BOTT'S THEOREM ON SAMELSON PRODUCTS

Kei SUGATA

1. Introduction

Let G be a topological group. For $x, y \in G$, if x or y is the unit element, so is the commutator $xyx^{-1}y^{-1}$. Hence the correspondence $(x, y) \mapsto xyx^{-1}y^{-1}$ induces a map $c : G \wedge G \to G$. For $\alpha \in \pi_n(G)$ and $\beta \in \pi_m(G)$, the composition

$$S^{n+m} = S^n \wedge S^m \xrightarrow{\alpha \wedge \beta} G \wedge G \xrightarrow{c} G$$

determines an element of $\pi_{n+m}(G)$, called the *Samelson product*, and denoted by $\langle \alpha, \beta \rangle$. Note that the Samelson product is bilinear and satisfies a kind of "anti-commutativity", $\langle \alpha, \beta \rangle = (-1)^{mn+1} \langle \beta, \alpha \rangle$.

Recall from [1] some results on homotopy groups of the unitary group:

$$\begin{aligned}
\pi_{2i+1}(U(t)) &\cong \mathbb{Z} &&\text{for } 0 \le i < t, \\
\pi_{2i}(U(t)) &\cong 0 &&\text{for } 0 \le i < t, \\
\pi_{2t}(U(t)) &\cong \mathbb{Z}/t!,
\end{aligned}$$

where the first and the second are in the stable range, and the third is in the beginning of the unstable range.

To begin with, let us recall a theorem due to R. Bott concerning the Samelson product:

Theorem ([2])**.** *If $\alpha \in \pi_{2r+1}(U(t))$, $\beta \in \pi_{2s+1}(U(t))$, $\gamma \in \pi_{2t}(U(t))$ with $t = r + s + 1$ are suitable generators, then*

$$\langle \alpha, \beta \rangle = r! s! \gamma.$$

This element does not vanish unless $\gamma = 0$, that is, unless $r = s = 0$.

Consider the homomorphisms induced by the standard embeddings:

$$\cdots \to \pi_{2t}(U(t-2)) \to \pi_{2t}(U(t-1)) \to \pi_{2t}(U(t)).$$

Our purpose is to pull back the Bott's result to $\pi_{2t}(U(t-1))$. Recall from Kervaire [3] that

$$(1.1) \qquad \pi_{2t}(U(t-1)) \cong \begin{cases} \mathbb{Z}/12\{\omega\} & \text{if } t = 3, \\ \mathbb{Z}/t!\{\gamma\} \oplus \mathbb{Z}/2\{\delta\} & \text{if } t : \text{odd}, \ t \ge 5, \\ \mathbb{Z}/(t!/2)\{\varepsilon\} & \text{if } t : \text{even}, \ t \ge 4, \end{cases}$$

where ω, γ, δ and ε are generators.

Let $\alpha \in \pi_{2r+1}(U(t-1))$ and $\beta \in \pi_{2s+1}(U(t-1))$ with $t = r + s + 1$ be generators in the stable range. Then we may set $1 \le r \le s$, since $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$. For the standard embedding $i : U(t-1) \to U(t)$, it is easily

seen that $i \circ \alpha$ and $i \circ \beta$ are also generators of $\pi_{2r+1}(U(t))$ and $\pi_{2s+1}(U(t))$ respectively. Then applying Bott's theorem, we have $\langle i \circ \alpha, i \circ \beta \rangle = r!s!\gamma$, where γ is a suitable generator of $\pi_{2t}(U(t))$. Further it is easily seen that the Samelson product enjoys the naturality with respect to the standard embedding i, that is, $i \circ \langle \alpha, \beta \rangle = \langle i \circ \alpha, i \circ \beta \rangle$. Therefore, from the homotopy exact sequence of the fibering $U(t-1) \to U(t) \to S^{2t-1}$, we can obtain the order of $\langle \alpha, \beta \rangle \in \pi_{2t}(U(t-1))$. However, the knowledge of the order of $\langle \alpha, \beta \rangle$ is not sufficient to see whether or not $\langle \alpha, \beta \rangle$ is divisible by 2, since $\pi_{2t}(U(t-1))$ has $\mathbb{Z}/2$-component when t is odd and $t \geq 5$. Our main result is the following which we show by extending the method of the proof of Bott's theorem:

