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QUATERNIONIC LINE BUNDLES OVER QUATERNIONIC PROJECTIVE SPACES

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1. Introduction

Let ξ be an $\mathbb{F}$-line bundle over a space X, the field $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. A natural question is to enumerate the set $Line_{\mathbb{F}}(X)$ of such bundles ξ for given classes of space. As it is well known, classical bundle theory of characteristic classes gives a complete answer for the real and complex case, where

$$Line_{\mathbb{R}}(X) = H^1(X;\mathbb{Z}/2), \qquad\qquad Line_{\mathbb{C}}(X) = H^2(X;\mathbb{Z}),$$

the bijection is given by the first Stiefel Withney and Chern classes, respectively. So far, the complete answer for the quaternionic case is still unknown, and as we will see in Section 2, a full general answer is unlikely. Ultimately, an answer to this problem would depend at least on the knowledge of the homotopy groups of the $3-$sphere. In this paper we deal with the case $X = \mathbb{H}P^n$. Thus, counting quaternionic line bundles over X is equivalent to count the element in the set $[\mathbb{H}P^n, \mathbb{H}P^n]_0$, of the based homotopy classes of self maps of the quaternionic projective n-space. We give a short review on what is known on this problem in Section 2. Our results give a complete answer for the low dimensional cases $n \leq 3$ and some indication on the general case. Here the subscript $+/-$ denotes the even/odd subset of a subset of integer numbers and $P(2), P(3), Q(2), Q(3)$ are sets of cardinalities $1, 2, 2, 4$, respectively.

Theorem 1. $[\mathbb{H}P^2, BS^3] = Line_{\mathbb{H}}(\mathbb{H}P^2) = R_{2,+} \times P(2) \cup R_{2,-} \times Q(2)$, *where* $R_2 = \{n \in \mathbb{Z}, n = 0, 1, 9, 16(\mathrm{mod}24)\}$.

An explicit description of the maps is given in Proposition 6.

Theorem 2. $Line_{\mathbb{H}}(\mathbb{H}P^3) = R_{3,+} \times P(3) \cup R_{3,-} \times Q(3)$, *where* $R_3 = \{n \in \mathbb{Z}, n = 0, 1, 9, 16, 25, 40, 49, 64, 81, 121, 136, 144, 145, 160, 169, 184, 216, 225, 241, 256, 265, 280, 289, 304(\mathrm{mod}360)\}$.

Theorem 3. *The number of non-equivalent line bundles in* $Line_{\mathbb{H}}(\mathbb{H}P^n)$ *with fixed first quaternionic characteristic class* λ*, only depends on the parity of* λ.

A direct consequence of the above result is:

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$Line_{\mathbb{H}}(\mathbb{H}P^n) = R_{n,+} \times P(n) \cup R_{n,-} \times Q(n)$, where R_n is defined in Section 2, and $P(n)$ and $Q(n)$ are finite sets whose cardinalities depend on n.

Also the above result together with the Feder and Gitler conjecture mentioned in Section 2, prompts the following conjecture (where one inclusion is proved).

Conjecture 1. *$Line_{\mathbb{H}}(\mathbb{H}P^n) = FG_{n,+} \times P(n) \cup FG_{n,-} \times Q(n)$, where FG_n is defined in Section 2, and $P(n)$ and $Q(n)$ are finite sets whose cardinalities depend on n.*

As a by-product of the techniques used we also compute $[\mathbb{K}P^2, \mathbb{K}P^2]$ where $\mathbb{K}P^2$ stands for the Cayley projective space or the cone of the Hopf map $h : S^{15} \to S^8$.

As mentioned in Section 2, further investigations using the methods of [12] or [8] would probably allow to extend the classification to dimension 4 and 5, but for the general problem the indications given in Theorem 3 appears to be the best general result we are likely to get, that is to say, it is unlikely to find out a general formula for the numbers $P(n)$ and $Q(n)$ appearing in the Conjecture 1.

The work is organized as follows. In the following section, we give a summary on the state of the art on the subject. In Section 3, we consider the quaternionic projective space and we give a proof of Theorem 2 by classical methods, to be superseded by the techniques introduced in Section 4, where we deal with the general case and with the proofs of Theorems 2 and 3. In Section 5 we compute the set of homotopy classes of self maps of the Cayley projective space.

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2. Self maps of quaternionic projective spaces

Let $f : \mathbb{F}P^n \to \mathbb{F}P^n$ be a self map of the projective space over the field $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In the complex and quaternionic cases we can identify free and based homotopy classes, since all the spaces involved are simple (thus we will omit the subscript 0 in the notation of $[X, Y]_0$, whenever Y is simple); furthermore, connectivity and dimension imply that $[\mathbb{F}P^n, \mathbb{F}P^n] = [\mathbb{F}P^n, \mathbb{F}P^\infty]$. Thus, in the complex case we get

$$[\mathbb{C}P^n, \mathbb{C}P^n] = H^2(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}.$$

The real case is a little bit more complicate. Using Theorems IIa and IIIa of P. Olum [14], we can state the following. Call $[\mathbb{R}P^n, \mathbb{R}P^n]_0^0$ the set of pointed homotopy classes of maps which induce the zero homomorphism on the fundamental group, $[\mathbb{R}P^n, \mathbb{R}P^n]_0^1$ the set of pointed homotopy classes of maps which induce the identity homomorphism on the fundamental group and $\tilde{\mathbb{Z}}$ the local orientation system over $\mathbb{R}P^n$. Then we have

$$[\mathbb{R}P^n, \mathbb{R}P^n]_0^0 = \begin{cases} 2\mathbb{Z} & n \text{ odd}, \\ \mathbb{Z}/2 & n \text{ even}, \end{cases} \qquad [\mathbb{R}P^n, \mathbb{R}P^n]_0^1 = 1 + 2\mathbb{Z},$$

with the same notation for free classes we have

$$[\mathbb{R}P^n, \mathbb{R}P^n]^0 = \begin{cases} 2\mathbb{Z} & n \text{ odd}, \\ \mathbb{Z}/2 & n \text{ even}, \end{cases} \qquad [\mathbb{R}P^n, \mathbb{R}P^n]^1 = \begin{cases} 1+2\mathbb{Z} & n \text{ odd}, \\ 1+2\mathbb{N} & n \text{ even}. \end{cases}$$

