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STABLE SELF-HOMOTOPY GROUPS OF COMPLEX PROJECTIVE PLANE

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Abstract. We compute the stable homotopy group $G_k(\mathbb{C}\mathrm{P}^2) = \{\Sigma^k \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2\}$ of the complex projective plane $\mathbb{C}\mathrm{P}^2$ for $k \leq 20$ by using the exact sequence associated with canonical cofiber sequence and Toda bracket.

1. Introduction

Throughout this paper, all spaces, maps and homotopies are assumed to be based, and we will not distinguish the map and its homotopy class. Moreover, we use freely the notations of homotopy groups of spheres given in the book of Toda [10]. For connected spaces A and B, we denote by $[A, B]$ the set consisting of all based homotopy classes of based maps from A to B, and by $\{A, B\}$ the stable homotopy groups $\lim_{n \longrightarrow \infty} [\Sigma^n A, \Sigma^n B]$. We would like to determine the stable self-homotopy groups $G_k(X) = \{\Sigma^k X, X\}$ for the case $X = \mathbb{C}\mathrm{P}^2$, where $\mathbb{C}\mathrm{P}^2$ is the complex projective plane. In general, $G_*(X) = \{G_k(X)\}$ is a graded ring by composition. We want to know the ring structure of $G_*(X)$. There is the necessity of knowing the group structure of $G_k(X)$ first for that purpose. If we denote by $\{\Sigma^k \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2; p\}$ the p-primary components of $\{\Sigma^k \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2\}$ and $p \geq 3$ is an odd prime, there is an isomorphism $\{\Sigma^k \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2; p\} \cong (G_{k-2}; p) \oplus (G_k; p) \oplus (G_k; p) \oplus (G_{k+2}; p)$, where G_k (resp. $(G_k; p)$) denotes the k-th (resp. p-primary) stable homotopy group of spheres. So in this paper we shall study the stable group $G_k(\mathbb{C}\mathrm{P}^2) = \{\Sigma^k \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2; 2\}$. More precisely, the main result of this paper is to determine the group $G_k(\mathbb{C}\mathrm{P}^2)$ for $k \leq 20$ and it is stated as follows.

Theorem 1.1. *The group $G_k(\mathbb{C}\mathrm{P}^2), k \leq 20$, are isomorphic to the corresponding groups in the following table:*

k	$G_k(\mathbb{C}\mathrm{P}^2)$	*generators*
-2	$\mathbb{Z}$	$i\pi$
-1	0	
0	$\mathbb{Z} \oplus \mathbb{Z}$	$1_{\mathbb{C}\mathrm{P}^2}$, $(\eta, 2\iota)\pi$
1	$\mathbb{Z}_2$	$i\nu\pi$
2	$\mathbb{Z}$	$(\eta, 2\iota)[2\iota, \eta]$

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3	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$(\eta,\nu)\pi,\ i[\nu,\eta]-(\eta,\nu)\pi$
4	0	
5	$\mathbb{Z}_{16} \oplus \mathbb{Z}_4$	$i\sigma\pi,\ (\eta,\nu)[2\iota,\eta]-2i\sigma\pi$
6	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$(\eta,\nu)\nu\pi,\ i\nu[\nu,\eta]-(\eta,\nu)\nu\pi$
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_{16}$	$(\eta,2\iota)\sigma\pi,\ [i\sigma,\eta]$
8	$\mathbb{Z}_4$	$(\eta,\nu)[\nu,\eta]$
9	$\mathbb{Z}_{32} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$[(\eta,2\iota)\sigma,\eta],\ (\eta,\eta\varepsilon)\pi-4[(\eta,2\iota)\sigma,\eta],$ $(\eta,\nu)\nu^2\pi$
10	0	
11	$\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$	$[(\eta,\eta\varepsilon),\eta],\ (\eta,\zeta)\pi,\ (\eta,\nu)\nu[\nu,\eta]$
12	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$i\sigma^2\pi,\ i\kappa\pi$
13	$\mathbb{Z}_{64} \oplus \mathbb{Z}_4$	$[(\eta,\zeta),\eta],\ (\eta,\zeta)[2\iota,\eta]$
14	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(\eta,\sigma^2)\pi,\ i[\sigma^2,\eta]-(\eta,\sigma^2)\pi,\ [i\kappa,\eta]$
15	$\mathbb{Z}_{32} \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}_4$	$(\eta,2\iota)\rho\pi,\ [i\rho,\eta],\ (\eta,\eta\kappa)\pi$
16	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$[(\eta,\sigma^2),\eta],\ i\nu^*\pi-2[(\eta,\sigma^2),\eta]$
17	$\mathbb{Z}_{32} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$[(\eta,2\iota)\rho,\eta],\ (\eta,\eta^2\rho)\pi,\ [(\eta,\eta\kappa),\eta],$ $(\eta,\nu)\kappa\pi,\ i\overline{\sigma}\pi$
18	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$	$(\eta,\nu^*)\pi,\ i\overline{\kappa}\pi,\ i[\nu^*,\eta]-(\eta,\nu^*)\pi$
19	$\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(\eta,\overline{\zeta})\pi,\ [(\eta,\eta^2\rho),\eta],\ (\eta,\overline{\sigma})\pi,\ i[\overline{\sigma},\eta]$
20	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$	$[i\overline{\kappa},\eta],\ (\eta,\nu^*)[2\iota,\eta],\ (\eta,2\iota)\overline{\kappa}\pi$

It was obtained by [1] when $k \leq 1$ in this table. Furtheremore, [2] showed the case of $k = 2$.

