Math. J. Okayama Univ. **48** (2006), 125–134

NOTE ON THE VECTOR-VALUED COHOMOLOGY EQUATION $f = h \circ T - h$

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Abstract. Let X be a Banach space and T be an ergodic endomorphism of a probability measure space $(\Omega, \mathcal{A}, \mu)$. Assuming that X is reflexive and has a countable (Schauder) basis $\{e_n : n \geq 1\}$, we show that a function f in $L_p(\Omega; X)$, where $1 \leq p \leq \infty$, has the form $f = h \circ T - h$ for some $h \in L_p(\Omega; X)$ if and only if there exists a set $A \in \mathcal{A}$ with $\mu(A) > 0$ such that $\liminf_{n\to\infty}(1/n)\sum_{j=1}^{n} \|\chi_A \cdot (\sum_{k=0}^{j-1} f \circ T^k)\|_p < \infty$. This is a vector-valued generalization of a scalar-valued result due to Alonso, Hong and Obaya.

1. Introduction and the result

Let $(X, \|\cdot\|_X)$ be a Banach space and $(\Omega, \mathcal{A}, \mu)$ be a probability measure space. We denote by $(L, \|\cdot\|_L)$ a Banach space of X-valued strongly measurable functions on $(\Omega, \mathcal{A}, \mu)$ under pointwise operations. Two functions f and g in L are not distinguished provided that $f(\omega) = g(\omega)$ for almost all $\omega \in \Omega$. In this note we assume the following properties:

(a) If $u, v \in L$ and $\|u(\omega)\|_X \leq \|v(\omega)\|_X$ for almost all $\omega \in \Omega$, then $\|u\|_L \leq \|v\|_L$.

(b) If v is an X-valued strongly measurable function on Ω and there exists a function $u \in L$ such that $\|v(\omega)\|_X \leq \|u(\omega)\|_X$ for almost all $\omega \in \Omega$, then $v \in L$.

(c) If (u_n) is a sequence of functions in L such that $\|u_1(\omega)\|_X \leq \|u_2(\omega)\|_X \leq \dots$ for almost all $\omega \in \Omega$, and $\sup_{n\geq 1} \|u_n\|_L < \infty$, then there exists a function $u \in L$ such that $\|u_n(\omega)\|_X \leq \|u(\omega)\|_X$ for almost all $\omega \in \Omega$ and all $n \geq 1$.

(d) If v is an X-valued strongly measurable function on Ω and $u \in L$ is such that

$$\mu(\{\omega : \|v(\omega)\|_X > a\}) = \mu(\{\omega : \|u(\omega)\|_X > a\})$$

for all $a \in \boldsymbol{R}$ with $a > 0$, then $v \in L$ and $\|v\|_L = \|u\|_L$.

It is interesting to note that, besides the usual X-valued L_p-spaces $L_p(\Omega; X)$ with $1 \leq p \leq \infty$, there are many important Banach spaces

Mathematics Subject Classification. Primary 28D05, 47A35.

Key words and phrases. Reflexive Banach space with a countable (Schauder) basis, probability measure space, endomorphism, automorphism, ergodic, vector-valued cohomology equation, coboundary cocycle.

$(L, \|\cdot\|_L)$ of X-valued strongly measurable functions on Ω which share properties (a)–(d). Examples are (X-valued) Orlicz spaces and Lorentz spaces, etc.

Let $T : \Omega \to \Omega$ be an *endomorphism* of $(\Omega, \mathcal{A}, \mu)$. Thus, if $A \in \mathcal{A}$ then $T^{-1}A \in \mathcal{A}$ and $\mu(T^{-1}A) = \mu(A)$. The endomorphism T is called an *automorphism* of $(\Omega, \mathcal{A}, \mu)$ if T is one-to-one and onto, and T^{-1} is again an endomorphism of $(\Omega, \mathcal{A}, \mu)$. If there does not exist a set A in $\mathcal{A}$ with $T^{-1}A = A$ and $0 < \mu(A) < 1$, then T is called *ergodic*. By property (d) every endomorphism T yields a linear isometry of $(L, \|\cdot\|_L)$ by the mapping $u \mapsto u \circ T$.