Turning to the quaternionic case, since $H^*(\mathbb{H}P^n; \mathbb{Z}) = \mathbb{Z}[x]/x^{n+1}$, where x is a generator in dimension 4, we can associate to each self map an integer $\lambda = \lambda(f)$ defined by $f^*x = \lambda(f)x$, and call it the degree of f. Note that $\lambda(f)$ corresponds to the usual degree of the induced self map of $\Omega\mathbb{H}P^\infty$ in the infinite case. Note also that the degree of f is the class, in $\pi_4(\mathbb{H}P^n)$, of the restriction of f on the 4-skeleton [13].

Definition 1. *An integer λ is called n-realizable if there is a self map of $\mathbb{H}P^n$ with degree λ.*

Let R_n be the subset of integers that are n-realizable, i.e. the image of the function $\lambda : [\mathbb{H}P^n, BS^3] \to \mathbb{Z}$. For $n \geq 1$, consider the congruences

$$C_n(\lambda) = 0 : \qquad \prod_{i=0}^{n-1} (\lambda - i^2) = 0 \bmod \begin{cases} (2n)! & n \text{ even} \\ \frac{(2n)!}{2} & n \text{ odd}, \end{cases}$$

and the set $FG_n = \{\lambda \in \mathbb{Z} \mid C_i(\lambda) = 0, 1 \leq i \leq n\}$. For $n < \infty$, the first congruences are:

$$C_1(\lambda) = 0 : \qquad \lambda = 0 (\mathrm{mod} 1),$$
$$C_2(\lambda) = 0 : \qquad \lambda(\lambda - 1) = 0 (\mathrm{mod} 24),$$
$$C_3(\lambda) = 0 : \qquad \lambda(\lambda - 1)(\lambda - 4) = 0 (\mathrm{mod} 360).$$

In general, the allowed degrees in dimension n are not known, but Feder and Gitler proved in [6], using complex and quaternionic K-theory, that $R_n \subset FG_n$ and they conjectured that the condition is also sufficient. This conjecture has been verified for $n = 1, 2, 3, 4, 5$, in [2], [11], [8] and $n = \infty$ in [6] using the results of [18].

$$R_1 = FG_1 = \mathbb{Z}, \qquad R_2 = FG_2 = \{0, 1, 9, 16\} (\mathrm{mod} 24).$$

$$R_3 = FG_3 = \begin{array}{l} \{0, 1, 9, 16, 25, 40, 49, 64, 81, 121, 136, 144, 145, 160, 169, 184 \\ \quad 216, 225, 241, 256, 265, 280, 289, 304\} (\mathrm{mod} 360), \end{array}$$

$$R_\infty = FG_\infty = \{0, (2n+1)^2, n \in \mathbb{Z}\}.$$

The other face of the problem is determining if two self maps with the same degree are in fact homotopic. Marcum and Randall [13] answered this question in the negative providing essential self maps with trivial degree for $n = 3, 4, 5$, and they conjectured that this is the general situation. They also showed that if a map has degree zero in $\mathbb{H}P^2$ then it is homotopic to the constant map.

Further results exist for the maps of degree 1, namely the group of self homotopy equivalence $\mathcal{E}(\mathbb{H}P^n)$. In fact, facing the problem under this point of view, we can state the general natural question of determining the group $\mathcal{E}(\mathbb{F}P^n)$ of all invertible elements in the set $[\mathbb{F}P^n, \mathbb{F}P^n]_0$ with monoid structure given by composition. Again, a complete answer for the real and complex cases easily follows from classical methods in homotopy theory [5] and [9].

$$\mathcal{E}(\mathbb{R}P^n) = \mathbb{Z}/2, \qquad\qquad \mathcal{E}(\mathbb{C}P^n) = \mathbb{Z}/2,$$

The quaternionic case was considered by Iwase, Maruyama and Oka in [9], where they use an unstable homotopy spectral sequence and homotopy operations to get

$$\mathcal{E}(\mathbb{H}P^2) = \mathbb{Z}/2, \qquad\qquad \mathcal{E}(\mathbb{H}P^3) = \mathbb{Z}/2 \times \mathbb{Z}/2,$$

$$\mathcal{E}(\mathbb{H}P^4) = 0 \text{ or } \mathbb{Z}/2,$$

and conjecture the first alternative for the last group.

Eventually, the complete answer for the infinite case was given by Mislin in [12], that using a generalization of the Sullivan conjecture of H. Miller and a theorem of Dwyer on finite p-groups, got the classification theorem

Theorem 4. *(Mislin) Self maps of $\mathbb{H}P^\infty$ are classified by their degree.*

3. Quaternionic projective plane

Let $j : S^4 \to BS^3$ be the inclusion of the 4-skeleton. The following result generalizes the known fact that each map from a sphere to BS^3 factors through S^4.

