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MULTIPLIERS AND CYCLIC VECTORS ON THE WEIGHTED BLOCH SPACE

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Abstract. In this paper we study the pointwise multipliers and cyclic vectors on the weighted Bloch space $\beta_L = \{f \in H(D) : \sup_D (1 - |z|^2)\ln(\frac{2}{1-|z|})|f'(z)| < +\infty\}$. We obtain a characterization of multipliers on β_L and little β_L^0. Also, a sufficient condition and a necessary condition are given for which f is a cyclic vector in β_L^0.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbf{C}$, and $H(D)$ denote the set of all analytic functions on D. For $f \in H(D)$, Let

$$\|f\|_{\beta_\alpha} = \sup\{(1 - |z|^2)^\alpha |f'(z)| : z \in D\}, \quad 0 < \alpha < +\infty,$$
$$\|f\|_{\beta_L} = \sup\{(1 - |z|^2)\ln(\frac{2}{1-|z|})|f'(z)| : z \in D\}.$$

As in [7], [9], the α-Bloch space β_α consists of all $f \in H(D)$ satisfying $\|f\|_{\beta_\alpha} < +\infty$ and the little α-Bloch space β_α^0 consists of all $f \in H(D)$ satisfying $\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0$; the logarithmic weighted Bloch space β_L consists of all $f \in H(D)$ satisfying $\|f\|_{\beta_L} < +\infty$ and the little logarithmic weighted Bloch space β_L^0 consists of all $f \in H(D)$ satisfying $\lim_{|z|\to 1^-}(1 - |z|^2)\ln(\frac{2}{1-|z|})|f'(z)| = 0$. It can easily proved that β_L is a Banach space under the norm

$$\|f\|_L = |f(0)| + \|f\|_{\beta_L}$$

and that β_L^0 is a closed subspace of β_L. It is well known that with the norm

$$\|f\|_\alpha = |f(0)| + \|f\|_{\beta_\alpha}$$

β_α is a Banach space and β_α^0 is a closed subspace of β_α. It is easily proved that for $0 < \alpha < 1$, $\beta_\alpha \subsetneqq \beta_L \subsetneqq \beta_1$. For more information about β_α, see, for example, [9].

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The space of analytic functions on D of bounded mean oscillation, denoted by $BMOA$, consists of f in H^2 for which

$$\|f\|_{BMOA} = \sup_I \frac{1}{|I|} \int_{S(I)} |f'(z)|^2 (1-|z|^2) dA(z) < +\infty,$$

where $dA(z)$ denotes the Lebesgue measure on D, I denotes a subarc of ∂D, $|I|$ denotes the arclength measure of I and $S(I) = \{re^{i\theta} : 1-r \leq |I|, e^{i\theta} \in I\}$. The subset of $BMOA$, denoted by $VMOA$, consists of f for which

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_{S(I)} |f'(z)|^2 (1-|z|^2) dA(z) = 0.$$

For more details, see [5].

Let X be an analytic function space. We say a function ϕ is a pointwise multiplier on X, if $\phi f \in X$ for all $f \in X$. Let $M(X)$ denote the space of all pointwise multipliers on X. By M_ϕ we denote the operator of multiplication by ϕ: $M_\phi f = \phi f, f \in X$. An application of the closed graph theorem shows that if $\phi \in M(X)$, then M_ϕ is a bounded linear transformation. Hence it has a finite norm $\|M_\phi\|$.

In [2], K. R. M. Attle showed that for $f \in L_a^2(D)$, the Hankel operator $H_f : L_a^1 \longrightarrow L^1$ is bounded if and only if $f \in \beta_L$, and in [3], L. Brown and A. L. Shields proved that M_ϕ is bounded on the classical Bloch space $\beta_1(\beta_1^0)$ if and only if $\phi \in \beta_L \bigcap H^\infty$. R. Yoneda [7] studied the composition operator in β_L space. In Section 2 we will characterize multiplier spaces $M(\beta_L)$ and $M(\beta_L^0)$.