Let f be an X-valued strongly measurable function on Ω. Define

$$S_0 f(\omega) := 0, \quad \text{and} \quad S_j f(\omega) := \sum_{k=0}^{j-1} f(T^k \omega) \quad \text{for } j \geq 1,$$

so that the cocycle identity $S_{j+k} f(\omega) = S_j f(\omega) + S_k f(T^j \omega)$ holds for every $j,\, k \geq 0$. The function f is called an (X-valued) *coboundary cocycle* if there exists an X-valued strongly measurable function h on Ω such that $f(\omega) = h(T\omega) - h(\omega)$ for almost all $\omega \in \Omega$. In this case we have

$$S_j f(\omega) = h \circ T^j(\omega) - h(\omega) \quad \text{for almost all } \omega \in \Omega.$$

Here, if h is in L, then $f \in L$ and furthermore

$$2\|h\|_L = \|h \circ T^j\|_L + \|h\|_L \geq \|S_j f\|_L \geq \|\chi_A \cdot S_j f\|_L$$

for every $A \in \mathcal{A}$ with $\mu(A) > 0$ by properties (b) and (a). Thus we have

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \|\chi_A \cdot S_j f\|_L < \infty. \tag{1}$$

The purpose of this note is to prove that the converse implication holds, under some additional assumptions on X and T. This may be regarded as a continuation of the paper [6]. For related topics we refer the reader to [1], [4] and [5] where scalar-valued functions are considered. (See also [7].) Our result is the following

Theorem (Cf. Remark 2 of [6]). *Assume that X is reflexive and has a countable (Schauder) basis $\{e_n : n \geq 1\}$, and that T is an ergodic endomorphism of $(\Omega, \mathcal{A}, \mu)$. Let f be an X-valued strongly measurable function on Ω. If (1) holds for some $A \in \mathcal{A}$, with $\mu(A) > 0$ and $\chi_A \cdot S_j f \in L$ for all $j \geq 1$, then there exists $h \in L$ such that $f = h \circ T - h$.*

Remarks. (i) Since every X-valued strongly measurable function on Ω is μ-almost separably valued, it is immediate that the conclusion of the above Theorem holds when X is a (not necessarily separable) Hilbert space.

(ii) It is known (cf. e.g. Singer [8], [9]) that, although many interesting concrete Banach spaces have countable (Schauder) bases, there are examples of separable reflexive Banach spaces which do not have countable (Schauder) bases. Thus one may wonder whether the above Theorem holds, without assuming the existence of a countable (Schauder) basis of X. The authors could not prove this, and it seems to us that this is an open problem.

2. Proof of Theorem

Since $\{e_n : n \geq 1\}$ is a (Schauder) basis of X, for every $x \in X$ there exists a unique sequence $\{\varphi_n(x) : n \geq 1\}$ of scalars such that

$$\lim_{n\to\infty} \|x - \sum_{j=1}^{n} \varphi_j(x)e_j\|_X = 0. \tag{2}$$

Here we may assume without loss of generality (see e.g. Chapter 1 of [8]) that X is a *real* Banach space, and that $\|e_n\|_X = 1$ for all $n \geq 1$. It is also known that $\varphi_n \in X^*$ for every $n \geq 1$. Thus, $f(\omega)$ can be written uniquely as

$$f(\omega) = \sum_{j=1}^{\infty} a_j(\omega)e_j = \sum_{j=1}^{\infty} \varphi_j(f(\omega))e_j, \tag{3}$$

and $a_n(\omega)$ becomes a real-valued measurable function on Ω for every $n \geq 1$.

