < 3^2 < S$	$3_+^{1+2} : 2^2$

TABLE 56. Conjecture 1.4 for Ru, $p = 3$

(d)	(3)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	9	3	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	9	3	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	9	2	−
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	9	2	+

TABLE 57. The radical 5-chains of Ru

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 5^2$	$5^2 : GL_2(5)$
$C_2 : 1 < 5$	$5 : 4 \times \mathfrak{A}_5$	$C_5 : 1 < 5^2 < S$	$5_+^{1+2} : 4^2$
$C_3 : 1 < 5 < 5^2$	$5 : 4 \times 5 : 2$	$C_6 : 1 < S$	$5_+^{1+2} : 4.D_8$

16. Ru

Let G be the simple Rudvalis group Ru. We assume that $p = 3$. By [6, Lemma 7.2], let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4\}$ in Table 55. The principal block B_0 of defect 3 and two blocks of defect 1 are 3-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{3}\right) = 1$ and t_2 such that $\left(\frac{t_2}{3}\right) = -1$. Then the value of $\mathrm{k}(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d, and thus the result follows from [6, Theorem 9.1]. For σ_{t_2}, we have Table 56.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

We assume that $p = 5$. The radical 5-chains of G (up to conjugacy) are given in Table 57. The principal block B_0 of defect 3 and the block of defect 1 are 5-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{5}\right) = 1$ and t_2 such that $\left(\frac{t_2}{5}\right) = -1$. Then we have Table 58.

Hence Conjecture 1.4 holds for G in the case of $p = 5$.

TABLE 58. Conjecture 1.4 for Ru, $p = 5$

$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_1})$	10	10	1	4	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_1})$			10	10	−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_1})$			10	10	+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_1})$	25		4		−
$\mathrm{k}(N_G(C_5), B_0, d, [\kappa], \sigma_{t_1})$	25		4		+
$\mathrm{k}(N_G(C_6), B_0, d, [\kappa], \sigma_{t_1})$	10	10	1	4	−
$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_2})$	10	10	1	2	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_2})$				10	−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_2})$				10	+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_2})$	25		4		−
$\mathrm{k}(N_G(C_5), B_0, d, [\kappa], \sigma_{t_2})$	25		4		+
$\mathrm{k}(N_G(C_6), B_0, d, [\kappa], \sigma_{t_2})$	10	10	1	2	−

TABLE 59. The radical 3-chains of Suz

$C_1 : 1$	$C_6 : 1 < 3 < 3^{2+4} < S$	$C_{11} :< 3^5 < S$
$C_2 : 1 < 3$	$C_7 : 1 < 3 < S$	$C_{12} :< 3^{2+4}$
$C_3 : 1 < 3 < 3^5$	$C_8 : 1 < 3^2$	$C_{13} :< 3^{2+4} < S$
$C_4 : 1 < 3 < 3^5 < S$	$C_9 : 1 < 3^2 < 3^4$	$C_{14} :< S$
$C_5 : 1 < 3 < 3^{2+4}$	$C_{10} : 1 < 3^5$	

17. Suz

Let G be the simple Suzuki group Suz. We assume that $p = 3$. The radical 3-chains of G (up to conjugacy) are given in Table 59. Let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, \cdots, C_{12}\}$. The principal block B_0 of defect 7, the block B_1 of defect 2 and the block of defect 1 are 3-blocks of G with positive defect. It suffices to consider B_0 and B_1.

Let t_1 be such that $\left(\frac{t_1}{3}\right) = 1$ and e is even, t_2 such that $\left(\frac{t_2}{3}\right) = -1$ and e is even, t_3 such that $\left(\frac{t_3}{3}\right) = 1$ and e is odd and t_4 such that $\left(\frac{t_4}{3}\right) = -1$ and e is odd in the condition (3.1) in §3. Then we have Table 60 and Table 61.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

We assume that $p = 5$. The radical 5-chains of G (up to conjugacy) are given in Table 62. The principal block B_0 of defect 2 and the block of defect 1 are 5-blocks of G with positive defect. It suffices to consider only B_0.

For all σ, we have Table 63 since all the relevant characters are invariant under any such σ.

Hence Conjecture 1.4 holds for G in the case of $p = 5$.