Theorem 1. *If $\alpha \in \pi_{2r+1}(U(t-1))$ and $\beta \in \pi_{2s+1}(U(t-1))$ with $t = r+s+1$ are generators in the stable range, then*

$$\langle \alpha, \beta \rangle = \begin{cases} \omega & \text{if } t = 3, \\ r!s!\gamma \oplus \delta & \text{if } t = 5, \\ r!s!\gamma & \text{if } t : \text{odd}, \ t \geq 7, \\ (r!s!/2)\varepsilon & \text{if } t : \text{even}, \ t \geq 4, \end{cases}$$

where ω, γ, δ and ε are suitable generators as seen in (1.1).

Most of the cases in this theorem are verified without difficulties from homotopy exact sequences of fiberings. In §2, we prove easy part of Theorem 1. In §3, we prove difficult part of Theorem 1 which is a generalization of Theorem II of [6], and the proof is parallel to that given in [6].

Acknowledgements. The author wishes to thank Professor M. Mimura for his useful advice, reading the manuscript of the paper.

2. PROOF OF EASY PART OF THEOREM 1

2.1. **The case $t = 3$ $(r = s = 1)$.** As in the proof of Lemma 1.6 of [3], we have a short exact sequence

$$0 \to \pi_7(S^5) \to \pi_6(U(2)) \xrightarrow{i_*} \pi_6(U(3)) \to 0,$$

which is expressed as

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/12 \to \mathbb{Z}/6 \to 0.$$

For the generators $\alpha, \beta \in \pi_3(U(2))$, the elements $i_*\alpha, i_*\beta \in \pi_3(U(3))$ are also generators. By Bott's theorem, $i_*\langle \alpha, \beta \rangle = \langle i_*\alpha, i_*\beta \rangle \in \pi_6(U(3))$ is a generator, and so is $\langle \alpha, \beta \rangle$.

2.2. **The case t is even.** As in the proof of Lemma 1.6 of [3], we have a short exact sequence

$$0 \to \pi_{2t}(U(t-1)) \xrightarrow{i_*} \pi_{2t}(U(t)) \to \pi_{2t}(S^{2t-1}) \to 0,$$

which is expressed as

$$0 \to \mathbb{Z}/(t!/2) \to \mathbb{Z}/t! \to \mathbb{Z}/2 \to 0.$$

For the generators $\alpha \in \pi_{2r+1}(U(t-1))$ and $\beta \in \pi_{2s+1}(U(t-1))$, the elements $i_*\alpha \in \pi_{2r+1}(U(t))$ and $i_*\beta \in \pi_{2s+1}(U(t))$ are also generators. By Bott's theorem, $i_*\langle \alpha, \beta \rangle = \langle i_*\alpha, i_*\beta \rangle$ is of order $t!/r!s!$, and so is $\langle \alpha, \beta \rangle$. Hence $\langle \alpha, \beta \rangle = r!s!/2 \in \mathbb{Z}/(t!/2)$.

2.3. **The case t is odd and $r > 1$.** Since $r > 1$, we have $r \leq s < t - 2$. Hence there exist generators in the stable range $\tilde{\alpha} \in \pi_{2r+1}(U(t-2))$ and $\tilde{\beta} \in \pi_{2s+1}(U(t-2))$ such that $i_*\tilde{\alpha} = \alpha$ and $i_*\tilde{\beta} = \beta$.