Let $P_n : X \to X$, $n \geq 1$, be the projection operators on X defined by

$$P_n x := \sum_{j=1}^{n} \varphi_j(x)e_j \qquad (x \in X).$$

Since $\lim_{n\to\infty} \|x - P_n x\|_X = 0$, it follows from the uniform boundedness principle that

$$M := \sup_{n\geq 1} \|P_n\| < \infty, \tag{4}$$

whence the X-valued functions

$$f_n(\omega) := \sum_{j=1}^{n} a_j(\omega)e_j \, (= P_n f(\omega)) \qquad (n \geq 1,\ \omega \in \Omega) \tag{5}$$

satisfy

$$\|f_n(\omega)\|_X \leq M\|f(\omega)\|_X \qquad (n \geq 1,\ \omega \in \Omega); \tag{6}$$

and $\chi_A \cdot f \in L$ implies $\chi_A \cdot f_n \in L$ for every $n \geq 1$, by (6) and property (b).

Here, we introduce a Banach space $\widetilde{L}$ of real-valued measurable functions on Ω as follows. Let $\widetilde{L}$ be the set of all real-valued measurable functions $\widetilde{u}$ on Ω such that $\widetilde{u} \cdot e_1 \in L$, and define

$$\|\widetilde{u}\|_{\widetilde{L}} := \|\widetilde{u} \cdot e_1\|_L. \tag{7}$$

By properties (a)–(d), $(\widetilde{L}, \|\cdot\|_{\widetilde{L}})$ becomes a Banach space under pointwise operations, and satisfies the following properties:

(A) If $\widetilde{u}, \widetilde{v} \in \widetilde{L}$ and $|\widetilde{u}(\omega)| \le |\widetilde{v}(\omega)|$ for almost all $\omega \in \Omega$, then $\|\widetilde{u}\|_{\widetilde{L}} \le \|\widetilde{v}\|_{\widetilde{L}}$.

(B) If $\widetilde{v}$ is a real-valued measurable function on Ω and there exists a function $\widetilde{u} \in \widetilde{L}$ such that $|\widetilde{v}(\omega)| \le |\widetilde{u}(\omega)|$ for almost all $\omega \in \Omega$, then $\widetilde{v} \in \widetilde{L}$.

(C) If $(\widetilde{u}_n)$ is a sequence of functions in $\widetilde{L}$ such that $|\widetilde{u}_1(\omega)| \le |\widetilde{u}_2(\omega)| \le \dots$ for almost all $\omega \in \Omega$, and $\sup_{n\ge 1} \|\widetilde{u}_n\|_{\widetilde{L}} < \infty$, then there exists a function $\widetilde{u} \in \widetilde{L}$ such that $|\widetilde{u}_n(\omega)| \le |\widetilde{u}(\omega)|$ for almost all $\omega \in \Omega$ and all $n \ge 1$.

(D) If $\widetilde{v}$ is a real-valued measurable function on Ω and $\widetilde{u} \in \widetilde{L}$ is such that

$$\mu(\{\omega : |\widetilde{v}(\omega)| > a\}) = \mu(\{\omega : |\widetilde{u}(\omega)| > a\})$$

for all $a \in \boldsymbol{R}$ with $a > 0$, then $\widetilde{v} \in \widetilde{L}$ and $\|\widetilde{v}\|_{\widetilde{L}} = \|\widetilde{u}\|_{\widetilde{L}}$.