An element $\alpha \in G_k(X)$ is called nilpotent, if $\alpha^n = 0$ for some integer n. By Nishida [8], any element of positive stem of the stable homotopy groups of spheres is nilpotent. Here, I consider the next conjecture.

Conjecture 1. *If $k \neq 0$, then any element of $G_k(\mathbb{C}\mathrm{P}^2)$ is nilpotent.*

Moreover, above table suggests the next relations.

Conjecture 2. *If $\alpha^n = 0$ for $\alpha \in G_k$, then*
(1) $((\eta,\alpha)\pi)^n = 0$,
(2) $[i\alpha,\eta]^n = 0$,
(3) $[(\eta,\alpha),\eta]^n = 0$.

2. Preliminaries

For elements $\alpha \in \{Y, Z\}, \beta \in \{X, Y\}, \gamma \in \{W, X\}$ with the conditions $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$, we denote by $\overline{\alpha} \in \{C_\beta, Z\}$ and $\widetilde{\gamma} \in \{\Sigma W, C_\beta\}$ the extension of α and coextension of γ satisfying the conditions $\overline{\alpha} \circ i = \alpha$ and $\pi \circ \widetilde{\gamma} = \Sigma\gamma$, where C_β denotes the mapping cone $C_\beta = Y \cup_\beta CX$ and the diagrams $X \xrightarrow{\beta} Y \xrightarrow{i} C_\beta \xrightarrow{\pi} \Sigma X$ is a cofiber sequence. We denote by

$\mathrm{Ext}(\alpha,\beta)$ (resp. $\mathrm{Coext}(\beta,\gamma)$) the set of all homotopy classes of extensions $\overline{\alpha}$ (resp. the set of all homotopy classes of coextensions $\widetilde{\gamma}$), and the Toda bracket $\langle\alpha,\beta,\gamma\rangle$ is defined by $\langle\alpha,\beta,\gamma\rangle = \mathrm{Ext}(\alpha,\beta)\circ\mathrm{Coext}(\beta,\gamma)$.

Denote by $G_k = \lim_{n\longrightarrow\infty}\pi_{n+k}(S^n)$ the k-th stable homotopy group of the sphere. We investigate stable Toda brackets $\langle\eta,\alpha,\beta\rangle$ for $\eta\in G_1$, $\alpha\in G_k$ and $\beta\in G_h$ with the conditios $\eta\circ\alpha=0, \alpha\circ\beta=0$.

Lemma 2.1.

(1) $0\in\langle\eta,\nu^*,8\iota\rangle \mod 8G_{20}$
(2) $0\in\langle\eta,\overline{\zeta},8\iota\rangle \mod \{\eta\overline{\kappa}\}$

Proof. (1) By use of the relation $4\nu^*=\eta^2\eta^*$ of Theorem 14.1 of [10],

$$\langle\nu^*,8\iota,\eta\rangle\supset\langle 4\nu^*,2\iota,\eta\rangle=\langle\eta^2\eta^*,2\iota,\eta\rangle\supset\eta\eta^*\langle\eta,2\iota,\eta\rangle\ni 2\eta\eta^*\nu=0.$$

Since $\langle\nu^*,8\iota,\eta\rangle$ is a coset of trivial subgroup, $\langle\nu^*,8\iota,\eta\rangle=0$. By Theorem 2.1 of [7], $\langle 8\iota,\eta,\nu^*\rangle\supset 4\langle 2\iota,\eta,\nu^*\rangle\ni 0$. Therefore, we have $0\in\langle\nu^*,8\iota,\eta\rangle$, by (3.9) of [10].

(2) By (3.7) of [10],

$$0\in\langle\langle\eta,\zeta,8\iota\rangle,2\sigma,8\iota\rangle-\langle\eta,\langle\zeta,8\iota,2\sigma\rangle,8\iota\rangle+\langle\eta,\zeta,\langle 8\iota,2\sigma,8\iota\rangle\rangle.$$

Since $0\in\langle\eta,\zeta,8\iota\rangle$ and $0\in\langle 8\iota,2\sigma,8\iota\rangle$,

$$0\in\langle\langle\eta,\zeta,8\iota\rangle,2\sigma,8\iota\rangle \mod 8G_{20},$$
$$0\in\langle\eta,\zeta,\langle 8\iota,2\sigma,8\iota\rangle\rangle \mod \eta(G_{19};2)=0.$$

Therefore, we have $0\in\langle\eta,\langle\zeta,8\iota,2\sigma\rangle,8\iota\rangle=\langle\eta,\overline{\zeta},8\iota\rangle$. □

Lemma 2.2.