Lemma 5. *Let G be a topological group, and X a space. Then, any map from ΣX in BG factors up to homotopy through the inclusion $i : \Sigma G \to BG$.*

Proof. Recall that the composition $(\Omega i)s : G \to \Omega BG$ is an homotopy equivalence, where s is the natural inclusion (the adjoint of the identity) and $i : \Sigma G \to BG$ is the inclusion given by the construction of BG. Call σ the homotopy inverse and consider the diagram

$$\begin{tikzcd} \Omega BG \arrow[rr, "\sigma"] & & G \arrow[dl, "s"] \\ & \Omega\Sigma G \arrow[ul, "\Omega i"] & \end{tikzcd}$$

For each $f : \Sigma X \to BG$, define

$$f' = \mathrm{coad}(s\sigma\mathrm{ad}(f));$$

then,

$$if' = i\mathrm{coad}(s\sigma\mathrm{ad}(f)) = \mathrm{coad}((\Omega i)s\sigma\mathrm{ad}(f)) \sim \mathrm{coad}(\mathrm{ad}(f)) = f.$$

□

This provides a right inverse ϕ in homotopy to $j_* : \pi_n(S^4) \to \pi_n(BS^3)$. The explicit form of ϕ is the composite of the suspension homomorphism Σ with the boundary ∂ appearing in the homotopy sequence associated to the universal fibration of BS^3.

We now consider the problem of counting the elements of $[\mathbb{H}P^2, \mathbb{H}P^2]$. Let $\lambda \in R_2$ be a 2-realizable degree, and consider the decomposition of $\mathbb{H}P^2 = S^4 \cup_{h_1} e^8$, where $h_1 : S^7 \to S^4$ is the Hopf map. Then, if $u_\lambda : S^4 \to BS^3$ has degree λ, it extends to a map $\hat{u} : \mathbb{H}P^2 \to BS^3$, and different extensions are given via the Hilton coaction by elements $\alpha \in \pi_8(BS^3)$ and denoted as $\hat{u}^\alpha$. Consider the right end of the Puppe sequence associated to the Hopf map h_1

$$\cdots \longrightarrow [\mathbb{H}P^2, BS^3] \xrightarrow{i_1^*} \mathbb{Z} \xrightarrow{h_1^*} \mathbb{Z}/12,$$

where $i_1 = j$ is the inclusion of S^4 in $\mathbb{H}P^2$. Then, by classical properties of Hilton coaction [17] VII.1, $[\mathbb{H}P^2, BS^3]$ is the union of the inverse images $(i_1^*)^{-1}([u_\lambda])$ of the classes of the maps $u_\lambda : S^4 \to BS^3$, and for each fixed u_λ, $(i_1^*)^{-1}([u_\lambda])$ contains as many different elements as there are non homotopic extensions $\hat{u}^\alpha_\lambda$. A construction of Barcus and Barratt [3], defines an homomorphism

$$\nu_{u_\lambda} : \pi_1(m_0(S^4, BS^3; u_\lambda)) \to \pi_8(BS^3),$$

such that $\hat{u}_\lambda \sim \hat{u}^\alpha_\lambda$ if and only if $\alpha \in \mathrm{Im}\nu_{u_\lambda}$. In the present case, since the 4-skeleton S^4 is a co-H-space, we can get the explicit form of this homomorphism. We use the commutative diagram

$$\begin{tikzcd} \pi_1(m_0(S^4, BS^3; u_\lambda)) \arrow[r, "\nu_{u_\lambda}"] \arrow[d, "(u_\lambda)_\natural"'] & \pi_8(BS^3) \\ \pi_5(BS^3)(= \pi_1(m_0(S^4, BS^3; u_0))) \arrow[ur, "\psi_{u_\lambda}"'] & \end{tikzcd}$$

where $(u_\lambda)_\natural$ is the isomorphism defined in 2.5, 2.6 of [3]. Using the composition Theorem 4.6 of [3], we get, for any $\zeta \in \pi_5(BS^3) = \mathbb{Z}/2$,

$$\psi_{u_\lambda}(\zeta) = \nu_{u_\lambda}(u_\lambda)_\natural^{-1}(\zeta) = \zeta \circ \Sigma\nu_4 + [(u_\lambda)_*(\iota_4), \zeta] \circ \Sigma H(\nu_4),$$

where ν_4 is the class of the Hopf map and $\Sigma H(\nu_4) = 1$. Here we are using the notation of Toda, and observe that $u_1 = j$. Now let's use the factorization given by Lemma 5, to reduce the computation in the homotopy groups of S^4. Then, let $\zeta = j_*(\chi)$ for some $\chi \in \pi_5(S^4) = \mathbb{Z}/2[\eta_4]$, where $\eta_4 = \Sigma^2\eta_2$, η_2 is the class of the Hopf map $h : S^3 \to S^2$ and observe that $u_1 = j$. By naturality of the Whitehead product and linearity of the composition with suspensions, the unique non trivial case is

$$[(u_\lambda)_*(\iota_4), j_*(\eta_4)] = \lambda(\mathrm{mod}2)j_*[\iota_4, \eta_4] = \lambda(\mathrm{mod}2)j_*[\iota_4, \iota_4] \circ \eta_7 =$$
$$= \lambda(\mathrm{mod}2)j_*((2\nu_4 \pm \Sigma\nu') \circ \eta_7) = \lambda(\mathrm{mod}2)j_*(\Sigma\nu' \circ \eta_7),$$

where $\Sigma\nu' \circ \eta_7$ is precisely the generator of $\pi_8(S^4)$ that is not in the kernel of j_*, we used [20] X.8.18, [19] 5.8, and ν' is the element of order 4 in $\pi_6(S^3)$. This gives

$$\psi_{u_\lambda}(j_*(\eta_4)) = (\lambda + 1)(\mathrm{mod}2)j_*(\Sigma\nu' \circ \eta_7).$$

Proposition 6. *Let $v_\lambda : \mathbb{H}P^2 \to BS^3$ be of 2-realizable degree λ, and $\alpha = j_*(\Sigma\nu' \circ \eta_7)$ the generator of $\pi_8(BS^3) = \mathbb{Z}/2[\alpha]$. Then, v_λ and v_λ^α are homotopic if and only if λ is even.*

Note that the above approach also allows to easily prove the Feder Gitler conjecture for $n = 2$, namely to find out the set R_2 of the 2-realizable integer (see [2] for the original proof). In fact, considering the Puppe sequence above, we see that $R_2 = \ker h_1^*$. Reducing the problem in S^4, we have

$$h_1^*(\lambda j_*(\iota_4)) = j_*\left(\lambda^2\nu_4 + \frac{\lambda(\lambda-1)}{2}\Sigma\nu'\right) = \frac{\lambda(\lambda-1)}{2}\Sigma\nu',$$

by [4] III.1.9, that gives for the kernel the condition $\lambda(\lambda - 1) = 0(\mathrm{mod}24)$.