Let Y be an analytic Banach function space and the polynomials are dense in it. For $f \in Y$ and let $[f]$ be the closure in Y of the polynomial multiples of f. Thus f is called a cyclic vector in Y if and only if $[f]$=Y. In [1], [3], L. Brown and A. L. Shields studied cyclic vectors in the classical Bloch space $\beta_1(\beta_1^0)$. In the $BMOA(VMOA)$ space, the author in [6] characterized the cyclic vectors. There are just the following theorem.

Theorem A.
(1) *For $f \in BMOA(VMOA)$, then f is a cyclic vector on $BMOA(VMOA)$ if and only if f is an outer function.*
(2) *If f is an outer function in $\beta_1(\beta_1^0)$, then f is cyclic in $\beta_1(\beta_1^0)$.*
(3) *There exists a singular inner function that is cyclic in β_1.*

In Section 3 we study cyclic vectors in β_L^0.

2. Multipliers in the weighted Bloch space

In this section we shall characterize the pointwise multipliers space $M(\beta_L)$ and $M(\beta_L^0)$. For this purpose, we need the following lemmas.

Lemma 2.1. *If $f \in \beta_L$, then*

(i) $|f(z)| \leq (2 + \ln(\ln \frac{2}{1-|z|}))\|f\|_L$;

(ii) $|f(z) - f(tz)| \leq \ln(\frac{ln\frac{2}{1-|z|}}{ln\frac{2}{1-|tz|}})\|f\|_{\beta_L}$, *for every t with $0 \leq t < 1$.*

Proof. Suppose $f \in \beta_L$ and $z \in D$, then

$$\begin{aligned} |f(z) - f(tz)| &= |z\int_t^1 f'(zt)dt| \leq \|f\|_{\beta_L}\int_t^1 \frac{|z|}{(1-|zt|^2)\ln\frac{2}{1-|zt|}}dt \\ &\leq \|f\|_{\beta_L}\int_{t|z|}^{|z|}\frac{dx}{(1-x)\ln\frac{2}{1-x}} \\ &= \|f\|_{\beta_L}(\ln\ln\tfrac{2}{1-|z|} - \ln\ln\tfrac{2}{1-|tz|}) \\ &\leq \ln(\frac{ln\frac{2}{1-|z|}}{ln\frac{2}{1-|tz|}})\|f\|_{\beta_L}. \end{aligned}$$

Especially, $|f(z) - f(0)| \leq \|f\|_{\beta_L}(\ln\ln\frac{2}{1-|z|} - \ln\ln 2)$, hence

$$|f(z)| \leq (2 + \ln\ln\frac{2}{1-|z|})\|f\|_L.$$

□

Lemma 2.2. *If $f \in \beta_L^0$, then* $\lim_{|z|\to 1^-}\frac{|f(z)|}{\ln(\ln\frac{2}{1-|z|})} = 0$.

The proof is similar to Lemma 2.1. The details are omitted.

Lemma 2.3. *Let* $f(z) = \frac{(1-|z|)\ln\frac{2}{1-|z|}}{|1-z|\ln\frac{4}{|1-z|}}$, $z \in D$. *Then* $|f(z)| < 2$.

Proof. Since $r(x) = x\ln\frac{2}{x}$ is increasing on $(0, \frac{2}{e}]$, decreasing on $[\frac{2}{e}, 1]$ and $r(\frac{2}{e}) = \frac{2}{e} < 1$, then $|f(z)| < 1$ where $z \in D_1 = \{z \in D : |1-z| < \frac{2}{e}\}$.

On the other hand, for $z \in D \setminus D_1$,

$$|f(z)| \leq \frac{(1-|z|)\ln\frac{2}{1-|z|}}{\frac{2}{e}\ln 2} \leq \frac{\frac{2}{e}}{\frac{2}{e}\ln 2} < 2,$$

hence $|f(z)| < 2$. □

Theorem 2.4. *The following are equivalent:*

(a) $\phi \in M(\beta_L)$;

(b) $\phi \in M(\beta_L^0)$;

(c) $\phi \in H^\infty$ *and*

$$\sup_D (1-|z|^2)\ln\frac{2}{1-|z|}\ln(\ln\frac{2}{1-|z|})|\phi'(z)| < +\infty. \tag{2.1}$$