By (5) we have

$$S_l f_n(\omega) = \sum_{k=0}^{l-1} f_n(T^k\omega) = \sum_{k=0}^{l-1}\sum_{j=1}^{n} a_j(T^k\omega)e_j = \sum_{j=1}^{n}\left(\sum_{k=0}^{l-1} a_j(T^k\omega)\right) e_j\,;$$

and since $\|e_1\|_X = \|e_j\|_X = 1$, it follows (cf. (d), (3) and properties (b) and (a)) that

$$\begin{aligned}\left\|\chi_A(\cdot)\left(\sum_{k=0}^{l-1} a_j(T^k\cdot)\right)\right\|_{\widetilde{L}} &= \left\|\chi_A(\cdot)\left(\sum_{k=0}^{l-1} a_j(T^k\cdot)\right)e_1\right\|_{L} \\ &= \left\|\chi_A(\cdot)\left(\sum_{k=0}^{l-1} a_j(T^k\cdot)\right)e_j\right\|_{L} = \left\|\chi_A(\cdot)\left(\sum_{k=0}^{l-1} \varphi_j\circ f(T^k\cdot)\right)e_j\right\|_{L} \\ &\le \|\varphi_j\|\left\|\chi_A(\cdot)\left(\sum_{k=0}^{l-1} f(T^k\cdot)\right)\right\|_{L} = \|\varphi_j\|\|\chi_A(\cdot)S_l f(\cdot)\|_L.\end{aligned}$$

Thus, (1) implies that for each fixed $j \ge 1$,

$$\begin{aligned}\liminf_{m\to\infty}\frac{1}{m}\sum_{l=1}^{m}&\left\|\chi_A(\cdot)\left(\sum_{k=0}^{l-1} a_j(T^k\cdot)\right)\right\|_{\widetilde{L}} \\ &\le \|\varphi_j\|\cdot\liminf_{m\to\infty}\frac{1}{m}\sum_{l=1}^{m}\|\chi_A\cdot S_l f\|_L < \infty.\end{aligned}$$

Since $(\widetilde{L}, \|\cdot\|_{\widetilde{L}})$ is a Banach lattice of equivalence classes of real-valued measurable functions on Ω satisfying Properties (A)–(D), it then follows from Theorem 2 of [5] that there exists $\widetilde{\xi}_j \in \widetilde{L}$ such that $a_j(\omega) = \widetilde{\xi}_j(T\omega) -$

$\widetilde{\xi}_j(\omega)$ for almost all $\omega \in \Omega$. Let $g_n = \sum_{j=1}^n \widetilde{\xi}_j \cdot e_j$. Since $\widetilde{\xi}_j \cdot e_j \in L$ by the definition of $\widetilde{L}$ and property (d), it follows that $g_n \in L$, and that

$$f_n(\omega) = P_n f(\omega) = \sum_{j=1}^n a_j(\omega) e_j = g_n(T\omega) - g_n(\omega) \quad \text{for almost all } \omega \in \Omega.$$

Thus we find

$$(8) \qquad f_n \in L, \quad \text{and} \quad S_k f_n = \sum_{i=0}^{k-1} f_n \circ T^i = g_n \circ T^k - g_n \quad \text{for } k \geq 1.$$

Clearly, by (4)–(6) we have $S_k f_n = S_k P_n f = P_n S_k f$, and

$$(9) \qquad \|S_k f_n(\omega)\|_X \leq M \|S_k f(\omega)\|_X \qquad (k,\, n \geq 1;\ \omega \in \Omega).$$

Since T is ergodic by assumption, we next apply the Birkhoff pointwise ergodic theorem (see e.g. Chapter 1 of [3]), together with (8) and (9), to infer that for almost all $\omega \in \Omega$

$$(10) \qquad \begin{aligned} \int_\Omega \|g_n(\cdot)\|_X \, d\mu &= \lim_{l \to \infty} \frac{1}{l} \sum_{k=1}^l \|g_n(T^k \omega)\|_X \\ &\leq \liminf_{l \to \infty} \frac{1}{l} \sum_{k=1}^l \|S_k f_n(\omega)\|_X + \|g_n(\omega)\|_X \\ &\leq \liminf_{l \to \infty} \frac{1}{l} M \sum_{k=1}^l \|S_k f(\omega)\|_X + \|g_n(\omega)\|_X . \end{aligned}$$