4. $\mathbb{H}P^3$ and the general case

The above technique allows to deal only with the self maps of odd degree on $\mathbb{H}P^3$. Although, we can use the following quite general method. The natural inclusion $i_{n-1} : \mathbb{H}P^{n-1} \to \mathbb{H}P^n$ induces a fibration

$$(i_{n-1})_\sharp : m_0(\mathbb{H}P^n, BS^3) \to m_0(\mathbb{H}P^{n-1}, BS^3).$$

Now we will define a subspace of $m_0(\mathbb{H}P^n, BS^3)$ (the space of pointed maps from $\mathbb{H}P^n$ to BS^3), denoted by $M_n(f_\lambda)$, which is suitable to study our problem. Then we will consider the fibration having $M_n(f_\lambda)$ as total space, the fibre map the restriction of $(i_{n-1})_\sharp$ and as base $M_{n-1}(f_\lambda)$ (the projection is not surjective in general).

Let f_λ be of degree λ, $m_n(f_\lambda) = m_0(\mathbb{H}P^n, BS^3; f_\lambda)$ denotes the component of f_λ and $M_n(f_\lambda) = \bigcup_g m_n(g)$, where g runs over all maps in

$m_0(\mathbb{H}P^n, BS^3)$ of degree λ. We take f_λ as base point of $M_n(f_\lambda)$. If p_n is the restriction of $(i_{n-1})_\sharp$ to $M_n(f_\lambda)$ then its fibre $F_n(f_\lambda)$ over $f_\lambda i_{n-1}$, with base point f_λ, has the type of $\Omega^{4n}BS^3$, and we have the exact sequence

(1)

$$\cdots \xrightarrow[\partial_{n,q+1}]{} \pi_q(F_n(f_\lambda)) \xrightarrow[j_{n,q}]{} \pi_q(M_n(f_\lambda)) \xrightarrow[p_{n,q}]{} \pi_q(M_{n-1}(f_\lambda i_{n-1})) \xrightarrow[\partial_{n,q}]{} \cdots$$

The sequences with index $n-1$ and n can be ' composed' as follows:

$$\begin{tikzcd}
 & & \cdots \arrow[d] & & \\
 & & \pi_q(F_{n-1}(f_\lambda i_{n-1})) \arrow[d, "j_{n-1,q}"'] \arrow[dr, "d_{n,q}"] & & \\
\cdots \arrow[r, "j_{n,q}"'] & \pi_q(M_n(f_\lambda)) \arrow[r, "p_{n,q}"'] & \pi_q(M_{n-1}(f_\lambda i_{n-1})) \arrow[r, "\partial_{n,q}"'] \arrow[d] & \pi_{q-1}(F_n(f_\lambda)) \arrow[r] & \cdots \\
 & & \cdots & &
\end{tikzcd}$$

where an explicit description of the homomorphism $d_{n,q} = \partial_{n,q} j_{n-1,q}$ can be obtained using a construction of James [10] as in [9]

$$d_{n,q} : \pi_q(F_{n-1}(f_\lambda i_{n-1})) = \pi_{4n+q-4}(BS^3) \to \pi_{q-1}(F_n(f_\lambda)) = \pi_{4n+q-1}(BS^3),$$

(2) $$d_{n,q} : \zeta \mapsto \pm(n-1)\zeta \circ \nu_{4n+q-4} \pm \lambda[\gamma, \zeta],$$

where $\gamma = [j]$ denotes the class of the inclusion $j : S^4 \to BS^3$. Notice in particular that the homomorphism

$$j_{1,q} : \pi_q(F_1(f_\lambda) = \pi_{q+4}(BS^3) \to \pi_q(M_1(f_\lambda)),$$

is always an isomorphism, since the two spaces have the same type. We can now apply this construction to determinate the number of connected components of the space $M_3(f_\lambda)$, for each λ. We will use Lemma 5 to reduce computation to homotopy groups of S^4. With $n = 2$, we get the sequence

$$\cdots \longrightarrow \mathbb{Z}/2[j_*(\eta_4^2)] \xrightarrow[\partial_{2,2}]{}$$

$$\xrightarrow[\partial_{2,2}]{} \mathbb{Z}/2[j_*(\Sigma\nu' \circ \eta_7^2)] \xrightarrow[j_{2,1}]{} \pi_1(M_2(f_\lambda)) \xrightarrow[p_{2,1}]{} \mathbb{Z}/2[j_*(\eta_4)] \xrightarrow[\partial_{2,1}]{}$$

$$\xrightarrow[\partial_{2,1}]{} \mathbb{Z}/2[j_*(\Sigma\nu' \circ \eta_7)] \xrightarrow[j_{2,0}]{} \pi_0(M_2(f_\lambda)) \xrightarrow[p_{2,0}]{} 0 \longrightarrow \pi_7(BS^3).$$

with connecting homomorphisms

$$d_{2,1} = \partial_{2,1} : \mathbb{Z}/2[j_*(\eta_4)] \to \mathbb{Z}/2[j_*(\Sigma\nu' \circ \eta_7)]$$

$$\partial_{2,1}(\zeta) = \zeta \circ \nu_5 + \lambda[j_*(\iota_4), \zeta],$$

that coincides with the homomorphism ψ_{u_λ} of Section 3, and

$$d_{2,2} = \partial_{2,2} : \mathbb{Z}/2[j_*(\eta_4^2)] \to \mathbb{Z}/2[j_*(\Sigma\nu' \circ \eta_7^2)]$$
$$\partial_{2,2}(\zeta) = \zeta \circ \nu_6 + \lambda[j_*(\iota_4), \zeta].$$