Proof. (c)⇒(a). Assume $\phi \in H^\infty$ and (2.1) holds. For every $f \in \beta_L$, by Lemma 2.1, we have

$$\begin{aligned}
&(1-|z|^2)\ln\frac{2}{1-|z|}|(M_\phi f)'(z)| \\
&\leq (1-|z|^2)\ln\frac{2}{1-|z|}|\phi(z)||f'(z)| + (1-|z|^2)\ln\frac{2}{1-|z|}|\phi'(z)||f(z)| \\
&\leq \|\phi\|_\infty \|f\|_{\beta_L} + (1-|z|^2)\ln\frac{2}{1-|z|}(2+\ln(\ln\frac{2}{1-|z|}))|\phi'(z)|\|f\|_L < +\infty.
\end{aligned}$$

Thus $f\phi \in \beta_L$.

(a)⇒(c). Suppose that ϕ is a multiplier of β_L. Then by [4, Proposition 3] $\phi \in H^\infty$ and $|\phi(z)| \leq \|M_\phi\|$. Let $z_0 = re^{i\theta}$. We take the test function

$$f(z) = \ln(\ln\frac{4}{1-e^{-i\theta}z}).$$

By Lemma 2.3 we know that $f \in \beta_L$ and $\|f\|_L \leq 5$. We have

$$\|f\phi\|_L \leq \|M_\phi\|\|f\|_L \leq 5\|M_\phi\|.$$

It follows that

$$\begin{aligned}
&(1-|z|^2)\ln\frac{2}{1-|z|}|\ln(\ln\frac{4}{1-e^{-i\theta}z})||\phi'(z)| \\
&\leq (1-|z|^2)\ln\frac{2}{1-|z|}|\phi(z)||f'(z)| + 5\|M_\phi\| \\
&\leq 5(\|\phi\|_\infty + \|M_\phi\|) < +\infty.
\end{aligned}$$

Let $z = z_0$. Hence

$$(1-|z_0|^2)\ln\frac{2}{1-|z_0|}|\ln(\ln\frac{4}{1-|z_0|})|\phi'(z_0)| \leq 5(\|\phi\|_\infty + \|M_\phi\|) < +\infty.$$

Thus

$$\sup_D (1-|z|^2)\ln\frac{2}{1-|z|}\ln(\ln\frac{2}{1-|z|})|\phi'(z)| < +\infty.$$

(b)⇒(c). Given $z_0 = re^{i\theta}$ and $\alpha \in (0,1)$. Let

$$f_\alpha(z) = (\ln(\ln\frac{4}{1-e^{-i\theta}z}))^\alpha.$$

A calculation shows that $f_\alpha \in \beta_L^0$ and $\sup_\alpha \|f\|_L = k < +\infty$. In a manner similar to the proof (a)⇒(c), one obtains that if ϕ is a multiplier of β_L^0, then for each α,

$$(1-|z_0|^2)\ln\frac{2}{1-|z_0|}|(\ln(\ln\frac{4}{1-|z_0|}))^\alpha||\phi'(z)| \le k(\|\phi\|_\infty + \|M_\phi\|) < +\infty.$$

Hence

$$(1-|z_0|^2)\ln\frac{2}{1-|z_0|}|\ln(\ln\frac{4}{1-|z_0|})||\phi'(z_0)| < +\infty,$$

which shows that (1) holds.

(c)⇒(b). Assume $\phi \in H^\infty$ and $\sup_D(1-|z|^2)\ln\frac{2}{1-|z|}\ln(\ln\frac{2}{1-|z|})|\phi'(z)| = M < +\infty$. For every $f \in \beta_L^0$, by Lemma 2.2, we have

$$\begin{aligned}&(1-|z|^2)\ln\frac{2}{1-|z|}|(M_\phi f)'(z)|\\ &\le (1-|z|^2)\ln\frac{2}{1-|z|}|\phi(z)||f'(z)| + (1-|z|^2)\ln\frac{2}{1-|z|}|\phi'(z)||f(z)|\\ &\le \|\phi\|_\infty(1-|z|^2)\ln\frac{2}{1-|z|}|f'(z)| + \frac{M}{\ln(\ln\frac{2}{1-|z|})}|f(z)| \to 0\,(|z|\to 1).\end{aligned}$$