To see that $\int_\Omega \|g_n(\cdot)\|_X \, d\mu < \infty$, we first prove that

$$(11) \qquad \liminf_{l \to \infty} \frac{1}{l} \sum_{k=1}^l \|S_k f(\omega)\|_X < \infty \quad \text{for almost all } \omega \in A.$$

To do this, let

$$(12) \qquad F(\omega) := \liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^n \chi_A(\omega) \|S_j f(\omega)\|_X ,$$

and

$$(13) \qquad F_n(\omega) := \inf_{m \geq n} \frac{1}{m} \sum_{j=1}^m \chi_A(\omega) \|S_j f(\omega)\|_X .$$

Then we have

$$(14) \qquad 0 \leq F_n(\omega) \uparrow F(\omega) \quad \text{as } n \to \infty, \quad \text{and} \quad F_n \in \widetilde{L},$$

where the last property comes from the assumption that $\chi_A \cdot S_j f \in L$ for $j \geq 1$ and the definition of $\widetilde{L}$, together with Property (B). Furthermore, we have

$$\begin{aligned}\|F_n\|_{\widetilde{L}} &\le \frac{1}{n}\left\|\sum_{j=1}^{n}\chi_A(\cdot)\|S_jf(\cdot)\|_X\right\|_{\widetilde{L}}\\ &\le \frac{1}{n}\sum_{j=1}^{n}\|\chi_A(\cdot)\|S_jf(\cdot)\|_X\|_{\widetilde{L}} = \frac{1}{n}\sum_{j=1}^{n}\|\chi_A\cdot S_jf\|_L \quad \text{(by (7) and (d))},\end{aligned}$$

whence

$$\tag{15} \begin{aligned}\sup_{n\ge1}\|F_n\|_{\widetilde{L}} &= \liminf_{n\to\infty}\|F_n\|_{\widetilde{L}}\\ &\le \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\|\chi_A\cdot S_jf\|_L < \infty \quad \text{(by (14) and (1))}.\end{aligned}$$

Using this, together with (14) and Properties (C) and (B), we find

$$\tag{16} F\in\widetilde{L},$$

which proves that $0\le F(\omega)<\infty$ for almost all $\omega\in\Omega$, and this completes the proof of (11).

Now, from (10), (11) and the assumption $\mu(A)>0$ we can take $\omega\in A$ such that

$$\int_\Omega \|g_n(\cdot)\|_X\,d\mu \le \liminf_{l\to\infty}\frac{1}{l}\,M\sum_{k=1}^{l}\|S_kf(\omega)\|_X + \|g_n(\omega)\|_X < \infty.$$

This implies that $g_n\in L_1(\Omega;X)$. Then we apply Theorem 4.2.1 of [3] to infer that the limit

$$\widehat{g}_n(\omega) := \lim_{l\to\infty}\frac{1}{l}\sum_{k=1}^{l}g_n(T^k\omega)$$

exists for almost all $\omega\in\Omega$. (Incidentally, we note that, by the ergodicity of T, we have $\widehat{g}_n(\omega)=\int_\Omega g_n(\cdot)\,d\mu$ for almost all $\omega\in\Omega$.) Using this and (8), we can define an X-valued strongly measurable function h_n on Ω as follows:

$$\tag{17} \begin{aligned}h_n(\omega) &:= \lim_{l\to\infty}\frac{-1}{l}\sum_{k=1}^{l}S_kf_n(\omega) = g_n(\omega) - \lim_{l\to\infty}\frac{1}{l}\sum_{k=1}^{l}g_n(T^k\omega)\\ &= g_n(\omega)-\widehat{g}_n(\omega) \quad \text{(for almost all } \omega\in\Omega).\end{aligned}$$

