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A NEW GENERALIZATION OF THE POISSON KERNEL

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Abstract. The purpose of this paper is to give a new generalization of the Poisson Kernel in two dimensions and discuss an integral formula for this.

1. Introduction

The Poisson Kernel in two dimensions is defined by

$$P_r(\theta) = \frac{1-r^2}{\left(1-re^{i\theta}\right)\left(1-re^{-i\theta}\right)}. \tag{1.1}$$

Here r is a real parameter satisfying $|r| < 1$, and $-\infty < \theta < \infty$. It is well-known that $P_r(\theta)$ is periodic in θ with period 2π and the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)d\theta = 1 \tag{1.2}$$

holds.

In [2], Haruki and Rassias gave the following new definitions (1.3) and (1.5):

First, they set

$$Q(\theta; a, b) \overset{\text{def}}{=} \frac{1-ab}{\left(1-ae^{i\theta}\right)\left(1-be^{-i\theta}\right)}, \tag{1.3}$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Remark 1. If we take $a = r$ and $b = r$ in (1.3), then we find that (1.3) is a generalization of (1.1).

Afterwards, they proved the following theorem.

Theorem 1.1.

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a, b)d\theta = 1, \tag{1.4}$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Remark 2. Note that, (1.4) is a generalization of (1.2).

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Second, they set

$$R(\theta;a,b,c,d)=\frac{L(a,b,c,d)}{(1-ae^{i\theta})(1-be^{-i\theta})(1-ce^{i\theta})(1-de^{-i\theta})}, \tag{1.5}$$

where a, b, c, d are complex parameters satisfying $|a|<1$, $|b|<1$, $|c|<1$, $|d|<1$ and

$$L(a,b,c,d)\stackrel{\text{def}}{=}\frac{(1-ab)(1-ad)(1-bc)(1-cd)}{1-abcd}. \tag{1.6}$$

Remark 3. If we take $c=0$ and $d=0$ in (1.5), then we find that (1.5) is a generalization of (1.3).

Afterwards, they proved the following theorem.

Theorem 1.2.

$$\frac{1}{2\pi}\int_0^{2\pi}R(\theta;a,b,c,d)d\theta=1, \tag{1.7}$$

where a, b, c, d are complex parameters satisfying $|a|<1$, $|b|<1$, $|c|<1$ and $|d|<1$.

In this paper, we shall generalize (1.5) and (1.7).

Set

$$S(\theta;x,y,z,t,u,v)=\frac{L(x,y,z,t,u,v)}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})}, \tag{1.8}$$

where x, y, z, t, u, v are complex parameters satisfying $|x|<1$, $|y|<1$, $|z|<1$, $|t|<1$, $|u|<1$, $|v|<1$ and

$$L(x,y,z,t,u,v)\stackrel{\text{def}}{=}\frac{(1-xy)(1-xt)(1-xv)(1-yz)(1-yu)(1-zt)(1-zv)(1-tu)(1-uv)}{K(x,y,z,t,u,v)}, \tag{1.9}$$

where

$$\begin{aligned} K(x,y,z,t,u,v) ={}& 1-[xz+xu+zu][yt+yv+tv] \\ &+xzu\left[y^2(t+v)+t^2(y+v)+v^2(y+t)\right] \\ &+ytv\left[x^2(z+u)+z^2(x+u)+u^2(x+z)\right] \\ &-\left[x^2zu+xz^2u+xzu^2\right]\left[y^2tv+yt^2v+ytv^2\right] \\ &+4xyztuv+x^2y^2z^2t^2u^2v^2. \end{aligned} \tag{1.10}$$

Remark 4. By taking $u=0$ and $v=0$ in (1.8), we find that (1.8) is a generalization of (1.5).

The purpose of this paper is to prove the following

Main Theorem.

$$\frac{1}{2\pi}\int_0^{2\pi} S(\theta; x, y, z, t, u, v)d\theta = 1, \tag{1.11}$$

where x, y, z, t, u, v are complex parameters satisfying $|x| < 1$, $|y| < 1$, $|z| < 1$, $|t| < 1$, $|u| < 1$, $|v| < 1$.

2. Proof of the Main Theorem

Proof. We get

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})} \tag{2.1}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\left(e^{i\theta}\right)^2}{(1-xe^{i\theta})(e^{i\theta}-y)(1-ze^{i\theta})(e^{i\theta}-t)(1-ue^{i\theta})(e^{i\theta}-v)} ie^{i\theta}d\theta.$$

If we substitute $w = e^{i\theta}$, then we have

$$ie^{i\theta}d\theta = dw. \tag{2.2}$$

We set

$$f(w) = \frac{w^2}{(1-xw)(w-y)(1-zw)(w-t)(1-uw)(w-v)}. \tag{2.3}$$

The function $f(w)$ is an analytic function in $|w| \leq 1$ except at $w = y$, $w = t$ and $w = v$ each of which is a pole of f.