Using $\eta_4^2 \circ \nu_6 = \eta_4 \circ \eta_5 \circ \nu_6 = 0$ [19] 5.9, this gives

$$\partial_{2,2}(j_*(\eta_4^2)) = \lambda j_*(\Sigma\nu' \circ \eta_7^2).$$

We can state the following facts:

(1) λ even
(1a) then $\partial_{2,1}$ is injective and thus is iso; hence $\ker j_{2,0} = \operatorname{im}\partial_{2,1} = \mathbb{Z}/2$, and since $j_{2,0}$ is onto, because $p_{2,0}$ is trivial, this means that $\pi_0(M_2(f_\lambda)) = 0$;
(1b) then $\partial_{2,1}$ is injective, so $\ker\partial_{2,1} = 0 = \operatorname{im} p_{2,1}$, and hence $p_{2,1}$ is trivial and $\pi_1(M_2(f_\lambda)) = \ker p_{2,1} = \operatorname{im} j_{2,1}$; with λ even, $\partial_{2,2}$ is trivial, and hence $\ker j_{2,1} = 0$ and $\pi_1(M_2(f_\lambda)) = \operatorname{im} j_{2,1} = \mathbb{Z}/2$.

(2) λ odd
(2a) then $\partial_{2,1}$ is trivial and hence $\ker j_{2,0} = 0$, thus $j_{2,0}$ is injective and $\operatorname{im} j_{2,0} = \ker p_{2,0} = \mathbb{Z}/2$; since $p_{2,0}$ is trivial, this implies $\pi_0(M_2(f_\lambda)) = \mathbb{Z}/2$
(2b) then $\partial_{2,1}$ is trivial so $\ker\partial_{2,1} = \mathbb{Z}/2 = \operatorname{im} p_{2,1}$; on the other side, $\partial_{2,2}$ is injective so $\ker j_{2,1} = \mathbb{Z}/2 = \operatorname{im}\partial_{2,2}$, and this implies that $j_{2,1}$ is trivial, $\ker p_{2,1} = \operatorname{im} j_{2,1} = 0$, hence $\pi_1(M_2(f_\lambda)) = \operatorname{im} p_{2,1}/\ker p_{2,1} = \mathbb{Z}/2$.

When $n = 3$, using the results of the case $n = 2$, we have the sequence

$$\cdots \longrightarrow \pi_2(M_2(f_\lambda i_2)) \xrightarrow[\partial_{3,2}]{} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[j_{3,1}]{} \pi_1(M_3(f_\lambda)) \xrightarrow[p_{3,1}]{}$$

(3)

$$\xrightarrow[p_{3,1}]{} \pi_1(M_2(f_\lambda i_2)) \xrightarrow[\partial_{3,1}]{} \mathbb{Z}/2 \xrightarrow[j_{3,0}]{} \pi_0(M_3(f_\lambda)) \xrightarrow[p_{3,0}]{} \pi_0(M_2(f_\lambda i_2)) \longrightarrow \mathbb{Z}/15.$$

(where recall that $\pi_1(M_2(f_\lambda i_2)) = \mathbb{Z}/2$) with the connecting homomorphisms

$$d_{3,1} : \pi_9(BS^3) = \mathbb{Z}/2[j^*(\Sigma\nu' \circ \eta_7^2)] \to \pi_{12}(BS^3) = \mathbb{Z}/2[\epsilon_4],$$
$$d_{3,1}(\zeta) = \pm 2\zeta \circ \nu_9 \pm \lambda[j_*(\iota_4), \zeta],$$

and

$$d_{3,2} : \pi_{10}(BS^3) = \mathbb{Z}/3 \to \pi_{13}(BS^3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

that is clearly trivial. Let's compute $d_{3,1}$, on the generator. By [19] 5.9

$$j^*(\Sigma\nu' \circ \eta_7^2) \circ \nu_9 = j_*(\Sigma\nu' \circ \eta_7 \circ \eta_8 \circ \nu_9) = 0.$$

$$[j_*(\iota_4), j_*(\Sigma\nu' \circ \eta_7^2)] = j_*((2\nu_4 \pm \Sigma\nu') \circ \Sigma^4(\nu' \circ \eta_6 \circ \eta_7)) = 0,$$

since $\Sigma^4(\nu' \circ \eta_6 \circ \eta_7) = \eta_7 \circ \nu_8 \circ \eta_{11} = 0$, and hence $d_{3,1}$ is always trivial. In order to proceed, we need to distinguish even and odd λ.

When λ is even, by point (1) above, $j_{2,1}$ is onto, and since $\partial_{3,1} j_{2,1} = d_{3,1} = 0$, it follows that $\partial_{3,1}$ is trivial in this case. Hence, $\ker j_{3,0} = \mathrm{im}\partial_{3,1} = 0$, and we have the exact sequence

$$0 \xrightarrow[\partial_{3,1}]{} \mathbb{Z}/2 \xrightarrow[j_{3,0}]{} \pi_0(M_3(f_\lambda)) \xrightarrow[p_{3,0}]{} \pi_0(M_2(f_\lambda i_2)) \xrightarrow[\partial_{3,0}]{} \mathbb{Z}/15.$$

Since when λ is even, by the computations with $n = 2$, $\pi_0(M_2(f_\lambda i_2)) = 0$, we infer that $j_{3,0}$ is a bijection and $\pi_0(M_3(f_\lambda)) = \mathbb{Z}/2$ in this case.