Since $\widehat{g}_n\circ T=\widehat{g}_n$, it follows from (8) that

$$\tag{18} h_n\circ T - h_n = g_n\circ T - g_n = f_n,$$

and from (9) and (12) that

$$\tag{19} \|\chi_A(\omega)h_n(\omega)\|_X \le \liminf_{l\to\infty}\frac{1}{l}\sum_{k=1}^{l}\chi_A(\omega)\|S_kf_n(\omega)\|_X$$

$$\leq \liminf_{l\to\infty} \frac{M}{l} \sum_{k=1}^{l} \chi_A(\omega) \|S_k f(\omega)\|_X = MF(\omega),$$

so that, by properties (a), (b), (7) and (16), we have

$$\chi_A \cdot h_n \in L, \quad \text{and} \quad \|\chi_A \cdot h_n\|_L \leq M \|F\|_{\widetilde{L}}. \tag{20}$$

At this point we remark that, by using the argument given on pp. 290–291 in [5], we may assume without loss of generality that T is an ergodic *automorphism* of $(\Omega, \mathcal{A}, \mu)$. Since $S_k f_n = h_n \circ T^k - h_n$ for $k \geq 1$ by (18) and $f_n \in L$ by (8), with this assumption we see that

$$\begin{aligned}
\frac{1}{l}\sum_{k=1}^{l} \|\chi_A \cdot S_k f_n\|_L + \|\chi_A \cdot h_n\|_L &\geq \frac{1}{l}\sum_{k=1}^{l} \|\chi_A \cdot (h_n \circ T^k)\|_L \\
&= \frac{1}{l}\sum_{k=1}^{l} \|(\chi_A \circ T^{-k}) \cdot h_n\|_L \qquad \text{(by (d))} \\
&\geq \left\| \left(\frac{1}{l}\sum_{k=1}^{l} \chi_A \circ T^{-k} \right) \cdot h_n \right\|_L ;
\end{aligned}$$

where, putting

$$d_n(\omega) := \inf_{m \geq n} \frac{1}{m} \sum_{k=1}^{m} \chi_A(T^{-k}\omega) \qquad (n \geq 1,\ \omega \in \Omega),$$

we have by the Birkhoff pointwise ergodic theorem that

$$0 \leq d_1(\omega) \leq d_2(\omega) \leq \ldots \longrightarrow \mu(A) > 0 \qquad \text{for almost all } \omega \in \Omega. \tag{21}$$

Therefore, from (9) and (20) we see (cf. also properties (a) and (b)) that

$$\begin{aligned}
\liminf_{l\to\infty} \|d_l \cdot h_n\|_L &\leq \liminf_{l\to\infty} \left\| \left(\frac{1}{l}\sum_{k=1}^{l} \chi_A \circ T^{-k} \right) \cdot h_n \right\|_L \\
&\leq \liminf_{l\to\infty} \frac{1}{l}\sum_{k=1}^{l} \|\chi_A \cdot S_k f_n\|_L + \|\chi_A \cdot h_n\|_L \\
&\leq M \left(\liminf_{l\to\infty} \frac{1}{l}\sum_{k=1}^{l} \|\chi_A \cdot S_k f\|_L + \|F\|_{\widetilde{L}} \right),
\end{aligned} \tag{22}$$

and from (21), property (a) and (1) that

$$\sup_{l \geq 1} \|d_l \cdot h_n\|_L = \liminf_{l\to\infty} \|d_l \cdot h_n\|_L < \infty,$$

and consequently from properties (c), (b) and (21) that $\mu(A)h_n \in L$, and thus

$$h_n \in L \qquad (n \geq 1). \tag{23}$$

For $n \geq 1$, we now define a real-valued nonnegative measurable function p_n on Ω by

$$p_n(\omega) := \inf_{m \geq n} \|h_m(\omega)\|_X \qquad (\omega \in \Omega). \tag{24}$$