Consider the homomorphisms induced by the standard embeddings:

$$\cdots \to \pi_{2t}(U(t-2)) \xrightarrow{i_*} \pi_{2t}(U(t-1)) \xrightarrow{i'_*} \pi_{2t}(U(t)),$$

where we recall from Matsunaga [4] that

$$\pi_{2t}(U(t-2)) \cong \mathbb{Z}/(t!(24, t+1)/24).$$

To investigate the map i_*, we consider the homotopy exact sequence of the fibering $U(t-2) \to U(t-1) \to S^{2t-3}$:

$$\cdots \to \pi_{2t+1}(S^{2t-3}) \to \pi_{2t}(U(t-2)) \xrightarrow{i_*} \pi_{2t}(U(t-1)) \to \pi_{2t}(S^{2t-3}) \to \cdots ,$$

where it is known that $\pi_{n+4}(S^n) \cong 0$ for $n \geq 6$ and $\pi_{n+3}(S^n) \cong \mathbb{Z}/24$ for $n \geq 5$ (see [7]), so the sequence is expressed as

$$0 \to \mathbb{Z}/(t!(24, t+1)/24) \xrightarrow{i_*} \mathbb{Z}/t! \oplus \mathbb{Z}/2 \to \mathbb{Z}/24.$$

For the generator $1 \in \mathbb{Z}/(t!(24, t+1)/24)$, we have $i_*(1) = 24/(24, t+1) \oplus 1 \in \mathbb{Z}/t! \oplus \mathbb{Z}/2$, since $\pi_{2t}(S^{2t-3})$ is cyclic.

By Bott's theorem, $i'_*i_*\langle \tilde{\alpha}, \tilde{\beta} \rangle = \langle i'_*i_*\tilde{\alpha}, i'_*i_*\tilde{\beta} \rangle$ is of order $t!/r!s!$. It follows that $\langle \tilde{\alpha}, \tilde{\beta} \rangle$ is also of order $t!/r!s!$, since i'_*i_* does not reduce the order of $\langle \tilde{\alpha}, \tilde{\beta} \rangle$. This implies $\langle \tilde{\alpha}, \tilde{\beta} \rangle = r!s!(24, t+1)/24 \in \mathbb{Z}/(t!(24, t+1)/24)$, so we have

$$\langle \alpha, \beta \rangle = \langle i_*\tilde{\alpha}, i_*\tilde{\beta} \rangle = i_*\langle \tilde{\alpha}, \tilde{\beta} \rangle = \begin{cases} 4 \oplus 1 & \text{if } r = s = 2 \ (t = 5), \\ r!s! \oplus 0 & \text{if } t : \text{odd}, \ t \geq 7 \text{ and } r > 1. \end{cases}$$

The other case "t is odd, $t \geq 5$ and $r = 1$" is verified in a similar way to the proof of Bott's theorem.

3. The case "t is odd, $t \geq 5$ and $r = 1$"

3.1. Method.

The Samelson product is compatible with the embedding $SU(n) \to U(n)$:

$$
\begin{array}{ccc}
\pi_k(SU(n)) \otimes \pi_l(SU(n)) & \xrightarrow{\langle -,- \rangle} & \pi_{k+l}(SU(n)) \\
\downarrow & & \downarrow \\
\pi_k(U(n)) \otimes \pi_l(U(n)) & \xrightarrow{\langle -,- \rangle} & \pi_{k+l}(U(n)),
\end{array}
$$

where it is clear that the vertical homomorphisms are isomorphic if $k \geq 2$ and $l \geq 2$. Therefore we may consider the homotopy groups of the special unitary groups instead of those of the unitary groups.

When $n < m$, there are two embeddings of $SU(n)$ in $SU(m)$; we denote by $i_{m,n}$ and $j_{m,n}$ the embeddings which identify $SU(n)$ with the following two subgroups of $SU(m)$ respectively:

$$
\left(\begin{array}{c|c} SU(n) & 0 \\ \hline 0 & 1 \end{array} \right), \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & SU(n) \end{array} \right).
$$

Consider the composite of the following maps:

$$
SU(2) \wedge SU(t-1) \xrightarrow{i_{t-1,2} \wedge 1} SU(t-1) \wedge SU(t-1) \xrightarrow{c} SU(t-1),
$$

where c is the commutator map defined in Introduction.