(1) $0=\langle\nu,\eta,\eta\varepsilon\rangle$
(2) $\sigma^2=\langle\nu,\sigma,\nu\rangle$

Proof. Since $\nu G_{11}+\eta\varepsilon G_5=0$, $\nu G_{11}=0$, these Toda brackets are cosets of trivial subgroup. Therefore, $\langle\nu,\eta,\eta\varepsilon\rangle$ and $\langle\nu,\sigma,\nu\rangle$ consist of a single element. We consider composites of these Toda brackets and η. Then we have

$$\eta\langle\nu,\eta,\eta\varepsilon\rangle=\langle\eta,\nu,\eta\rangle\eta\varepsilon=\nu^2\eta\varepsilon=0 \quad\text{and}\quad \eta\langle\nu,\sigma,\nu\rangle=\langle\eta,\nu,\sigma\rangle\nu=0.$$

Since $\langle\nu,\eta,\eta\varepsilon\rangle,\langle\nu,\sigma,\nu\rangle$ are subsets of the same group $(G_{14};2)=\{\sigma^2\}\oplus\{\kappa\}\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$ and $\eta\sigma^2=0,\eta\kappa\neq 0$, these Toda brackets are equal to 0 or σ^2. Next we consider the composites of these Toda brackets and σ. Then we have

$$\sigma\langle\nu,\eta,\eta\varepsilon\rangle=\langle\sigma,\nu,\eta\rangle\eta\varepsilon=0 \quad\text{and}\quad \sigma\langle\nu,\sigma,\nu\rangle=\langle\sigma,\nu,\sigma\rangle\nu=\nu^*\nu=\sigma^3.$$

Since the element σ^3 is not zero in the group $(G_{21};2)$, we have $\langle\nu,\eta,\eta\varepsilon\rangle=0$ and $\langle\nu,\sigma,\nu\rangle=\sigma^2$. □

Lemma 2.3.

$$0 \in \langle \eta, \nu, \zeta \rangle = \langle \eta, \zeta, \nu \rangle \mod \eta(G_{15}; 2) = \{\eta\rho\}$$

Proof. We apply (3.7) of [10] to the case $(\alpha, \beta, \gamma, \delta, \varepsilon) = (\eta, \nu, \sigma, 16\iota, \nu)$. By (9.3) of [10], $x\zeta \in \langle \nu, \sigma 16\iota \rangle$ (x:odd). And by (9.1) of [10], $\zeta = \langle \sigma, 16\iota, \nu \rangle$. Therefore, we have

$$\langle \eta, \zeta, \nu \rangle = \langle \eta, \nu, \zeta \rangle.$$

Next, we apply (3.7) of [10] to the case $(\alpha, \beta, \gamma, \delta, \varepsilon) = (\eta, \nu, \eta, \eta\varepsilon, 2\iota)$. Since $\nu^2 = \langle \eta, \nu, \eta \rangle$, we have $\langle \langle \eta, \nu, \eta \rangle, \eta\varepsilon, 2\iota \rangle = \langle \nu^2, \eta\varepsilon, 2\iota \rangle \supset \langle \eta\nu^2, \varepsilon, 2\iota \rangle \ni 0$. Since $\langle \nu^2, \eta\varepsilon, 2\iota \rangle$ is a coset of trivial subgroup, we have $0 = \langle \langle \eta, \nu, \eta \rangle, \eta\varepsilon, 2\iota \rangle$.

By (1) of Lemma 2.2, we have $0 \in \langle \eta, \langle \nu, \eta, \eta\varepsilon \rangle, 2\iota \rangle \mod \eta(G_{15}; 2) + 2G_{16} = \{\eta\rho\}$. And by Lemma 9.1 of [10], $\langle \eta, \nu, \langle \eta, \eta\varepsilon, 2\iota \rangle \rangle = \langle \eta, \nu, \zeta \rangle$. Since this Toda bracket is a coset of subgroup $\eta(G_{15}; 2) + \zeta(G_5; 2) = \{\eta\rho\}$, we have $0 \in \langle \eta, \nu, \zeta \rangle$. □

Lemma 2.4.

(1) $2\zeta \in \langle \eta, 2\iota, \mu \rangle \mod \eta(G_{10}; 2) = \{4\zeta\}$
(2) $\eta\kappa = \langle \nu, \varepsilon, \nu \rangle$
(3) $2\bar{\zeta} \in \langle \eta, 2\iota, \bar{\mu} \rangle \mod \eta(G_{18}; 2) = \{4\bar{\zeta}\}$
(4) $0 \in \langle \eta, 8\iota, \bar{\kappa} \rangle \mod \eta(G_{21}; 2) = \{\eta^2\bar{\kappa}\}$