Thus $f\phi \in \beta_L^0$. □

3. Cyclic vectors in the little weighted Bloch space

Lemma 3.1. *Let $g(x) = (1-x)\ln\frac{2}{1-x}, x \in [0,1)$. Then $\frac{g(x)}{g(tx)} \le 2$ for each $t \in [0,1]$.*

Proof. Let $x_0 = 1 - \frac{2}{e}$. A calculation shows that $\frac{4}{3}x_0 < 1$. We know that $g(x)$ is increasing on $[0, x_0]$, and decreasing on $[x_0, 1)$.

First, suppose $t > \frac{3}{4}$ and $x > \frac{4}{3}x_0$. Then $x \ge tx > x_0$, hence $g(x) \le g(tx)$.

Next, suppose $t > \frac{3}{4}$ and $x \le \frac{4}{3}x_0$. Then

$$\frac{g(x)}{g(tx)} \le \frac{g(x_0)}{\min(g(0), g(\frac{4}{3}x_0))} = \frac{2/e}{\ln 2} < 2.$$

Finally, suppose $t \le \frac{3}{4}$. A calculation shows that

$$\frac{3}{4}\ln 2 \le g(tx) \le \frac{2}{e}.$$

Then

$$\frac{g(x)}{g(tx)} \leq \frac{2/e}{\frac{3}{4}\ln 2} < 2.$$

□

Lemma 3.2. *Let* $h(x) = (1-x)\ln^2 \frac{2}{1-x}, x \in [0,1)$. *Then there exists a constant* $M > 0$ *such that* $\dfrac{h(x)}{h(tx)} \leq M$ *for each* $t \in [0,1]$.

The proof is similar to Lemma 3.1. We omit the details.

Lemma 3.3. *Suppose* $f \in \beta_L$, *then* $f \in \beta_L^0$ *if and only if* $\|f_t - f\|_L \longrightarrow 0 (t \to 1^-)$, *where* $f_t(z) = f(tz)$.

Proof. Suppose $f \in \beta_L^0$, then given any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that $(1-|z|)\ln \dfrac{2}{1-|z|}|f'(z)| \leq (1-|z|^2)\ln \dfrac{2}{1-|z|}|f'(z)| < \epsilon$ for all $\delta^2 < |z| < 1$. Consider

$$\begin{aligned}
\|f_t - f\|_L &= \sup_D (1-|z|^2)\ln \frac{2}{1-|z|}|tf'(tz) - f'(z)| \\
&\leq \sup_{|z|>\delta} (1-|z|^2)\ln \frac{2}{1-|z|}|tf'(tz) - f'(z)| \\
&\quad + \sup_{|z|\leq\delta} (1-|z|^2)\ln \frac{2}{1-|z|}|tf'(tz) - f'(z)| \\
&\triangleq I_1 + I_2.
\end{aligned}$$

If $|z| > \delta$ and $t > \delta$, then $|tz| > \delta^2$. By Lemma 3.1 we have

$$\begin{aligned}
I_1 &\leq \sup_{|z|>\delta} (1-|z|^2)\ln \frac{2}{1-|z|}|f'(z)| + \sup_{|z|>\delta} (1-|z|^2)\ln \frac{2}{1-|z|}|tf'(tz)| \\
&\leq 2\sup_{|z|>\delta} (1-|z|)\ln \frac{2}{1-|z|}|f'(z)| + 2\sup_{|z|>\delta} (1-|z|)\ln \frac{2}{1-|z|}|f'(tz)| \\
&\leq 2\epsilon + +4\sup_{|z|>\delta} (1-|zt|)\ln \frac{2}{1-|zt|}|f'(tz)| \\
&\leq 2\epsilon + 4\epsilon = 6\epsilon.
\end{aligned}$$

On the other hand, $I_2 \longrightarrow 0$ as $t \longrightarrow 1^-$ since $tf'(tz) \longrightarrow f'(z)$ uniformly for $|z| \leq \delta$. Thus $\lim_{t\to 1^-} \|f_t - f\|_L = 0$.