It follows from (23), property (b), the definition of $\widetilde{L}$, and Property (B) that $p_n \in \widetilde{L}$, and $d_n \cdot p_n \in \widetilde{L}$. Then, as in (22), we can see (cf. (7), property (b), and (1)) that

$$\begin{aligned}\liminf_{n\to\infty} \|d_n \cdot p_n\|_{\widetilde{L}} \leq \liminf_{n\to\infty} \|d_n \cdot h_n\|_L &\leq \liminf_{n\to\infty} \left\| \left(\frac{1}{n} \sum_{k=1}^{n} \chi_A \circ T^{-k} \right) \cdot h_n \right\|_L \\ &\leq M \left(\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|\chi_A \cdot S_k f\|_L + \|F\|_{\widetilde{L}} \right) < \infty.\end{aligned}$$

Here, the relation

$$0 \leq d_n(\omega)p_n(\omega) \leq d_{n+1}(\omega)p_{n+1}(\omega) \qquad (n \geq 1,\ \omega \in \Omega),$$

together with Property (A), implies that $\|d_n \cdot p_n\|_{\widetilde{L}} \leq \|d_{n+1} \cdot p_{n+1}\|_{\widetilde{L}}$. Hence we have $\sup_{n\geq 1} \|d_n \cdot p_n\|_{\widetilde{L}} = \liminf_{n\to\infty} \|d_n \cdot p_n\|_{\widetilde{L}} < \infty$; and from Property (C) there exists $\widetilde{u} \in \widetilde{L}$ such that $d_n(\omega)p_n(\omega) \leq \widetilde{u}(\omega)$ for almost all $\omega \in \Omega$ and all $n \geq 1$. Using $p_n(\omega) \leq p_{n+1}(\omega)$ (cf. (24)), we then find that if $l \geq n \geq 1$, then $d_l(\omega)p_l(\omega) \geq d_l(\omega)p_n(\omega)$. Therefore, letting $n \geq 1$ fixed, we have by (21) that

$$\begin{aligned}\mu(A)p_n(\omega) &= \lim_{l\to\infty} d_l(\omega)p_n(\omega) \\ &\leq \lim_{l\to\infty} d_l(\omega)p_l(\omega) \leq \widetilde{u}(\omega) \quad \text{for almost all } \omega \in \Omega.\end{aligned} \tag{25}$$

Lastly, let $\omega \in \Omega$ be such that $p_n(\omega) \leq \widetilde{u}(\omega)/\mu(A) < \infty$ for all $n \geq 1$. (By (25) and the fact $\widetilde{u} \in \widetilde{L}$ we see that the relation $p_n(\omega) \leq \widetilde{u}(\omega)/\mu(A) < \infty$ holds for almost all $\omega \in \Omega$ and all $n \geq 1$.) Then we have

$$\liminf_{n\to\infty} \|h_n(\omega)\|_X = \lim_{n\to\infty} p_n(\omega) \leq \widetilde{u}(\omega)/\mu(A) < \infty \qquad \text{(cf. (24))}.$$

Since X is a reflexive Banach space by assumption, the closed ball $\{x \in X : \|x\|_X \leq (\widetilde{u}(\omega)+1)\mu(A)^{-1}\}$ is weakly compact, and hence weakly sequentially compact by Theorem V.6.1 of [2]. Since the set $\{n \geq 1 : \|h_n(\omega)\|_X \leq (\widetilde{u}(\omega) + 1)\mu(A)^{-1}\}$ is infinite, it then follows that there exists $h(\omega) \in X$

which is a weak-limit point of a subsequence of the sequence $\{h_n(\omega) : n \geq 1\}$ in X. Using the Schauder basis $\{e_j : j \geq 1\}$ of X, we write

$$h(\omega) = \sum_{j=1}^{\infty} c_j(\omega) e_j\,, \quad \text{where } c_j(\omega) = \varphi_j(h(\omega)) \qquad \text{(cf. (2))}. \tag{26}$$