The case λ odd is different since $j_{2,1}$ is trivial now (by point (2) above). However, by exactness of the sequence in (3) and since $p_{3,1}$ is surjective by point (4) of Proposition 2.5 of [9] (notice that $M_n(f_\lambda)$ corresponds to C_{f_n} in the notation of [9]), we infer that $\partial_{3,1}$ is trivial as well. Next, since when λ is odd, $\pi_0(M_2(f_\lambda i_2)) = \mathbb{Z}/2$, we can consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & \pi_0(F_2(f_\lambda i_2)) = \mathbb{Z}/2 \arrow[d, "j_{2,0}"'] \arrow[dr, "d_{3,0}"] & \\
\cdots \arrow[r, "j_{3,0}"'] & \pi_0(M_3(f_\lambda)) \arrow[r, "p_{3,0}"'] & \pi_0(M_2(f_\lambda i_2)) = \mathbb{Z}/2 \arrow[r, "\partial_{3,0}"'] \arrow[d] & \pi_{11}(BS^3) = \mathbb{Z}/15 \\
 & & 0 &
\end{tikzcd}$$

and the function $\partial_{3,0}$ can be identified with the homomorphism $d_{3,0}$ that is clearly trivial. Observing that the function $p_{3,0}$ is an homomorphism with respect to the group structure induced by composition in the case of maps of degree 1, we see that $\pi_0(M_3(f_\lambda))$ has four elements. This concludes the proof of Theorem 2.

For the general case we prove the following proposition.

Proposition 7. *The number of non-homotopic classes of maps of given degree λ in $[\mathbb{H}P^n, \mathbb{H}P^n]$, only depends on the parity of λ.*

Proof. From sequence (1), we get (for $n > 2$)

$$\pi_{4n-4}(BS^3) \xrightarrow[j_{n,0}]{} \pi_0(M_n(f_\lambda)) \xrightarrow[p_{n,0}]{} \pi_0(M_{n-1}(f_\lambda i_{n-1})) \xrightarrow[\partial_{n,0}]{} \pi_{4n-1}(BS^3),$$

where it is clear that $j_{n,0}$ and $p_{n,0}$ do not depend on the degree. We proceed by induction on n, and assume in dimension $n-1$ the number of connected components of the space $M_{n-1}(f_\lambda i_{n-1})$ only depends on the parity of λ. In order to prove that the number of the connected components of the space

$M_n(f_\lambda)$ only depends on the parity of λ, it is enough to show that the boundary homomorphism $\partial_{n,q}$ only depends on this parity for at least $q = 0$ and 1 (in fact this happens to be true for all q). Now, λ only appears in the second term $\lambda[\gamma, \zeta]$ in the given formula (2) for $\partial_{n,q}$. We can use the following facts:

(1) for the odd components of the homotopy groups of S^4, we have the Serre isomorphism [16] (p odd)

$$\pi_{q-1}(S^3; p) \oplus \pi_q(S^7; p) \to \pi_q(S^4; p)$$
$$(\alpha, \beta) \mapsto \Sigma\alpha + [\iota_4, \iota_4] \circ \beta;$$

(2) the maximum order of the elements of the $2-$component of the homotopy groups of S^4 is 4 [15].

By Lemma 5, all elements $\zeta \in \pi_{4n-4}(BS^3)$ can be written as $\zeta = j_*(y)$, for some $y = \phi(x) = \Sigma\partial(x)$ in $\pi_{4n-4}(S^4)$. Thus,

$$[\gamma, x] = [j_*(\iota_4), j_*(y)] = j_*([\iota_4, y]),$$

by naturality of the Whitehead product and since y is a suspension

$$[\iota_4, y] = [\iota_4, \Sigma z] = [\iota_4, \iota_4] \circ \Sigma^4 z,$$

with $\Sigma^4_* z \in \pi_{4n-1}(S^7)$. In the odd components, the element $[\iota_4, \iota_4] \circ \Sigma^4 z$ belongs to the kernel of j_* by the Serre isomorphisms, and hence $[\gamma, \zeta]$ is trivial. Thus, we can restrict computation to the 2-component. But in the 2-component $\lambda x = \lambda(\mathrm{mod} 2)x$, for all x, since the realizable integers λ, according to the lists given in Section 2 are divisible by 4, or the same happens for $\lambda - 1$. This proves that the unique term depending on the degree actually depends only on the parity of the degree, and thus gives the thesis. □

5. The Cayley projective plane

In this section we consider the case of the Cayley projective plane that we denote by $\mathbb{K}P^2$. We compute the realizable degrees for self maps on the Cayley projective plane and the cardinality of the set $[\mathbb{K}P^2, \mathbb{K}P^2]$ for each realizable number λ. For degree 1 we compute the group structure. One should ask if Lemma 5 generalizes to the present case, i.e. if given a homotopy class $\Sigma X \to \mathbb{K}P^2$, does there exist a map which factors through S^8? The following example proves that this is not the case. An easy computation shows that $[\Sigma X, S^8] \to [\Sigma X, \mathbb{K}P^2]$ is onto if $X = S^n$ for $n < 22$. However, $\pi_{23}(S^8) \to \pi_{23}(\mathbb{K}P^2)$ is not onto, since the first group is finite while the second is infinite. However, certainly we can say that given a homotopy class $\Sigma X \to \mathbb{K}P^2$ there exists a map which factors through S^8 if $X = S^n$ and $n < 15$.