The image group $j_{t-1,t-3}(SU(t-3))$ commutes with $i_{t-1,2}(SU(2))$ elementwisely in $SU(t-1)$. Hence if we write $W_{t-1,2}$ for $SU(t-1)/j_{t-1,t-3}(SU(t-3))$ and denote by p the natural projection, the composition above induces a map $\lambda : S^3 W_{t-1,2} \to SU(t-1)$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
SU(2) \wedge SU(t-1) & \xrightarrow{c \circ (i_{t-1,2} \wedge 1)} & SU(t-1). \\
{\scriptstyle 1 \wedge p} \downarrow & \nearrow {\scriptstyle \lambda} & \\
S^3 W_{t-1,2} & &
\end{array}
$$

According to Bott [2], the map λ is suspendable, that is, there is a map

$$
\lambda^E : S^4 W_{t-1,2} \longrightarrow SU(t+1)/SU(t-1)
$$

with the commutative diagram

$$
\begin{array}{ccc}
\pi_{2t+1}(S^4 W_{t-1,2}) & \xleftarrow{S} & \pi_{2t}(S^3 W_{t-1,2}) \\
{\scriptstyle \lambda^E_*} \downarrow & & \downarrow {\scriptstyle \lambda_*} \\
\pi_{2t+1}(SU(t+1)/SU(t-1)) & \xrightarrow{\Delta} & \pi_{2t}(SU(t-1)),
\end{array}
$$

where Δ is the boundary homomorphism in the homotopy exact sequence of the fibering $SU(t-1) \xrightarrow{i_{t+1,t-1}} SU(t+1) \longrightarrow SU(t+1)/SU(t-1)$.

Although Bott constructed the map λ^E in [2], we will reconstruct, for our purpose, a map of $S^4 W_{t-1,2}$ to $SU(t+1)/SU(t-1)$ which also makes the above diagram commutative, and so we denote by $\tilde{\lambda}$ the reconstructed map.

Consider the following commutative diagram:

$$
\begin{array}{ccc}
\pi_3(SU(2)) \otimes \pi_{2t-3}(SU(t-1)) & \xrightarrow{1_* \otimes p_*} & \pi_3(S^3) \otimes \pi_{2t-3}(W_{t-1,2}) \\
\downarrow{h} & & \downarrow{h} \\
\pi_{2t}(SU(2) \wedge SU(t-1)) & \xrightarrow{(1 \wedge p)_*} & \pi_{2t}(S^3 W_{t-1,2}), \\
\downarrow{(c \circ (i_{t-1,2} \wedge 1))_*} & \nearrow{\lambda_*} & \\
\pi_{2t}(SU(t-1)) & &
\end{array}
$$

where h is the obvious homomorphism which sends $f \otimes g$ to $f \wedge g$. For the generator $\alpha \in \pi_3(SU(t-1))$, there exists a generator $\alpha' \in \pi_3(SU(2))$ such that $i_{t-1,2_*}(\alpha') = \alpha$. Then the Samelson product $\langle \alpha, \beta \rangle$ is expressed as follows:

$$
\begin{aligned}
\langle \alpha, \beta \rangle &= c \circ (\alpha \wedge \beta) \\
&= (c \circ (i_{t-1,2} \wedge 1))_* \circ h(\alpha' \otimes \beta) \\
&= \lambda_* \circ h \circ (1_* \otimes p_*)(\alpha' \otimes \beta) \\
&= \Delta \circ \tilde{\lambda}_* \circ S \circ h \circ (1_* \otimes p_*)(\alpha' \otimes \beta).
\end{aligned}
$$

(3.1)

In this expression, we have the following:

Proposition 3.2.
(i) h and S are isomorphisms.
(ii) Δ is an epimorphism.

Proof. As for (i), we consider the following commutative diagram:

$$
\begin{array}{ccc}
& & \pi_{2t-3}(W_{t-1,2}) \\
& \nearrow{n} & \downarrow{S^3} \\
\pi_3(S^3) \otimes \pi_{2t-3}(W_{t-1,2}) & \xrightarrow{h} & \pi_{2t}(S^3 W_{t-1,2}),
\end{array}
$$

where n is the natural isomorphism. Since $W_{t-1,2} = S^{2t-5} \times S^{2t-3}$ is $(2t-6)$-connected, S^3 is an isomorphism by the suspension theorem (cf. [8]). Hence h is an isomorphism. Moreover S in (3.1) is an isomorphism again by using the suspension theorem.