Here there are five cases:

1) $y \neq t \neq v$
2) $y = t \neq v$
3) $y = v \neq t$
4) $t = v \neq y$
5) $y = t = v$

Case 1. Let $y \neq t \neq v$.

Then, by (2.1), (2.2) and (2.3) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})} \tag{2.4}$$

$$= \frac{1}{2\pi i}\int_{|w|=1} f(w)dw,$$

where the complex integral of the function $f(w)$ along the unit circle $|w| = 1$ is in the positive direction.

Let R_1, R_2 and R_3 denote the residues of $f(w)$ at $w = y$, $w = t$ and $w = v$ each of which is a simple pole of f, respectively. So, by the Residue Theorem ([1]) we have

$$\frac{1}{2\pi i} \int\limits_{|w|=1} f(w)dw = R_1 + R_2 + R_3. \tag{2.5}$$

In the following, we shall calculate this residues R_1, R_2 and R_3.

We get

$$\begin{aligned} R_1 &= \lim_{w\to y} \left[(w-y)\, f(w)\right] \\ &= \lim_{w\to y} \frac{w^2}{(1-xw)(1-zw)(w-t)(1-uw)(w-v)} \quad \text{(by (2.3))} \\ &= \frac{y^2}{(1-xy)(1-zy)(y-t)(1-uy)(y-v)}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} R_2 &= \lim_{w\to t} \left[(w-t)\, f(w)\right] \\ &= \lim_{w\to t} \frac{w^2}{(1-xw)(w-y)(1-zw)(1-uw)(w-v)} \quad \text{(by (2.3))} \\ &= \frac{t^2}{(1-xt)(t-y)(1-zt)(1-ut)(t-v)} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} R_3 &= \lim_{w\to v} \left[(w-v)\, f(w)\right] \\ &= \lim_{w\to v} \frac{w^2}{(1-xw)(w-y)(1-zw)(w-t)(1-uw)} \quad \text{(by (2.3))} \\ &= \frac{v^2}{(1-xv)(v-y)(1-zv)(v-t)(1-uv)}. \end{aligned} \tag{2.8}$$

So, from (2.5), (2.6), (2.7) and (2.8) we obtain

$$\frac{1}{2\pi i} \int\limits_{|w|=1} f(w)dw = \frac{1}{L(x,y,z,t,u,v)} \quad \text{(by (1.9) and (1.10))}. \tag{2.9}$$

Therefore, by (1.8), (2.4) and (2.9) we get

$$\frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, t, u, v)d\theta = 1.$$

Case 2. Let $y = t \neq v$.

Hence, (2.3) is of the form

$$f(w) = \frac{w^2}{(1 - xw)(w - y)^2 (1 - zw)(1 - uw)(w - v)}. \tag{2.10}$$

Thus, by (2.1), (2.2) and (2.10) we obtain

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})} \\ = &\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})^2(1-ze^{i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})} \\ = &\frac{1}{2\pi i}\int_{|w|=1} f(w)dw, \end{aligned} \tag{2.11}$$

where the complex integral of the function $f(w)$ along the unit circle $|w| = 1$ is in the positive direction.

Here, note that $f(w)$ is an analytic function in $|w| \leq 1$ except at $w = y$ which is a double pole of f and $w = v$ which is a simple pole of f.

Let R_1 denotes the residue of $f(w)$ at $w = y$ and R_2 denotes the residue of $f(w)$ at $w = v$. By the Residue Theorem, we have

$$\frac{1}{2\pi i}\int_{|w|=1} f(w)dw = R_1 + R_2. \tag{2.12}$$

First, we shall calculate R_1. By Cauchy's Integral Formula for the derivative ([1]), we have

$$\begin{aligned} R_1 &= \frac{1}{2\pi i}\int_{|w|=1} \frac{w^2}{(1-xw)(1-zw)(1-uw)(w-v)}/(w-y)^2 dw \\ &= \left[\frac{d}{dw}\left(\frac{w^2}{(1-xw)(1-zw)(1-uw)(w-v)}\right)\right]_{w=y} \\ &= \frac{y}{(1-xy)(1-zy)(1-uy)(y-v)}\left[2+\frac{xy}{1-xy}+\frac{zy}{1-zy}+\frac{yu}{1-yu}-\frac{y}{y-v}\right]. \end{aligned} \tag{2.13}$$