Proposition 8. *The set R_2 of integers which are 2-realizable, i.e. the image of the function $\lambda : [\mathbb{K}P^2, \mathbb{K}P^2] \to \mathbb{Z}$ is given by the integers λ which satisfy the congruence*

$$C_2(\lambda) = 0 : \ \lambda(\lambda-1)/2 = 0(\text{mod}120),$$
$$\lambda = 0, 1, 16, 81, 96, 145, 160, 225(\text{mod}240).$$

Proof. The proof follows the line of the argument described below Proposition 6, in order to determinate the set of 2-realizable integer for $\mathbb{H}P^2$ and uses the observation above to reduce the problem to computations in the homotopy groups of S^8 and the fact that the suspension $\Sigma\sigma'$ has order 120. □

Now we will show:

Theorem 9. $[\mathbb{K}P^2, \mathbb{K}P^2] = R_{2,+} \times \{1, 2, ..., 4\} \cup R_{2,-} \times \{1, 2, ..., 8\}$ *where* $R_2 = \{n \in \mathbb{Z}, n = 0, 1, 16, 81, 96, 145, 160, 225(\text{mod}240)\}$.

Proof. We will follow the same steps as in the quaternionic case. Using the composition Theorem 4.6 of [3], we get, for any $\zeta \in \pi_9(\mathbb{K}P^2) = \mathbb{Z}/2$,

$$\psi_{u_\lambda}(\zeta) = \nu_{u_\lambda}(u_\lambda)_\natural^{-1}(\zeta) = \zeta \circ \Sigma\sigma_8 + [(u_\lambda)_*(\iota_8), \zeta] \circ \Sigma H(\sigma_8),$$

where σ_8 is the class of the Hopf map ($S^{15} \to S^8$) and $\Sigma H(\sigma_8) = 1$. Given a homotopy class $\Sigma X \to \mathbb{K}P^2$ there exists a map which factors through S^8 if $X = S^n$ and $n < 15$. So we can reduce our problem to a problem of homotopy groups of spheres and we can perform a similar calculation as the one done before Proposition 6. Namely, by naturality of the Whitehead product, linearity of the composition with suspensions (see [20] X.8.18), and the fact that the group $\pi_{16}(S^8)$ is 2-elementary, the only possible non trivial case is

$$[(u_\lambda)_*(\iota_8), j_*(\eta_8)] = \lambda(\text{mod}2) j_*([\iota_8, \eta_8]) = \lambda(\text{mod}2) j_*([\iota_8, \iota_8] \circ \eta_{15}) =$$
$$= \lambda(\text{mod}2) j_*((2\sigma_8 \pm \Sigma\sigma') \circ \eta_{15}) = \lambda(\text{mod}2) j_*(\Sigma\sigma' \circ \eta_{15}),$$

where $\Sigma\sigma' \circ \eta_{15}$ is a generator of $\pi_{16}(S^8)$ that is not in the kernel of j_* (see [19] VII.7.1), and σ' is the element of order 120 in $\pi_{14}(S^7)$. This yields

$$\psi_{u_\lambda}(j_*(\eta_8)) = (\lambda+1)(\text{mod}2) j_*(\Sigma\sigma' \circ \eta_{15}).$$

Therefore for λ odd the image is trivial and for λ even the image is isomorphic to $\mathbb{Z}/2$ and the result follows. □

Now we consider the group structure for the case $\lambda = 1$. So we show some result about the coaction which arises from the Barratt-Puppe sequence associated to some cell complexes.

In general for A and B spaces and $f : A \to B$ a continuous map we have:

Lemma 10. *The following diagram is commutative*

$$\begin{tikzcd} A \vee A \arrow[r, "f\vee f"] \arrow[d, "\nabla"'] & B \vee B \arrow[d, "\nabla"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Let L be $(n-1)$ connected, with $n > 2$. Let $K = L \sqcup_h e^m$, $S = \partial e^m = S^{m-1}$, $m > dim(L)$. From the Lemma above we obtain:

Corollary 11. *The following diagram is commutative*

$$\begin{tikzcd} K \vee K \arrow[r, "\theta\vee\theta"] \arrow[d, "\nabla"'] & K \vee \Sigma S \vee K \vee \Sigma S \arrow[d, "\nabla"] \\ K \arrow[r, "\theta"'] & K \vee \Sigma S \end{tikzcd}$$

Lemma 12. *Let L be $(n-1)$ connected, with $n > 2$. Let $K = L \sqcup_h e^m$, $S = \partial e^m = S^{m-1}$. Then, the following diagram is homotopy commutative*

$$\begin{tikzcd} K \arrow[r, "\theta"] \arrow[d, "\theta"'] & K \vee \Sigma S \arrow[d, "1\vee\nu"] \\ K \vee \Sigma S \arrow[r, "\theta\vee 1"'] & K \vee \Sigma S \vee \Sigma S \end{tikzcd}$$

where $\nu : S^m \to S^m \vee S^m$ is the coproduct and $\theta : K \to K \vee S$ the pinching map.

Proposition 13. *Let L be $(n-1)$ connected, with $n > 2$. Let $K = L \sqcup_h e^m$, $S = \partial e^m = S^{m-1}$. Let $f = 1^\alpha$, with $\alpha \in \pi_m(K)$. Then,*

$$f \circ f \sim 1^{2\alpha + \alpha \circ q_*(\alpha)},$$

where $q : K \to K/L = \Sigma S$ is the natural projection.