As for (ii), we consider the homotopy exact sequence of the fibering $SU(t-1) \to SU(t+1) \to SU(t+1)/SU(t-1)$:

$$\cdots \longrightarrow \pi_{2t+1}(SU(t+1)/SU(t-1)) \xrightarrow{\Delta} \pi_{2t}(SU(t-1))$$
$$\longrightarrow \pi_{2t}(SU(t+1)) \longrightarrow \cdots,$$

where we have $\pi_{2t}(SU(t+1)) = 0$; so Δ is an epimorphism. $\square$

Consequently the factorization in (3.1) is expressed as follows:

$$\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{1_* \otimes p_*} \mathbb{Z} \otimes (\mathbb{Z} \oplus \mathbb{Z}/2) \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z}/2$$
$$\xrightarrow{S} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\tilde{\lambda}_*} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\Delta} \mathbb{Z}/t! \oplus \mathbb{Z}/2.$$

However, for the generator $\beta \in \pi_{2t-3}(SU(t-1))$, we have by [5] the following:

$$p_*(\beta) = \begin{cases} 6 \oplus 1 & \text{if } r = 1, \, s = 3 \, (t = 5), \\ s! \oplus 0 & \text{if } t : \text{odd}, \, t \geq 7 \text{ and } r = 1, \end{cases}$$

in $\pi_{2t-3}(W_{t-1,2}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Therefore, in order to complete the proof of Theorem 1, it is sufficient to prove the following:

Proposition 3.3. $\tilde{\lambda}_*$ *is an isomorphism.*

Now we will construct the map $\tilde{\lambda}$.

3.2. Construction of $\tilde{\lambda}$.

As was defined before, λ is the map induced by $c \circ (i_{t-1,2} \wedge 1)$:

$$SU(2) \wedge SU(t-1) \xrightarrow{c \circ (i_{t-1,2} \wedge 1)} SU(t-1).$$

with vertical map $1 \wedge p$ down to $S^3 W_{t-1,2}$ and diagonal map λ.

Let $X = S^3 W_{t-1,2}$. For the sake of our convenience we represent SX as the quotient $X \times [0, \pi/2]/(X \times \{0\} \cup X \times \{\pi/2\})$. For each $\theta \in [0, \pi/2]$ and for $1 \leq i < j \leq n$, we define an element $T_n(\theta; i, j) \in SU(n)$ as follows:

$$T_n(\theta; i, j)\mathbf{e}_k = \begin{cases} \cos\theta\,\mathbf{e}_i + \sin\theta\,\mathbf{e}_j & \text{if } k = i, \\ -\sin\theta\,\mathbf{e}_i + \cos\theta\,\mathbf{e}_j & \text{if } k = j, \\ \mathbf{e}_k & \text{otherwise}, \end{cases}$$

where $\mathbf{e}_k$ $(1 \leq k \leq n)$ is the n-tuple with k-th coordinate 1 and all others 0.

We also define a map $\mathcal{T}_\theta : SU(t+1) \to SU(t+1)$ as follows:

$$
\mathcal{T}_\theta = \begin{cases}
\overline{T_{t+1}(3\theta; 2, t)} & \text{if } 0 \le \theta \le \frac{\pi}{6}, \\
\overline{T_{t+1}(3\theta - \frac{\pi}{2}; 1, 2)} \circ \overline{T_{t+1}(\frac{\pi}{2}; 2, t)} & \text{if } \frac{\pi}{6} \le \theta \le \frac{\pi}{3}, \\
\overline{T_{t+1}(3\theta - \pi; 2, t+1)} \circ \overline{T_{t+1}(\frac{\pi}{2}; 1, 2)} \circ \overline{T_{t+1}(\frac{\pi}{2}; 2, t)} & \text{if } \frac{\pi}{3} \le \theta \le \frac{\pi}{2},
\end{cases}
$$

where $\overline{P}$ denotes the inner automorphism induced by $P \in SU(n)$:

$$\overline{P}(A) = PAP^{-1} \quad \text{for } A \in SU(n).$$

Then the two groups $i_{t+1,2}(SU(2))$ and $\mathcal{T}_{\pi/2} \circ i_{t+1,t-1}(SU(t-1))$ commute elementwisely. Further, $\mathcal{T}_\theta \circ i_{t+1,t-1}(g) = i_{t+1,t-1}(g)$ for any $\theta \in [0, \pi/2]$ and any $g \in j_{t-1,t-3}(SU(t-3))$, since the elements in $i_{t+1,t-1} \circ j_{t-1,2}(SU(2))$ commute with $T_{t+1}(\theta; 2, t)$, $T_{t+1}(\theta; 1, 2)$ and $T_{t+1}(\theta; 2, t+1)$. Hence the two groups $i_{t+1,2}(SU(2))$ and $\mathcal{T}_\theta \circ i_{t+1,t-1} \circ j_{t-1,t-3}(SU(t-3))$ commute elementwisely. Thus the map

$$\varphi : SU(2) \wedge SU(t-1) \times [0, \pi/2] \longrightarrow SU(t+1)$$

defined by

$$\varphi(f, g, \theta) = c \circ \big(i_{t+1,2} \wedge (\mathcal{T}_\theta \circ i_{t+1,t-1}) \big)(f, g)$$

for $f \in SU(2)$, $g \in SU(t-1)$ and $\theta \in [0, \pi/2]$ induces a map

$$\overline{\varphi} : CX = X \times [0, \pi/2] / X \times \{\pi/2\} \longrightarrow SU(t+1).$$

Since $\mathcal{T}_0$ is the identity map, $\overline{\varphi}$ restricted to $X \times \{0\}$ is precisely $i_{t+1,t-1} \circ \lambda$. Hence, for the bundle projection $\tau : SU(t+1) \to SU(t+1)/i_{t+1,t-1}(SU(t-1))$, the composite $\tau \circ \overline{\varphi}$ maps $X \times \{0\}$ to the base point. Thus we may take the map induced by $\tau \circ \overline{\varphi}$ for the required map

$$\tilde{\lambda} : SX \longrightarrow SU(t+1)/SU(t-1).$$

By these constructions, we have the following consequence:

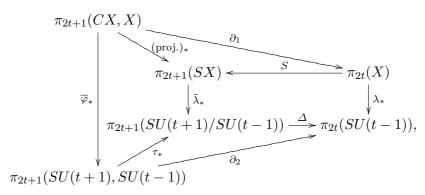**Proposition 3.4.** *The map $\tilde{\lambda}$ makes the following diagram commutative*:

$$
\begin{array}{ccc}
\pi_{2t+1}(S^4 W_{t-1,2}) & \xleftarrow{\;\;S\;\;} & \pi_{2t}(S^3 W_{t-1,2}) \\
\Big\downarrow{\scriptstyle \tilde{\lambda}_*} & & \Big\downarrow{\scriptstyle \lambda_*} \\
\pi_{2t+1}(SU(t+1)/SU(t-1)) & \xrightarrow{\;\;\Delta\;\;} & \pi_{2t}(SU(t-1)),
\end{array}
$$

where Δ is the boundary homomorphism in the homotopy exact sequence of the fibering $SU(t-1) \to SU(t+1) \to SU(t+1)/SU(t-1)$.

Proof. Consider the following diagram:

$$
\begin{array}{c}
\pi_{2t+1}(CX, X) \\
\end{array}
$$

where ∂_i $(i = 1, 2)$ are the boundary homomorphisms in the homotopy exact sequence of the pair. It follows from the constructions that the left-hand side and the outside quadrilaterals are commutative, and it is clear that the triangles are also commutative. Since ∂_1 is clearly isomorphic, the rectangle is also commutative. □