Table 60. Conjecture 1.4 for Suz, $p = 3$

(d)	(7)	(6)	(5)	(4)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_1})$	18	9	3	4		+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_1})$	12	6	12	5		−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_1})$	12	3	12			+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_1})$	12	3	18			−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_1})$	12	6	18	5		+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_1})$	12	3	18			−
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_1})$	12	3	18			+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_1})$				30		−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_1})$				30		+
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_1})$	18	3	3			−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_1})$	18	3	18			+
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_1})$	18	9	18	4		−
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_1})$					6	+
$\mathrm{k}(N_G(C_8), B_1, d, [1], \sigma_{t_1})$					6	−
(d)	(7)	(6)	(5)	(4)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	12	9	1	4		+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$	8	6	8	5		−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$	8	3	8			+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$	8	3	10			−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$	8	6	10	5		+
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_2})$	8	3	10			−
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_2})$	8	3	10			+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_2})$				30		−
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_2})$				30		+
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_2})$	12	3	1			−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_2})$	12	3	6			+
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_2})$	12	9	6	4		−
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_2})$					6	+
$\mathrm{k}(N_G(C_8), B_1, d, [1], \sigma_{t_2})$					6	−

TABLE 61. Conjecture 1.4 for Suz, $p = 3$

(d)	(7)	(6)	(5)	(4)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_3})$	18	7	1	4		$+$
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_3})$	10	6	10	5		$-$
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_3})$	10	3	10			$+$
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_3})$	10	3	14			$-$
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_3})$	10	6	14	5		$+$
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_3})$	10	3	14			$-$
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_3})$	10	3	14			$+$
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_3})$				28		$-$
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_3})$				28		$+$
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_3})$	18	3	1			$-$
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_3})$	18	3	16			$+$
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_3})$	18	7	16	4		$-$
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_3})$					6	$+$
$\mathrm{k}(N_G(C_8), B_1, d, [1], \sigma_{t_3})$					6	$-$
(d)	(7)	(6)	(5)	(4)	(2)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_4})$	12	7	3	4		$+$
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_4})$	10	6	10	5		$-$
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_4})$	10	3	10			$+$
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_4})$	10	3	14			$-$
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_4})$	10	6	14	5		$+$
$\mathrm{k}(N_G(C_6), B_0, d, [1], \sigma_{t_4})$	10	3	14			$-$
$\mathrm{k}(N_G(C_7), B_0, d, [1], \sigma_{t_4})$	10	3	14			$+$
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_4})$				28		$-$
$\mathrm{k}(N_G(C_9), B_0, d, [1], \sigma_{t_4})$				28		$+$
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_4})$	12	3	3			$-$
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_4})$	12	3	6			$+$
$\mathrm{k}(N_G(C_{12}), B_0, d, [1], \sigma_{t_4})$	12	7	6	4		$-$
$\mathrm{k}(N_G(C_1), B_1, d, [1], \sigma_{t_4})$					6	$+$
$\mathrm{k}(N_G(C_8), B_1, d, [1], \sigma_{t_4})$					6	$-$

TABLE 62. The radical 5-chains of Suz

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_4 : 1 < 5'$	$(5 \times \mathfrak{A}_6) : 4$
$C_2 : 1 < 5$	$(5 \times \mathfrak{A}_5) : 4$	$C_5 : 1 < 5' < S$	$(5^2 : 4) : 2$
$C_3 : 1 < 5 < S$	$(5^2 : 4) : 2$	$C_6 : 1 < S$	$((5^2 : 3) : 4) : 2$

Table 63. Conjecture 1.4 for Suz, $p = 5$

$(d, [\kappa])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma)$	8	8	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma)$	2	12	−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma)$	2	12	+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma)$	2	12	−
$\mathrm{k}(N_G(C_5), B_0, d, [\kappa], \sigma)$	2	12	+
$\mathrm{k}(N_G(C_6), B_0, d, [\kappa], \sigma)$	8	8	−