Second, we have

$$\begin{aligned} R_2 &= \lim_{w\to v} \left[(w - v) f(w)\right] \\ &= \lim_{w\to v} \frac{w^2}{(1 - xw)(w - y)^2 (1 - zw)(1 - uw)} \end{aligned} \tag{2.14}$$

$$= \frac{v^2}{(1-xv)(v-y)^2(1-zv)(1-uv)}.$$

So, by (2.12), (2.13) and (2.14) we obtain

$$\frac{1}{2\pi i}\int\limits_{|w|=1} f(w)dw = \frac{K(x,y,z,y,u,v)}{(1-xy)^2(1-zy)^2(1-yu)^2(1-xv)(1-uv)(1-zv)} \tag{2.15}$$

$$= \frac{1}{L(x,y,z,y,u,v)} \text{ (by (1.9) and (1.10))}.$$

Thus, by (1.8), (2.11) and (2.15) we have

$$\frac{1}{2\pi}\int\limits_0^{2\pi} S(\theta;x,y,z,y,u,v)d\theta = 1.$$

Case 3 and Case 4. The proofs of the integral formulas

$$\frac{1}{2\pi}\int\limits_0^{2\pi} S(\theta;x,y,z,t,u,y)d\theta = 1$$

and

$$\frac{1}{2\pi}\int\limits_0^{2\pi} S(\theta;x,y,z,t,u,t)d\theta = 1$$

for Case 3 and Case 4, respectively, are similar to the proof of the Case 2.

Case 5. Let $y = t = v$.

In this case, (2.3) is of the form

$$f(w) = \frac{w^2}{(1-xw)(w-y)^3(1-zw)(1-uw)}. \tag{2.16}$$

Thus, by (2.1), (2.2) and (2.16) we obtain

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})} \tag{2.17}$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})^3(1-ze^{i\theta})(1-ue^{i\theta})}$$

$$= \frac{1}{2\pi i}\int\limits_{|w|=1} f(w)dw,$$

where the complex integral of the function $f(w)$ along the unit circle $|w| = 1$ is in the positive direction.

So, $f(w)$ is an analytic function in $|w| \leq 1$ except at $w = y$. Note that $f(w)$ has a pole of order 3 at $w = y$. Let R denotes the residue of f at $w = y$. Then, by the Residue Theorem we get

$$\frac{1}{2\pi i} \int\limits_{|w|=1} f(w)dw = R. \tag{2.18}$$

Therefore, we must calculate R. By Cauchy's Integral Formula for the derivative, we have

$$\begin{aligned} \frac{1}{2\pi i} \int\limits_{|w|=1} f(w)dw &= \frac{1}{2\pi i} \int\limits_{|w|=1} \frac{w^2}{(1-xw)(1-zw)(1-uw)} / (w-y)^3 dw \\ &= \frac{1}{2!} \left[\frac{d^2}{dw^2} \left(\frac{w^2}{(1-xw)(1-zw)(1-uw)} \right) \right]_{w=y} \\ &= \frac{K(x,y,z,y,u,y)}{(1-xy)^3(1-zy)^3(1-uy)^3} \\ &= \frac{1}{L(x,y,z,y,u,y)} \text{ (by (1.9) and (1.10)).} \end{aligned} \tag{2.19}$$

Thus, by (1.8), (2.17) and (2.19) we have

$$\frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, y, u, y) d\theta = 1.$$

From the Cases 1, 2, 3, 4 and 5, we get the desired result (1.11). □

Corollary 2.1. *If we set $z = x$, $u = x$ and $t = y$, $v = y$ in the Main Theorem, then we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1-xy)^3}{(1-xe^{i\theta})^3 (1-ye^{-i\theta})^3} d\theta = \frac{1+4xy+x^2y^2}{(1-xy)^2}. \tag{2.20}$$

Proof. By the Main Theorem, we know that

$$1 = \frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, t, u, v) d\theta.$$

Then, for $z = x$, $u = x$ and $t = y$, $v = y$ we obtain

$$\begin{aligned}
1 &= \frac{1}{2\pi}\int_0^{2\pi} S(\theta; x, y, x, y, x, y) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{L(x, y, x, y, x, y)}{\left(1 - xe^{i\theta}\right)^3 \left(1 - ye^{-i\theta}\right)^3} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - xy)^9}{\left(1 - xe^{i\theta}\right)^3 \left(1 - ye^{-i\theta}\right)^3 K(x, y, x, y, x, y)} d\theta \\
&= \frac{(1 - xy)^6}{K(x, y, x, y, x, y)} \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - xy)^3}{\left(1 - xe^{i\theta}\right)^3 \left(1 - ye^{-i\theta}\right)^3} d\theta \\
&= \frac{(1 - xy)^6}{(1 - xy)^4 (1 + 4xy + x^2y^2)} \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - xy)^3}{\left(1 - xe^{i\theta}\right)^3 \left(1 - ye^{-i\theta}\right)^3} d\theta \\
&= \frac{(1 - xy)^2}{(1 + 4xy + x^2y^2)} \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - xy)^3}{\left(1 - xe^{i\theta}\right)^3 \left(1 - ye^{-i\theta}\right)^3} d\theta,
\end{aligned}$$

where

$$K(x, y, x, y, x, y) = (1 - xy)^4 \left(1 + 4xy + x^2y^2\right).$$

Thus, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - xy)^3}{\left(1 - xe^{i\theta}\right)^3 \left(1 - ye^{-i\theta}\right)^3} d\theta = \frac{1 + 4xy + x^2y^2}{(1 - xy)^2}.$$

□

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