3.3. **Proof of Proposition 3.3.**

Consider the following two fiberings

$$SU(t-2)/SU(t-3) \xrightarrow{i_1} SU(t-1)/SU(t-3) \xrightarrow{p_1} SU(t-1)/SU(t-2),$$

$$SU(t)/SU(t-1) \xrightarrow{i_2} SU(t+1)/SU(t-1) \xrightarrow{p_2} SU(t+1)/SU(t)$$

induced respectively by the embeddings

$$SU(t-3) \xrightarrow{j_{t-2,t-3}} SU(t-2) \xrightarrow{j_{t-1,t-2}} SU(t-1),$$

$$SU(t-1) \xrightarrow{i_{t,t-1}} SU(t) \xrightarrow{i_{t+1,t}} SU(t+1).$$

Now we consider the following diagram:

(3.5)

$$
\begin{array}{ccc}
\pi_{2t+1}(S(S^3 \wedge S^{2t-5})) & \xrightarrow{\lambda^E_*} & \pi_{2t+1}(SU(t)/SU(t-1)) \\
\downarrow {\scriptstyle (S^4 i_1)_*} & & \downarrow {\scriptstyle i_{2*}} \\
\pi_{2t+1}(S^4 W_{t-1,2}) & \xrightarrow{\tilde{\lambda}_*} & \pi_{2t+1}(SU(t+1)/SU(t-1)) \\
\downarrow {\scriptstyle (S^4 p_1)_*} & & \downarrow {\scriptstyle p_{2*}} \\
\pi_{2t+1}(S(S^3 \wedge S^{2t-3})) & \xrightarrow{\lambda^E_*} & \pi_{2t+1}(SU(t+1)/SU(t)),
\end{array}
$$

where the vertical sequences are clearly exact, and λ^E is the map defined by Bott in [2]. When λ^E is a map between spheres of the same dimension, λ^E

is of degree one by Proposition 2.2 of [2]. Hence it is sufficient to prove that the above diagram is commutative.

(i) As for the commutativity of the first square of the diagram (3.5), it is sufficient to prove that the following diagram is homotopy commutative:

(3.6)

$$\begin{array}{ccc} (C(S^3 \wedge S^{2t-5}), S^3 \wedge S^{2t-5}) & \xrightarrow{\ \bar{s}\ } & (SU(t), SU(t-1)) \\ \downarrow{\scriptstyle C(1 \wedge i_1)} & & \downarrow{\scriptstyle i_{t+1,t}} \\ (C(S^3 W_{t-1,2}), S^3 W_{t-1,2}) & \xrightarrow{\ \bar{\varphi}\ } & (SU(t+1), SU(t-1)), \end{array}$$

where $\bar{s}$ is the map constructed from a map

$$s : SU(2) \wedge SU(t-2) \times [0, \pi/2] \longrightarrow SU(t)$$

defined in [2] by

$$s(f, g, \theta) = c \circ \left(i_{t,2} \wedge (\overline{T_t(\theta; 2, t)} \circ i_{t,t-1} \circ j_{t-1,t-2}) \right)(f, g)$$

for $f \in SU(2)$, $g \in SU(t-2)$ and $\theta \in [0, \pi/2]$. Then observe that the map $\bar{s}$ induces λ^E.

From the definitions, it follows immediately that

$$\varphi \circ (1 \wedge j_{t-1,t-2} \times 1)(f, g, \theta) = \begin{cases} i_{t+1,t} \circ s(f, g, 3\theta) & \text{if } 0 \le \theta \le \pi/6, \\ \text{base point} & \text{if } \pi/6 \le \theta \le \pi/2. \end{cases}$$

Here we have that $\varphi \circ (1 \wedge j_{t-1,t-2} \times 1)$ is homotopic, as a map of pairs, to $i_{t+1,t} \circ s$ by an obvious homotopy, and we can easily see that this homotopy induces a homotopy which makes the diagram (3.6) commutative.