Here we note, by the definition of h_n (see (17) and (5)), that for almost all $\omega \in \Omega$ we have

$$\begin{aligned} h_n(\omega) &= \lim_{l\to\infty} \frac{-1}{l} \sum_{k=1}^{l} S_k f_n(\omega) = \lim_{l\to\infty} \frac{-1}{l} \sum_{k=1}^{l}\sum_{m=0}^{k-1} f_n(T^m\omega) \\ &= \lim_{l\to\infty} \frac{-1}{l} \sum_{k=1}^{l}\sum_{m=0}^{k-1} \left(\sum_{j=1}^{n} a_j(T^m\omega)e_j\right) \\ &= \lim_{l\to\infty} \sum_{j=1}^{n} \left(\frac{-1}{l}\sum_{k=1}^{l}\sum_{m=0}^{k-1} a_j(T^m\omega)\right) e_j\,. \end{aligned}$$

Therefore, if $n \geq j \geq 1$, then for almost all $\omega \in \Omega$ we can define

$$\begin{aligned} b_j(\omega) &:= \lim_{l\to\infty} \frac{-1}{l} \sum_{k=1}^{l}\sum_{m=0}^{k-1} a_j(T^m\omega) \\ &\quad \left(= \lim_{l\to\infty} \varphi_j\left(\frac{-1}{l}\sum_{k=1}^{l} S_k f_n(\omega)\right) = \varphi_j(h_n(\omega))\right), \end{aligned}$$

where the last equality comes from the fact that $\varphi_j \in X^*$. That is, we have gotten a sequence $\{b_j : j \geq 1\}$ of real-valued measurable functions on Ω such that for almost all $\omega \in \Omega$ and all $n \geq 1$, the following equality holds:

$$h_n(\omega) = \sum_{j=1}^{n} b_j(\omega)e_j. \tag{27}$$

Since $h(\omega) = \sum_{j=1}^{\infty} c_j(\omega)e_j$ is a weak-limit point of a subsequence of the sequence $\{h_n(\omega) : n \geq 1\}$=$\{\sum_{j=1}^{n} b_j(\omega)e_j : n \geq 1\}$ in X for almost all $\omega \in \Omega$, it then follows that

$$c_k(\omega) = \varphi_k(h(\omega)) = \lim_{n'\to\infty} \varphi_k(h_{n'}(\omega)) = \lim_{n'\to\infty} \varphi_k\left(\sum_{j=1}^{n'} b_j(\omega)e_j\right) = b_k(\omega)$$

for almost all $\omega \in \Omega$ and all $k \geq 1$. Consequently, we conclude that $h(\omega) = \sum_{j=1}^{\infty} b_j(\omega)e_j$ for almost all $\omega \in \Omega$. That is,

$$\lim_{n\to\infty} \|h(\omega) - \sum_{j=1}^{n} b_j(\omega)e_j\|_X = \lim_{n\to\infty} \|h(\omega) - h_n(\omega)\|_X = 0 \tag{28}$$

for almost all $\omega \in \Omega$. By this, we may consider h to be an X-valued strongly measurable function on Ω.

On the other hand, since $\mu(A) > 0$ and $\widetilde{u} \in \widetilde{L}$, it follows from (25) and Property (B) that the function

$$p(\omega) := \lim_{n\to\infty} p_n(\omega) \,(= \liminf_{n\to\infty} \|h_n(\omega)\|_X) \qquad (\omega \in \Omega) \tag{29}$$

belongs to $\widetilde{L}$. Using this and the fact that $\|h(\omega)\|_X = \lim_{n\to\infty} \|h_n(\omega)\|_X = p(\omega)$ for almost all $\omega \in \Omega$, which comes from (28), we observe (cf. the definition of $\widetilde{L}$ and property (d)) that $h \in L$. Furthermore, by (18) and (5), we have that $h(T\omega) - h(\omega) = \lim_{n\to\infty}(h_n(T\omega) - h_n(\omega)) = \lim_{n\to\infty} f_n(\omega) = \lim_{n\to\infty} P_n f(\omega) = f(\omega)$ for almost all $\omega \in \Omega$. This completes the proof.

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(*Received May 19, 2005*)