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CHARACTERIZATION OF FROBENIUS GROUPS OF SPECIAL TYPE

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Abstract. We define a *Con-Cos* group G to be one having a proper normal subgroup N whose cosets other than N itself are conjugacy classes. It follows easily that $N = G'$, the derived group of G. Most of the paper is devoted to trying to classify finite Con-Cos groups satisfying the additional requirement that N has just two conjugacy classes. We show that for such groups the center $Z(G)$ has order at most 2, and if $Z(G) = \{1\}$, then G is a Frobenius group of a rather special type.

1. Introduction

Definition 1.1. A finite group G is called a Con-Cos group if there exists a proper normal subgroup N in G such that $xN = cl(x)$ for all $x \in G \setminus N$, where $cl(x)$ is the conjugacy class of x.

Theorem 1.1. *If G is a Con-Cos group and N is the normal subgroup as in the above definition then $N = G'$, the commutator subgroup of G.*

Proof. We first show that G/N is abelian, so that $G' \subseteq N$. Let $x, y \in G$. If $x \in N$ then $xN = N$, hence yN commutes with xN. If x is not in N then $y^{-1}xy$ is not in N. In this case , we have $(y^{-1}N)(xN)(yN) = y^{-1}xyN = cl(y^{-1}xy) = cl(x) = xN$, so that xN and yN commute with each other. Conversely, let $n \in N$ and $x \in G \setminus N$. Then, $xn \in cl(x)$, so that there exists $y \in G$ such that $xn = y^{-1}xy$. Hence , $n = [x, y]$, and $n \in G'$ □

Theorem 1.2. *Let G be a Con-Cos group. Then $G' = \{[x, y] : y \in G\}$, for any $x \in G \setminus G'$. In particular, any element of G' is a commutator.*

Proof. Each element of G' is a commutator $[x, y]$ where x is arbitrarily fixed in $G \setminus G'$. □

Theorem 1.3. *If G is a non-abelian Con-Cos group then $Z(G) \subseteq G'$.*

Proof. Let $a \in Z(G)$ and $a \neq 1$. Then $cl(a) = \{a\}$. If a is not in G' then $aG' = cl(a)$, so that $\mid G' \mid = 1$, contrary to the assumption that G is not abelian. □

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2. 2-Con-Cos groups

Definition 2.1. A finite group G is called a 2-Con-Cos group if the following conditions are satisfied for a proper commutator subgroup G' of G;
i) $G'x = cl(x)$, for all x in $G \setminus G'$
ii) $G' = 1 \cup cl(a)$, for some a in G.

Note that, every nonidentity element of G' is of same order as G' consists of only two conjugacy classes. Now, if p is a prime divisor of $\mid G' \mid$ then G' has an element of order p. So, all nonidentity elements of G' are of order p. Therefore G' is a p-group and hence $G'' \neq G'$. Thus G'' is properly contained in G'. Hence, $G'' = 1$ as G' is the minimal normal subgroup of G. Hence G' is abelian. Thus G' is elementary abelian.

Now, as both G' and G/G' are abelian, it immediately follows that G is metabelian.

3. Examples

Symmetric group S_3 of degree 3 is a 2-Con-Cos group with $S_3' = \{1, (123), (123)^2\}$. Alternating group A_4 as the set of even permutations on 4 elements is a 2-Con-Cos group with $A_4' = \{1, (12)(34), (13)(24), (14)(23)\}$. Quaternion group Q of order 8 described via generators a, b with relations $a^4 = 1, b^2 = a^2, ba = a^{-1}b$, is a 2-Con-Cos group with $Q' = \{1, a^2\}$. Frobenius group $F_{42} = Z_7 \rtimes Z_6$ with kernel Z_7 and complement Z_6 ,where Z_6 acts fixed point freely on $Z_7 \setminus \{1\}$, is a 2-Con-Cos group with $G' = Z_7$. $G = Z_p \rtimes Aut(Z_p)$ is a 2-Con-Cos group with $G' = Z_p$. Frobenius group of order $p^r(p^r - 1)$ with kernel $(Z_p)^r$ (elementary abelian) and complement Z_{p^r-1} (cyclic) is a 2-Con-Cos group. However, there do exist Frobenius groups of order $p^r(p^r - 1)$ which are not 2-Con-Cos. e.g. Frobenius group $F_{702} = Z_{351} \rtimes Z_2$ with kernel Z_{351} and complement Z_2 where Z_2 acts fixed point freely on $Z_{351} \setminus \{1\}$, has order $3^3(3^3 - 1)$. But F_{702} is not a 2-Con-Cos group.