Proof. (1) The Toda bracket $\langle \eta, 2\iota, \mu \rangle$ is a coset of the subgroup $\{4\zeta\}$. Since $2\langle \eta, 2\iota, \mu \rangle = \langle 2\iota, \eta, 2\iota \rangle \mu = \eta^2\mu = 4\zeta$, we have

$$2\zeta \in \langle \eta, 2\iota, \mu \rangle.$$

(2) By (3.10) of [10], $\langle \nu, \varepsilon, \nu \rangle$ and $\langle \varepsilon, \nu, 2\nu \rangle$ have a common element. Since these Toda brackets are the coset of trivial subgroup, they consist of a single element. By p.189 of [10], $\eta\kappa = \langle \varepsilon, 2\iota, \nu^2 \rangle$. $\langle \varepsilon, 2\iota, \nu^2 \rangle \subset \langle \varepsilon, 2\nu, \nu \rangle \supset \langle \varepsilon, \nu, 2\nu \rangle$ with the indeterminacy $\varepsilon(G_7; 2) + \nu(G_{12}; 2) = 0$, and hence we have

$$\eta\kappa = \langle \varepsilon, \nu, 2\nu \rangle = \langle \nu, \varepsilon, \nu \rangle.$$

(3) The Toda bracket $\langle \eta, 2\iota, \bar{\mu} \rangle$ is a coset of the subgroup $\eta(G_{18}; 2) = \{4\bar{\zeta}\}$. Since $2\langle \eta, 2\iota, \bar{\mu} \rangle = \langle 2\iota, \eta, 2\iota \rangle \bar{\mu} = \eta^2\bar{\mu} = 4\bar{\zeta}$, we have $2\bar{\zeta} + \ell\bar{\sigma} \in \langle \eta, 2\iota, \bar{\mu} \rangle \mod \{4\bar{\zeta}\}$ for some integer ℓ. Moreover, $\nu\langle \eta, 2\iota, \bar{\mu} \rangle = \langle \nu, \eta, 2\iota \rangle \bar{\mu} = 0$ and $\nu\bar{\sigma}$ is non-trivial in G_{22}. Therefore, ℓ must be 0.

(4) $\langle \eta, 8\iota, \bar{\kappa} \rangle$ is a coset of the subgroup $\eta(G_{21}; 2) + \bar{\kappa}(G_2; 2) = \{\eta^2\bar{\kappa}\}$. By [10] p.189, $\varepsilon \in \langle \eta, 2\iota, \nu^2 \rangle$. Then we have

$$\langle \eta, 8\iota, \bar{\kappa} \rangle \supset \langle \eta, 2\iota, 4\bar{\kappa} \rangle = \langle \eta, 2\iota, \nu^2\kappa \rangle \supset \langle \eta, 2\iota, \nu^2 \rangle \kappa \ni \varepsilon\kappa = \eta^2\bar{\kappa}.$$

Therefore, $0 \in \langle \eta, 8\iota, \bar{\kappa} \rangle \mod \{\eta^2\bar{\kappa}\}$. □

3. The homotopy groups $\pi_k^s(\mathbb{C}\mathrm{P}^2)$

Since $\mathbb{C}\mathrm{P}^2 = S^2 \cup_{\eta_2} e^4$, we have the cofiber sequence

$$S^3 \xrightarrow{\eta_2} S^2 \xrightarrow{i} \mathbb{C}\mathrm{P}^2 \xrightarrow{\pi} S^4,$$

this also induces the short exact sequence

$$0 \longrightarrow G_{k-2}/\eta \circ G_{k-3} \xrightarrow{i_*} \pi_k^s(\mathbb{C}\mathrm{P}^2) \xrightarrow{\pi_*} G_{k-4} \cap \ker \eta_* \longrightarrow 0,$$

where $\pi_k^s(\mathbb{C}\mathrm{P}^2) = \{S^k, \mathbb{C}\mathrm{P}^2\}$.

Lemma 3.1. *The* 2*-primary components* $\pi_k^s(\mathbb{C}\mathrm{P}^2;2)$ *of the homotopy groups* $\pi_k^s(\mathbb{C}\mathrm{P}^2)$ *and their generators are listed in the following table;*

k	$\pi_k^s(\mathbb{C}\mathrm{P}^2;2)$	*generators*
2	$\mathbb{Z}$	i
3	0	
4	$\mathbb{Z}$	$(\eta, 2\iota)$
5	$\mathbb{Z}_4$	$i\nu$
6	0	
7	$\mathbb{Z}_8$	(η, ν)
8	$\mathbb{Z}_2$	$i\nu^2$
9	$\mathbb{Z}_{16}$	$i\sigma$
10	$\mathbb{Z}_4$	$(\eta, \nu)\nu$
11	$\mathbb{Z}_{16}$	$(\eta, 2\iota)\sigma$
12	0	
13	$\mathbb{Z}_2 \oplus \mathbb{Z}_8$	$(\eta, \nu)\nu^2, (\eta, \eta\varepsilon)$
14	0	
15	$\mathbb{Z}_8$	(η, ζ)
16	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$i\sigma^2, i\kappa$
17	$\mathbb{Z}_{32}$	$i\rho$
18	$\mathbb{Z}_4$	(η, σ^2)
19	$\mathbb{Z}_{32} \oplus \mathbb{Z}_4$	$(\eta, 2\iota)\rho, (\eta, \eta\kappa)$
20	$\mathbb{Z}_4$	$i\nu^*$
21	$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(\eta, \eta^2\rho), (\eta, \nu)\kappa, i\bar{\sigma}$
22	$\mathbb{Z}_8 \oplus \mathbb{Z}_8$	$(\eta, \nu^*), i\bar{\kappa}$
23	$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(\eta, \bar{\zeta}), (\eta, \bar{\sigma}), i\sigma^3$
24	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$(\eta, 2\iota)\bar{\kappa}, i\nu\bar{\sigma}$