Proof. By definition

$$f = \nabla \circ (1 \vee a) \circ \theta,$$

where $\alpha = [a]$. Using Corollary 11, Lemma 10 and some diagram chasing, we can transform the following diagram

$$K \xrightarrow[\theta]{} K \vee \Sigma S \xrightarrow[1\vee a]{} K \vee K \xrightarrow[\nabla]{} K \xrightarrow[\theta]{} K \vee \Sigma S \xrightarrow[1\vee a]{} K \vee K \xrightarrow[\nabla]{} K,$$

into the diagram

$$K \xrightarrow[\theta]{} K \vee \Sigma S \xrightarrow[1\vee a]{} K \vee K \xrightarrow[\theta\vee\theta]{} (K \vee \Sigma S) \vee (K \vee \Sigma S) \xrightarrow[(1\vee a)\vee(1\vee a)]{}$$

$$\xrightarrow[(1\vee a)\vee(1\vee a)]{} (K \vee K) \vee (K \vee K) \xrightarrow[\nabla\vee\nabla]{} K \vee K \xrightarrow[\nabla]{} K.$$

Now consider the map $g = \theta a : S^m \to K \vee S^m$, where $S^m = \Sigma S = \Sigma S^{m-1}$. The class $[g]$ is in $\pi_m(K \vee S^m)$, and hence decomposes as

$$[g] = j_{1*}q_{1*}([g]) + j_{2*}q_{2*}([g]) + \partial\beta,$$

where $j_i : X_i \to X_1 \vee X_2$ are the inclusions and $q_i : X_1 \vee X_2 \to X_i$ the projections, $\beta \in \pi_{m+1}(K \times S^m, K \vee S^m)$, and ∂ is the boundary of the homotopy exact sequence of the pair. Projecting on the summands, $q_{1*}([g]) = [a]$ and $q_{2*}([g]) = [qa]$, where $q : K \to K/L = S^m$ is the natural projection. Now, by [20] XI.11.7, ∂ is trivial whenever $m < m + n - 1$, i.e. whenever $n > 1$. Thus,

$$g \sim (j_1 a \vee j_2 qa)\nu,$$

and by Lemma 12, we have got the thesis.

□

Remark One can show a similar formula in a more general situation. Namely given f_1, f_2 two self homotopy equivalences and $\alpha_1, \alpha_2 \in \pi_m(K)$, then one can show that

$$(f_2^{\alpha_2}) \circ (f_1^{\alpha_1}) = (f_2 \circ f_1)^{\alpha_2 + f_{2\#}(\alpha_1) + \alpha_2 \circ q \circ (\alpha_1)}.$$

Lemma 14. *Let $\mathbb{K}P^2 = S^8 \sqcup_h e^{16}$ be the Cayley projective plane. Then, $q_*(\alpha) = 0$ for all $\alpha \in \pi_{16}(\mathbb{K}P^2)$.*

Proof. The projection $q : \mathbb{K}P^2 \to \mathbb{K}P^2/S^8 = S^{16}$ induces the homomorphism $q_* : \pi_{16}(\mathbb{K}P^2) \to \pi_{16}(S^{16}) = \mathbb{Z}$, thus to prove that q_* is trivial, it is enough to prove that $|\pi_{16}(\mathbb{K}P^2)| < \infty$. For, consider the sequence of the pair

$$\longrightarrow H_{16}(S^8) = 0 \longrightarrow H_{16}(\mathbb{K}P^2) \longrightarrow H_{16}(\mathbb{K}P^2, S^8) \longrightarrow H_{15}(S^8) = 0 \longrightarrow$$

This and the relative Hurewicz isomorphism imply that

$$\pi_{16}(\mathbb{K}P^2, S^8) = H_{16}(\mathbb{K}P^2, S^8) = \mathbb{Z}.$$

Next, consider the homotopy sequence

$$\pi_{16}(S^8) = (\mathbb{Z}/2)^4 \xrightarrow[\phi]{} \pi_{16}(\mathbb{K}P^2) \xrightarrow[\psi]{}$$

$$\xrightarrow[\psi]{} \pi_{16}(\mathbb{K}P^2, S^8) = \mathbb{Z} \xrightarrow[\mu]{} \pi_{15}(S^8) = \mathbb{Z} \oplus \mathbb{Z}/120.$$

This shows that μ is multiplication by some integer k, and $\mathrm{ker}\mu = \mathbb{Z}/k = \mathrm{Im}\psi$. Hence,

$$\pi_{16}(\mathbb{K}P^2)/\mathrm{ker}\psi = \mathrm{Im}\psi,$$

shows that $|\pi_{16}(\mathbb{K}P^2)| = |\mathrm{Im}\psi||\mathrm{Im}\phi| < \infty$.

□

Corollary 15. *Given $[f_1], [f_2] \in [\mathbb{K}P^2, \mathbb{K}P^2]$ where $f_i = 1^{\alpha_i}$, the composite $f_1 \circ f_2$ satisfies $[f_1 \circ f_2] = [1^{\alpha_1 + \alpha_2}]$.*

Theorem 16. *The group of the self-homotopy equivalences of* $\mathbb{K}P^2$ *is isomorphic to* $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

Proof. From the previous corollary we have that $[f] \circ [f]$ is the class of the identity $[1]$ for every f. So the group is isomorphic to a sum of $\mathbb{Z}/2$ and the result follow. □

Remarks 1- If W is an $H-$space, as a result of Lemma 5 we have the following natural question: let P^2W be the projective space constructed from the H structure of W. Given a homotopy class $\Sigma X \to P^2W$ does the map factors through ΣW? The following example shows that the answer is negative in general, even when W is a topological group: just consider the projection $p : S^{11} \to S^{11}/Sp_1 = \mathbb{H}P^2$. This map can not factor through S^4.

2- We observe that it is known that $[\iota_n, \eta_n] = 0$ if and only if $n = 3$ mod(4) or n=2,6. This information is not enough to study the case above because the group $\pi_{16}(S^8)$ is bigger than $\mathbb{Z}_2$. This is why we need to make some extra calculation which does not appear in the case of the sphere S^4.

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