Conversely, suppose $f \in \beta_L^0$ and $\lim_{t\to 1} \|f_t - f\|_L = 0$. Then for $\epsilon > 0$ there exists $t \in (0,1)$ such that $\|f_t - f\|_L < \epsilon$. It follows that

$$\begin{aligned}(1-|z|^2)\ln\tfrac{2}{1-|z|}|f'(z)| &\le \|f_t - f\|_L + (1-|z|^2)\ln\tfrac{2}{1-|z|}|(f_t)'(z)| \\ &< \epsilon + (1-|z|^2)\ln\tfrac{2}{1-|z|}|(f_t)'(z)|.\end{aligned}$$

Now let $|z| \longrightarrow 1$ then $(1-|z|^2)\ln\frac{2}{1-|z|}|(f_t)'(z)| \longrightarrow 0$ because $f_t \in \beta_L^0$. Hence $f \in \beta_L^0$. □

Proposition 3.4. *The polynomials are dense in* β_L^0.

Proof. Let $f \in \beta_L^0$ and $t_n = 1 - \dfrac{1}{n}$, then $f(t_n z)$ is analytic in $|z| \le 1$. Hence there exists a polynomial $p_n(z)$ such that

$$|f(t_n z) - p_n(z)| < \frac{1}{n}, \quad |f'(t_n z) - p_n'(z)| < \frac{1}{n}$$

for all $|z| \le 1$. Then by Lemma 3.3 we get

$$\begin{aligned}\|f(z) - p_n(z)\|_L &\le \|f(z) - f(t_n z)\|_L + \|f(t_n z) - p_n(z)\|_L \\ &< \|f(z) - f(t_n z)\|_L + \frac{(1+\frac{4}{e})}{n} \longrightarrow 0\,(n\to\infty).\end{aligned}$$

Thus the polynomials are dense in β_L^0. □

Proposition 3.5. $\beta_L \subset VMOA$.

Proof. Let I is an arc in ∂D and $S(I)$ is the Carleson box based on I, i.e, $S(I) = \{re^{i\theta} : 1 - r \le |I|, e^{i\theta} \in I\}$. For $f \in \beta_L$, it follows that

$$\begin{aligned}&\int_{S(I)} |f'(z)|^2(1-|z|^2)\,dA(z) \\ &\le \int_{S(I)} \frac{\|f\|_{\beta_L}^2}{(1-|z|^2)\ln^2\frac{2}{1-|z|}}\,dA(z) \\ &\le \|f\|_{\beta_L}^2 |I| \int_{1-|I|}^1 \frac{1}{(1-r)\ln^2\frac{2}{1-r}}\,dr = \|f\|_{\beta_L}^2 \frac{|I|}{\ln\frac{2}{|I|}}.\end{aligned}$$

Then

$$\frac{1}{|I|}\int_{S(I)} |f'(z)|^2(1-|z|^2)\,dA(z) \le \frac{\|f\|_{\beta_L}^2}{\ln\frac{2}{|I|}} \longrightarrow 0\,(|I|\to 0).$$

Hence $f \in VMOA$.

□

Since $\beta_\alpha \subset \beta_L$ for $0 < \alpha < 1$, we have the following corollary.

Corollary 3.6. *For $0 < \alpha < 1$, $\beta_\alpha \subset VMOA$.*

This fact was proved in [8, Theorem 3]. However this proof is much easier than the one in [8].

Theorem 3.7.
(1) *Let $f \in \beta_L^0$, if $|f(z)| \geq \sigma > 0\,(|z| < 1)$, then f is a cyclic vector in β_L^0.*
(2) *If f is a cyclic vector in β_L^0, then f is an outer function.*

Proof. (1) For $0 < t < 1$, $f_t(z) = f(tz)$. Since $\frac{1}{f_t}$ is analytic in $|z| \leq 1$, we can easily prove that there exists polynomials p_n such that $\|p_n f - \dfrac{f}{f_t}\|_L \longrightarrow 0$ as $n \to \infty$. Thus we have $\dfrac{f}{f_t} \in [f]$. If $\|\dfrac{f}{f_t} - 1\|_L \longrightarrow 0$ as $t \to 1^-$, then $1 \in [f]$, hence by Proposition 3.4, f is cyclic in β_L^0. Now we are going to show that $\|\dfrac{f}{f_t} - 1\|_L \longrightarrow 0 (t \to 1^-)$.