Here, (η, α) *is an element of* $\mathrm{Coext}(\eta, \alpha)$*. Moreover, these elements satisfy the next relations.*

Relations; $\quad i\varepsilon = 2(\eta, \nu)\nu, \ i\mu = 8(\eta, 2\iota)\sigma, \ i\zeta = 2(\eta, \eta\varepsilon), \ i\eta^* = 2(\eta, \sigma^2),$

$$i\nu\kappa = 2(\eta, \eta\kappa), \ i\bar{\mu} = 16(\eta, 2\iota)\rho, \ i\bar{\zeta} = 2(\eta, \eta^2\rho).$$

Proof. For $k = 3, 4, 6, 7, 12$ and 14, $\pi_* : \pi_k^s(\mathbb{C}\mathrm{P}^2) \longrightarrow G_{k-4} \cap \ker \eta_*$ is an isomorphism. For $k = 2, 5, 8, 9, 16, 17$ and 20, $i_* : G_{k-2}/\eta \circ G_{k-3} \longrightarrow \pi_k^s(\mathbb{C}\mathrm{P}^2)$ is an isomorphism. Now we compute $\pi_{22}^s(\mathbb{C}\mathrm{P}^2)$. Other cases can be computed by trying the similar argument.

Consider the short exact sequence

$$0 \longrightarrow G_{20}/\eta \circ G_{19} \xrightarrow{i_*} \pi_{22}^s(\mathbb{C}\mathrm{P}^2) \xrightarrow{\pi_*} G_{18} \cap \ker \eta_* \longrightarrow 0,$$

where $G_{20}/\eta \circ G_{19} = \{\bar{\kappa}\} \cong \mathbb{Z}_8$ and $G_{18} \cap \ker \eta_* = \{\nu^*\} \cong \mathbb{Z}_8$. By Lemma 2.1 (1), $0 \in \langle \eta, \nu^*, 8\iota \rangle$. Hence we have $0 \in i_*\langle \eta, \nu^*, 8\iota \rangle = 8\,\mathrm{Coext}(\eta, \nu^*)$, by Proposition 1.8 of [10]. Therefore, we can choose an element (η, ν^*) of $\mathrm{Coext}(\eta, \nu^*)$ whose order is 8. This implies that the above short exact sequence splits. Therefore we have $\pi_{22}^s(\mathbb{C}\mathrm{P}^2) = \{(\eta, \nu^*)\} \oplus \{i\bar{\kappa}\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_8$. □

Moreover, we have following relations, by Lemma 2.3 and Theorem 2.1 of [7].

$$\textit{Relations};\quad \begin{aligned} 0 &= (\eta,\nu)\zeta = (\eta,\zeta)\nu \\ i\eta^* &= (\eta, 2\iota)\sigma^2 \\ 2i\nu^* &= (\eta, 2\iota)\eta^*. \end{aligned}$$

By Spanier-Whitehead duality, we have an isomorphism

$$\pi_k^s(\mathbb{C}\mathrm{P}^2) \cong \{\Sigma^{k-6}\mathbb{C}\mathrm{P}^2, S^0\}.$$

Lemma 3.2. *The* 2*-primary components* $\{\Sigma^k C_\eta, S^0; 2\}$ *of the homotopy groups* $\{\Sigma^k C_\eta, S^0\}$ *and its generator are listed in the following table;*

k	-4	-3	-2	-1	0	1	2	3	4	5
$\{\Sigma^k C_\eta, S^0; 2\}$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	$\mathbb{Z}_4$	$\mathbb{Z}_{16}$
generators	π		$[2\iota,\eta]$	$\nu\pi$		$[\nu,\eta]$	$\nu^2\pi$	$\sigma\pi$	$\nu[\nu,\eta]$	$\sigma[2\iota,\eta]$

6	7	8	9	10	11	12	13	14
0	$\mathbb{Z}_2\oplus\mathbb{Z}_8$	0	$\mathbb{Z}_8$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_{32}$	$\mathbb{Z}_4$	$\mathbb{Z}_{32}\oplus\mathbb{Z}_4$	$\mathbb{Z}_4$
	$\nu^2[\nu,\eta], [\eta\varepsilon,\eta]$		$[\zeta,\eta]$	$\sigma^2\pi, \kappa\pi$	$\rho\pi$	$[\sigma^2,\eta]$	$\rho[2\iota,\eta], [\eta\kappa,\eta]$	$\nu^*\pi$