4. Some classification theorems

Theorem 4.1. *If G is a 2-Con-Cos group such that $|G'| \neq 2$ then $Z(G) = 1$.*

Proof. Let G be a 2-Con-Cos group with $|G'| > 2$. Then $G' = cl(a) \cup \{1\}$ with $|cl(a)| > 1$. So, if $z \in Z(G)$ then $z = 1$ from $Z(G) \subseteq G'$. □

Theorem 4.2. *If G is a 2-Con-Cos group then $|Z(G)| \leq 2$, and if $Z(G) = \{1\}$ then G is Frobenius with its complement abelian.*

Proof. $|Z(G)| \leq 2$ follows from theorem 4.1. Let G be a 2-Con-Cos group with $G' = \{1\} \cup cl(h)$. Suppose there exists a normal subgroup K in G such that $1 \neq K$. Now, as $Z(G) = \{1\}, 1 \neq [G, K] \subseteq G' \cap K$ so $|G' \cap K| > 1$. As

G' and K are normal subgroups of G , $G' \cap K$ is also a normal subgroup of G. But G' is also a minimal normal subgroup of G. Therefore $G' \cap K = G'$. Hence $G' \subseteq K$. Thus, G' is a unique minimal normal subgroup of G. Let $|G'| = p^m$ with p a prime and $q \neq p$ be another prime such that $q \mid |G|$. Let $Q \in Syl_q(G)$. Then $G'Q \leq G$ and $G' \subseteq G'Q$, so $G'Q \trianglelefteq G$. By Frattini argument we can write $G'N_G(Q) = G$. Set $H = N_G(Q)$, then $G' \cap H \trianglelefteq H$. Also $G' \cap H \trianglelefteq G'$ as G' is abelian, and hence $G' \cap H \trianglelefteq G'H = G$. By unique minimality of G' we have $G' \cap H = \{1\}$ or $G' \cap H = G'$. If $G' \cap H = G'$ then $G' \subseteq H$ which implies $G = G'H = H$, hence $Q \trianglelefteq G$. But G' is the unique minimal normal subgroup of G, so $G' \subseteq Q$. Hence $p \mid q$, which is a contradiction. Therefore $G' \cap H = \{1\}$, so H is abelian.

Let $x \in H, x \neq 1$. Then for any $h \in H, hC_{G'}(x)h^{-1} = C_{G'}(x)$ as H is abelian. But $C_{G'}(x) \trianglelefteq G'$ hence $C_{G'}(x) \trianglelefteq G'H = G$. Therefore $C_{G'}(x) = G'$ or $C_{G'}(x) = \{1\}$. If $C_{G'}(x) = G'$ then x commutes with every element of G'. As H is abelian $x \in Z(G'H) = Z(G) = \{1\}$ which is a contradiction. Hence $C_{G'}(x) = \{1\}$. This shows that G is a Frobenius group with kernel G' and complement H. □

Corollary 4.1. *If G is a 2-Con-Cos group and G is a Frobenius group with kernel G' and complement H, then H is cyclic.*

Proof. Since G is 2-Con-Cos, G' is elementary abelian. So, H is a group of fixed point free automorphisms of an abelian group, hence by [5] theorem IX.4.f. H is cyclic. □

Theorem 4.3. *Let G be a 2-Con-Cos group with $Z(G) = \{1\}$. Then G is a Frobenius group of the kind $(Z_p)^r \rtimes Z_{p^r-1}$ for some prime p and some $r \geq 1$. Conversely, Frobenius groups of such kind with kernel $(Z_p)^r$ and complement Z_{p^r-1} are 2-Con-Cos groups and have trivial center.*

Proof. G is Frobenius with elementary abelian kernel G'. Complement H is cyclic. Hence $G' = (Z_p)^r$ for some prime p and $H = Z_h$ for some $h \geq 2$. Since $G' \setminus \{1\}$ is a single conjugacy class the action of H on G' by conjugation has to be transitive and fixed point free, so that $|H| = p^r - 1$. On the other hand, for any Frobenius group $G = (Z_p)^r \rtimes Z_{p^r-1}$ with kernel $(Z_p)^r$ and complement Z_{p^r-1} , G' is the kernel, $Z(G) = \{1\}$ and action of the complement on $(Z_p)^r \setminus \{1\}$ is regular. Hence G is 2-Con-Cos. □

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References

[1] Mackdonald Ian. D. "Theory of groups" Oxford University Press, 1968.
[2] Muktibodh A.S., "Con-Cos groups", Review Bulletin of Calcutta Mathematical Society. Vol.11 , 53–56, 2003.
[3] Muktibodh A.S., "Generalized Con-Cos groups", Advances in Algebra,Proceedings of the ICM Sattelite Conference in Algebra and related topics. 434–441, 2003.
[4] Passman D., "Permutation groups", W.A.Benjamin Inc, New York Amsterdam, 1968.
[5] Schenkman Eugene, "Group Theory" East West Edition, 1971.

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