Table 64. The radical 3-chains of Co_2

$C_1 : 1$
$C_2 : 1 < 3$
$C_3 : 1 < 3 < 3^4$
$C_4 : 1 < 3 < 3^4 < 3 \times (3^3 : 3)$
$C_5 : 1 < 3 < 3 \times 3^{1+2}_+$
$C_6 : 1 < 3 < 3 \times 3^{1+2}_+ < 3 \times (3^3 : 3)$
$C_7 : 1 < 3 < 3 \times (3^3 : 3)$
$C_8 : 1 < 3^4$ $C_9 : 1 < 3^4 < S$
$C_{10} : 1 < 3^{1+4}_+$
$C_{11} : 1 < 3^{1+4}_+ < S$
$C_{12} : 1 < S$

Table 65. Conjecture 1.4 for Co_2, $p = 3$

(d)	(6)	(5)	(4)	(3)	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [1], \sigma_{t_2})$	27	6	9	1	+
$\mathrm{k}(N_G(C_2), B_0, d, [1], \sigma_{t_2})$		27	39	3	−
$\mathrm{k}(N_G(C_3), B_0, d, [1], \sigma_{t_2})$		27	39		+
$\mathrm{k}(N_G(C_4), B_0, d, [1], \sigma_{t_2})$		27	24		−
$\mathrm{k}(N_G(C_5), B_0, d, [1], \sigma_{t_2})$		27	24	3	+
$\mathrm{k}(N_G(C_8), B_0, d, [1], \sigma_{t_2})$	27	6	9		−
$\mathrm{k}(N_G(C_{10}), B_0, d, [1], \sigma_{t_2})$	27	6	12	1	−
$\mathrm{k}(N_G(C_{11}), B_0, d, [1], \sigma_{t_2})$	27	6	12		+

18. Co_2

Let G be the simple Conway's second group Co_2. We assume that $p = 3$. The radical 3-chains of G (up to conjugacy) are given in Table 64. By [7, (5B)], let $\mathcal{R}_0(G) \subset \mathcal{R}(G)$ such that $\mathcal{R}_0(G)/G = \{C_1, C_2, C_3, C_4, C_5, C_8, C_{10}, C_{11}\}$ The principal block B_0 of defect 6 and two blocks of defect 1 are 3-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{15}\right) = 1$ and t_2 such that $\left(\frac{t_2}{15}\right) = -1$. Then the value of $\mathrm{k}(N_G(C), B_0, d, [\kappa], \sigma_{t_1})$ is equal to that without κ and σ_{t_1} for every C and d, and thus the result follows from [7, (6A)]. For σ_{t_2}, we have Table 65.

Hence Conjecture 1.4 holds for G in the case of $p = 3$.

TABLE 66. The radical 5-chains of Co_2

Chain C	$N_G(C)$	Chain C	$N_G(C)$
$C_1 : 1$	G	$C_3 : 1 < 5 < 5^2$	$F_5^4 \times F_5^4$
$C_2 : 1 < 5$	$F_5^4 \times \mathfrak{S}_5$	$C_4 : 1 < S$	$5_+^{1+2} : (4\mathfrak{S}_4)$

TABLE 67. Conjecture 1.4 for Co_2, $p = 5$

$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_1})$	10	10	3	4	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_1})$			25		−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_1})$			25		+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_1})$	10	10	3	4	−
$(d, [\kappa])$	$(3, [1])$	$(3, [2])$	$(2, [1])$	$(2, [2])$	Parity
$\mathrm{k}(N_G(C_1), B_0, d, [\kappa], \sigma_{t_2})$	10	10	1	2	+
$\mathrm{k}(N_G(C_2), B_0, d, [\kappa], \sigma_{t_2})$			25		−
$\mathrm{k}(N_G(C_3), B_0, d, [\kappa], \sigma_{t_2})$			25		+
$\mathrm{k}(N_G(C_4), B_0, d, [\kappa], \sigma_{t_2})$	10	10	1	2	−

We assume that $p = 5$. The radical 5-chains of G (up to conjugacy) are given in Table 66. The principal block B_0 of defect 3 and two blocks of defect 1 are 5-blocks of G with positive defect. It suffices to consider only B_0.

Let t_1 be such that $\left(\frac{t_1}{15}\right) = 1$ and t_2 such that $\left(\frac{t_2}{15}\right) = -1$. Then we have Table 67. Hence Conjecture 1.4 holds for G in the case of $p = 5$.

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