15	16	17	18
$\mathbb{Z}_8\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_8$	$\mathbb{Z}_8\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_4\oplus\mathbb{Z}_2$
$[\eta^2\rho,\eta], \kappa[\nu,\eta], \bar{\sigma}\pi$	$[\nu^*,\eta], \bar{\kappa}\pi$	$[\bar{\zeta},\eta], [\bar{\sigma},\eta], \sigma^3\pi$	$\bar{\kappa}[2\iota,\eta], \nu\bar{\sigma}\pi$

where $[\alpha,\eta]$ *is an element of* $\mathrm{Ext}(\alpha,\eta)$*, and these elements satisfy the following relations.*

$$\begin{aligned} &\varepsilon\pi = 2\nu[\nu,\eta],\ \mu\pi = 8\sigma[2\iota,\eta],\ \zeta\pi = 2[\eta\varepsilon,\eta],\ \eta^*\pi = 2[\sigma^2,\eta], \\ &\nu\kappa\pi = 2[\eta\kappa,\eta],\ \bar{\mu}\pi = 16\rho[2\iota,\eta],\ \bar{\zeta}\pi = 2[\eta^2\rho,\eta], \\ &0 = \zeta[\nu,\eta] = \nu[\zeta,\eta],\ \eta^*\pi = \sigma^2[2\iota,\eta],\ 2\nu^*\pi = \eta^*[2\iota,\eta]. \end{aligned}$$

4. The homotopy groups $\{\Sigma^k\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2\}$

By considering the following exact sequence, we will calculate $G_k(\mathbb{C}\mathrm{P}^2)$.

$$0 \to \pi_{k+4}^s(\mathbb{C}\mathrm{P}^2)/\pi_{k+3}^s(\mathbb{C}\mathrm{P}^2)\circ\eta \xrightarrow{\pi^*} G_k(\mathbb{C}\mathrm{P}^2) \xrightarrow{i^*} \pi_{k+2}^s(\mathbb{C}\mathrm{P}^2)\cap\ker\eta^* \to 0$$

If we have $2m\,\mathrm{Ext}(\alpha,\eta)$ for any $\alpha\in\pi_{k+2}^s(\mathbb{C}\mathrm{P}^2)\cap\eta^*$ and its order $2m$, then we can determine $G_k(\mathbb{C}\mathrm{P}^2)$ by similar argument to the proof of Lemma 3.1. So the rest of this paper is devoted to the determination of $2m\,\mathrm{Ext}(\alpha,\eta)$.

Lemma 4.1.

(1) $4i[\nu,\eta] = 4(\eta,\nu)\pi$
(2) $4i[\zeta,\eta] = 4(\eta,\zeta)\pi$
(3) $4i[\nu^*,\eta] = 4(\eta,\nu^*)\pi$

(4) $4i[\overline{\zeta}, \eta] = 4(\eta, \overline{\zeta})\pi$

Proof. (3) For the exact sequence

$$\pi^s_{22}(\mathbb{C}\mathrm{P}^2) \xrightarrow{\pi^*} G_{18}(\mathbb{C}\mathrm{P}^2) \xrightarrow{i^*} \pi^s_{20}(\mathbb{C}\mathrm{P}^2),$$

we have $i^*(4i[\nu^*, \eta]) = 0$. Then there is an element of $\alpha \in \pi^s_{22}(\mathbb{C}\mathrm{P}^2)$, such that $\alpha\pi = 4i[\nu^*, \eta]$. By Theorem 2.1 of [7],

$$2\alpha = \alpha\pi(2\iota, \eta) = 4i\langle \nu^*, \eta, 2\iota\rangle = 0.$$

Therefore, we have $\alpha \in 4\pi^s_{22}(\mathbb{C}\mathrm{P}^2)$ and $\alpha\pi = 4(\eta, \nu^*)\pi$.

(4) For the exact sequence

$$\pi^s_{23}(\mathbb{C}\mathrm{P}^2) \xrightarrow{\pi^*} G_{19}(\mathbb{C}\mathrm{P}^2) \xrightarrow{i^*} \pi^s_{21}(\mathbb{C}\mathrm{P}^2),$$

we have $i^*(4i[\overline{\zeta}, \eta]) = 0$. Therefore, $4i[\overline{\zeta}, \eta] = x(\eta, \overline{\zeta})\pi + y(\eta, \overline{\sigma})\pi$ for some integer x and y. Since

$$0 = \pi_*(4i[\overline{\zeta}, \eta]) = x\overline{\zeta}\pi + y\overline{\sigma}\pi,$$

x and y must be 4 and 2, respectively.

(1) and (2) are proved similarly. □

The stable class of identity map of $\mathbb{C}\mathrm{P}^2$ can be considered as an extension of i. Similarly, it is considered also as an coextension of π. Therefore, we obtain the next Lemma.

Lemma 4.2.

(1) *If* $\alpha \circ \eta = 0$ *for* $\alpha \in \pi^s_k(\mathbb{C}\mathrm{P}^2)$*, then* $\mathrm{Ext}(\alpha, \eta) \subset \langle \alpha, \eta, \pi\rangle$.

(2) *If* $\eta \circ \alpha = 0$ *for* $\alpha \in \{\Sigma^k\mathbb{C}\mathrm{P}^2, S^0\}$*, then* $\mathrm{Coext}(\eta, \alpha) \subset \langle i, \eta, \alpha\rangle$.