We have

$$\begin{aligned}\|\frac{f}{f_t} - 1\|_L &\leq \frac{1}{\sigma}\|f - f_t\|_L + \frac{1}{\sigma^2}\sup_D (1-|z|^2)\ln\frac{2}{1-|z|}|f(z) - f(tz)||tf'(tz)| \\ &\triangleq \frac{1}{\sigma} I_3 + \frac{1}{\sigma^2} I_4.\end{aligned}$$

By Lemma 3.3 we know $I_3 \longrightarrow 0\,(t \to 1^-)$, then we only prove $I_4 \longrightarrow 0\,(t \to 1^-)$.

Since $f \in \beta_L^0$, for a given any $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$(1-|z|^2)\ln\frac{2}{1-|z|}|f'(z)| < \epsilon$$

for all $\delta^2 < |z| < 1$. If $|z| > \delta$ and $t > \delta$, then $|tz| > \delta^2$. By Lemmas 2.1 and 3.2 it follows that

$$\begin{aligned}&\sup_{|z|>\delta}(1-|z|^2)\ln\frac{2}{1-|z|}|f(z) - f(tz)||tf'(tz)| \\ &\leq \epsilon \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|tz|^2)\ln\frac{2}{1-|tz|}}|f(z) - f(tz)| \\ &\leq \epsilon\|f\|_{\beta_L}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|tz|^2)\ln\frac{2}{1-|tz|}}\ln\left(\frac{ln\frac{2}{1-|z|}}{ln\frac{2}{1-|tz|}}\right) \\ &\leq \epsilon\|f\|_{\beta_L}\frac{2(1-|z|)\ln^2\frac{2}{1-|z|}}{(1-|tz|)\ln^2\frac{2}{1-|tz|}} \\ &\leq 2M\|f\|_{\beta_L}\epsilon.\end{aligned}$$

On the other hand,

$$\sup_{|z|\le\delta}(1-|z|^2)\ln\frac{2}{1-|z|}|f(z)-f(tz)||tf'(tz)|$$

$$\le \|f\|^2_{\beta_L}\sup_{|z|\le\delta}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|tz|^2)\ln\frac{2}{1-|tz|}}\ln\Big(\frac{ln\frac{2}{1-|z|}}{ln\frac{2}{1-|tz|}}\Big)\longrightarrow 0\,(t\longrightarrow 1^-).$$

Hence $I_4 \longrightarrow 0\,(t\to 1^-)$. Thus f is a cyclic vector in β^0_L.

(2) If f is a cyclic vector in β^0_L, then, according to Proposition 3.5 and [4, Proposition 6], f is a cyclic vector in $VMOA$. Hence f is an outer function by Theorem A. This completes the proof of Theorem 3.1. □

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References

[1] J. M. Anderson, J. L. Fernandez and A. L. Shields, Inner functions and cyclic vertors in the *Bloch* space, Trans. Amer. Math. Soc. 323(1991), 429-448.
[2] K. R. M. Attele, Toeplitz and Hankel on Bergman one space, Hokkaido Math J. 21(1992), 279-293.
[3] L. Brown and A. L. Shields, Multipliers and cyclic vectors in the *Bloch* space, Michigan Math. J. 38(1991), 141-146.
[4] L. Brown and A. L. Shields, Cyclic vectors in the Dirichlet space, Trans. Amer. Math. Soc. 285(1984), 269-304.
[5] J. Garnett, Bounded analytic functions, New York 1981.
[6] S. L. Ye, Cyclic vectors in the $BMOA$ space, Acta Math. Sci. 17(1997), 356-360(in Chinese).
[7] R. Yoneda, The composition operators on weighted *Bloch* space, Arch. Math. 78(2002), 310-317.
[8] R. H. Zhao, On $\alpha-$*Bloch* functions and $VMOA$, Acta Math. Sci. 16(1996), 349-360.
[9] K. H. Zhu, Bloch type spaces of analytic functions, Rocky Mountain J. Math. 23(1993), 1143-1177.

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