(ii) As for the commutativity of the second square of the diagram (3.5), it is sufficient to prove that the following diagram is homotopy commutative:

(3.7)

$$\begin{array}{ccc} (C(S^3 W_{t-1,2}), S^3 W_{t-1,2}) & \xrightarrow{\ \bar{\varphi}\ } & (SU(t+1), SU(t-1)) \\ \downarrow{\scriptstyle C(1 \wedge p_1)} & & \downarrow{\scriptstyle 1} \\ (C(S^3 \wedge S^{2t-3}), S^3 \wedge S^{2t-3}) & \xrightarrow{\ \bar{s}\ } & (SU(t+1), SU(t)), \end{array}$$

where $\bar{s}$ is the map constructed from a map

$$s : SU(2) \wedge SU(t-1) \times [0, \pi/2] \longrightarrow SU(t+1)$$

defined in [2] by

$$s(f, g, \theta) = c \circ \left(i_{t+1,2} \wedge (\overline{T_{t+1}(\theta; 2, t+1)} \circ i_{t+1,t} \circ j_{t,t-1}) \right)(f, g),$$

for $f \in SU(2)$, $g \in SU(t-1)$ and $\theta \in [0, \pi/2]$.

By the definition of φ, the image $\varphi(f, g, \theta)$ is included in $i_{t+1,t}(SU(t))$ if $0 \leq \theta \leq \pi/3$. Hence we can deform φ along $0 \leq \theta \leq \pi/3$, that is, there is a homotopy starting with φ:

$$F_k(f, g, \theta) = c \circ \big(i_{t+1,2} \wedge (\mathcal{T}_\mu \circ i_{t+1,t-1})\big)(f, g) \quad k \in [0, 1],$$

where $\mu = (1 - k)\theta + k((\theta + \pi)/3)$. Denote by φ' the final map of this homotopy, then φ' is a map defined by

$$\varphi'(f, g, \theta) = c \circ \big(i_{t+1,2} \wedge (\overline{T_{t+1}(\theta; 2, t+1)} \circ \overline{T_{t+1}(\pi/2; 1, 2)}$$
$$\circ \overline{T_{t+1}(\pi/2; 2, t)} \circ i_{t+1,t-1})\big)(f, g).$$

Thus φ is homotopic to φ' as a map of pairs into $(SU(t + 1), SU(t))$.

Next, we consider a composite map

$$\rho = \overline{T_{t-1}(\pi/2; 2, 3)} \circ \overline{T_{t-1}(\pi/2; 3, 4)} \circ \cdots \circ \overline{T_{t-1}(\pi/2; t-2, t-1)},$$

which is homotopic to the identity, since each $\overline{T_n(\pi/2; i, j)} : SU(n) \to SU(n)$ is clearly homotopic to the identity. Then it follows by an easy calculation that

$$\overline{T_{t+1}(\pi/2; 1, 2)} \circ \overline{T_{t+1}(\pi/2; 2, t)} \circ i_{t+1,t-1} \circ \rho = i_{t+1,t} \circ j_{t,t-1}.$$

Therefore we have

$$\varphi \simeq \varphi' \simeq \varphi' \circ (1 \wedge \rho \times 1) = s$$

as a map of pairs, and we can easily see that the homotopy between φ and s induces a homotopy which makes the diagram (3.7) commutative.

References

[1] R. Bott, *The space of loops on a Lie group*, Michigan Math. J., **5**(1958), 35–61.

[2] R. Bott, *A note on the Samelson product in the classical groups*, Comment. Math. Helv., **34**(1960), 249–256.

[3] M. A. Kervaire, *Some nonstable homotopy groups of Lie groups*, Ill. J. Math., **4**(1960), 161–169.

[4] H. Matsunaga, *The homotopy groups $\pi_{2n+i}(U(n))$ for $i = 3$, 4 and 5*, Mem. Fac. Sci. Kyushu Univ., **15**(1961), 72–81.

[5] Y. Nomura and Y. Furukawa, *Some homotopy groups of complex Stiefel manifolds $W_{n,2}$ and $W_{n,3}$*, Sci. Rep. College Gen. Ed. Osaka Univ., **25**(1976), 1–17.

[6] K. Sugata, *On the group $[X, SU(4)]$ where X is an S^3-bundle over S^7*, Math. J. Okayama Univ., **45**(2003), 99–110.

[7] H. Toda, *Composition methods in homotopy groups of spheres*, Ann. Math. Studies, **49**(1962).

[8] G. W. Whitehead *Elements of Homotopy Theory*, Springer, New York, 1978.

Kei Sugata
The Graduate School of Natural Science and Technology
Okayama University
Okayama 700-8530, Japan