Lemma 4.3. *If* $i \circ G_{k+2} \subset 2\pi^s_{k+4}(\mathbb{C}\mathrm{P}^2)$*, then*

$$\langle i, \eta, 2\alpha\rangle \subset \langle i\alpha, \eta, 2\iota\rangle \mod 2\pi_{k+4}(\mathbb{C}\mathrm{P}^2)$$

for any $\alpha \in G_k$*.*

Proof. Assumption of Lemma shows that $\langle i\alpha, \eta, 2\iota\rangle$ and $\langle i, \eta\alpha, 2\iota\rangle$ are cosets of the same subgroup $2\pi^s_{k+4}(\mathbb{C}\mathrm{P}^2)$. Therefore we have

$$\langle i\alpha, \eta, 2\iota\rangle = \langle i, \eta\alpha, 2\iota\rangle \supset \langle i, \eta, 2\alpha\rangle.$$

□

We can apply this Lemma to the case $\alpha = \sigma, \kappa$ and ρ.

Corollary 4.4.

(1) $(\eta, 2\iota)\sigma \in \langle i\sigma, \eta, 2\iota\rangle \mod 2\pi^s_{11}(\mathbb{C}\mathrm{P}^2)$

(2) $0 \in \langle i\kappa, \eta, 2\iota\rangle \mod 2\pi^s_{18}(\mathbb{C}\mathrm{P}^2)$

(3) $(\eta, 2\iota)\rho \in \langle i\rho, \eta, 2\iota\rangle \mod 2\pi^s_{19}(\mathbb{C}\mathrm{P}^2)$

Proposition 4.5.

(1) $(\eta, \zeta) \in \langle(\eta, \eta\varepsilon), \eta, 2\iota\rangle \mod 2\pi_{15}^s(\mathbb{C}\mathrm{P}^2)$
(2) $i\rho \in \langle(\eta, \zeta), \eta, 2\iota\rangle \mod 2\pi_{17}^s(\mathbb{C}\mathrm{P}^2)$
(3) $0 \in \langle(\eta, \sigma^2), \eta, 2\iota\rangle \mod 2\pi_{20}^s(\mathbb{C}\mathrm{P}^2)$
(4) $0 \in 2\langle(\eta, \eta\kappa), \eta, 2\iota\rangle \mod 4\pi_{21}^s(\mathbb{C}\mathrm{P}^2)$
(5) $(\eta, \overline{\zeta}) + \ell i\sigma^3 \in \langle(\eta, \eta^2\rho), \eta, 2\iota\rangle \mod 2\pi_{23}^s(\mathbb{C}\mathrm{P}^2)$ *(ℓ=0 or 1)*
(6) $0 \in \langle(\eta, \nu^*), 8\iota, \eta\rangle \mod \{i\nu\overline{\sigma}\}$

Proof. (1) We apply (3.7) of [10] to the case $(\alpha, \beta, \gamma, \delta, \varepsilon) = (i, \eta, \eta\varepsilon, \eta, 2\iota)$, with the facts : $(\eta, \eta\varepsilon) \in \langle i, \eta, \eta\varepsilon\rangle, \zeta \in \langle\eta\varepsilon, \eta, 2\iota\rangle \mod 2G_{11}$ by Lemma 9.1 of [10]. Thus, the result can be obtained.

(2) We apply (3.7) of [10] to the case $(\alpha, \beta, \gamma, \delta, \varepsilon) = ((\eta, \eta\varepsilon), \eta, 2\iota, \eta, 8\iota)$.

$$\langle\langle(\eta, \eta\varepsilon), \eta, 2\iota\rangle, \eta, 8\iota\rangle = \langle(\eta, \zeta), \eta, 8\iota\rangle$$

is valid by using the result of (1) of this Proposition. Moreover, we have

$$\begin{aligned}\langle(\eta, \eta\varepsilon), \langle\eta, 2\iota, \eta\rangle, 8\iota\rangle &= \langle(\eta, \eta\varepsilon), 2\nu, 8\iota\rangle \supset \langle 2(\eta, \eta\varepsilon), \nu, 8\iota\rangle \\ &= \langle i\zeta, \nu, 8\iota\rangle \supset i\langle\zeta, \nu, 8\iota\rangle \ni 4i\rho,\end{aligned}$$

by Proposition 2.3 (7) of [9]. Thus, we obtain

$$4i\rho \in \langle(\eta, \zeta), \eta, 8\iota\rangle \mod 8\pi_{17}^s(\mathbb{C}\mathrm{P}^2) = \{8i\rho\}.$$

Since the Toda bracket $\langle(\eta, \zeta), \eta, 2\iota\rangle$ is a coset of $2\pi_{17}^s(\mathbb{C}\mathrm{P}^2) = \{2i\rho\}$, it must contain $i\rho$.

(3) First we show $(\eta, \sigma^2) = \langle y(\eta, \nu), \sigma, \nu\rangle$ for some odd integer y. We apply (3.7) of [10] to the case $(\alpha, \beta, \gamma, \delta, \varepsilon) = (i, \eta, \nu, \sigma, \nu)$ with the facts : $(\eta, \nu) \in \langle i, \eta, \nu\rangle, 0 = \langle\eta, \nu, \sigma\rangle$ and $\sigma^2 = \langle\nu, \sigma, \nu\rangle$ by Lemma 2.2 (2). Then we have

$$\langle\langle i, \eta, \nu\rangle, \sigma, \nu\rangle = \langle x(\eta, \nu), \sigma, \nu\rangle$$

for some odd integer x, $0 \in \langle i, \langle\eta, \nu, \sigma\rangle, \nu\rangle \mod iG_{16} + \pi_{15}^s(\mathbb{C}\mathrm{P}^2; 2)\nu = \{i\eta^*\} + \{2(\eta, \sigma^2)\}$ and $\langle i, \eta, \langle\nu, \sigma, \nu\rangle\rangle = \langle i, \eta, \sigma^2\rangle \ni (\eta, \sigma^2) \mod iG_{16} + \{(\eta, 2\iota)\sigma^2\} = \{2(\eta, \sigma^2)\}$. Therefore we obtain

$$\pm(\eta, \sigma^2) = \langle x(\eta, \nu), \sigma, \nu\rangle,$$

and we put $y = \pm x$. Next we apply (3.7) of [10] to the case $(\alpha, \beta, \gamma, \delta, \varepsilon) = (y(\eta, \nu), \sigma, \nu, \eta, 2\iota)$. Since $\langle\sigma, \nu, \eta\rangle = \langle\nu, \eta, 2\iota\rangle = 0$,

$$\begin{aligned}&0 \in \langle y(\eta, \nu), \langle\sigma, \nu, \eta\rangle, 2\iota\rangle + \langle y(\eta, \nu), \sigma, \langle\nu, \eta, 2\iota\rangle\rangle \\ &\qquad \mod y(\eta, \nu)(G_{13}; 2) + 2\pi_{20}^s(\mathbb{C}\mathrm{P}^2) = 2\pi_{20}^s(\mathbb{C}\mathrm{P}^2).\end{aligned}$$

Therefore, we have $0 \in \langle(\eta, \sigma^2), \eta, 2\iota\rangle \mod 2\pi_{20}^s(\mathbb{C}\mathrm{P}^2)$.

(4) First we show that

$$(\eta, \eta\kappa) + \ell i\overline{\mu} \in \langle(\eta, \nu), \varepsilon, \nu\rangle \mod \{i\nu\kappa\} = \{2(\eta, \eta\kappa)\},$$

where ℓ is 0 or 1. We apply (3.7) of [10] to the case $(\alpha,\beta,\gamma,\delta,\varepsilon) = (i,\eta,\nu,\varepsilon,\nu)$ with the facts : $(\eta,\nu) \in \langle i,\eta,\nu\rangle$, $\langle\eta,\nu,\varepsilon\rangle = 0$ and $\langle\nu,\varepsilon,\nu\rangle = \eta\kappa$ by (2) of Lemma 2.4. Then we have $\langle\langle i,\eta,\nu\rangle,\varepsilon,\nu\rangle = \langle(\eta,\nu),\varepsilon,\nu\rangle$, $0 \in \langle i,\langle\eta,\nu,\varepsilon\rangle,\nu\rangle \mod iG_{17}+\pi^s_{16}(\mathbb{C}\mathrm{P}^2;2)\nu = \{i\nu\kappa\}\oplus\{i\overline{\mu}\}$ and $\langle i,\eta,\langle\nu,\varepsilon,\nu\rangle\rangle = \langle i,\eta,\eta\kappa\rangle \ni (\eta,\eta\kappa) \mod iG_{17} + \{(\eta,2\iota)\eta\kappa\} = \{i\nu\kappa\}\oplus\{i\overline{\mu}\}$. Therefore we have $(\eta,\eta\kappa)+\ell i\overline{\mu} \in \langle(\eta,\nu),\varepsilon,\nu\rangle \mod \{i\nu\kappa\}$.

Next we apply (3.7) of [10] to the case $(\alpha,\beta,\gamma,\delta,\varepsilon) = ((\eta,\nu),\varepsilon,\nu,\eta,2\iota)$ with the facts :

$$\langle i\overline{\mu},\eta,2\iota\rangle = \langle 16(\eta,2\iota)\rho,\eta,2\iota\rangle \supset 8(\eta,2\iota)\rho\langle 2\iota,\eta,2\iota\rangle = 8(\eta,2\iota)\rho\eta^2 = 0,$$
$$\langle i\nu\kappa,\eta,2\iota\rangle = \langle 2(\eta,\eta\kappa),\eta,2\iota\rangle \supset (\eta,\eta\kappa)\langle 2\iota,\eta,2\iota\rangle = (\eta,\eta\kappa)\eta^2 = 0.$$

Then we have

$$\langle\langle(\eta,\nu),\varepsilon,\nu\rangle,\eta,2\iota\rangle = \langle(\eta,\eta\kappa),\eta,2\iota\rangle,$$
$$0 \in \langle(\eta,\nu),\langle\varepsilon,\nu,\eta\rangle,2\iota\rangle \mod (\eta,\nu)(G_{14};2)+2\pi^s_{21}(\mathbb{C}\mathrm{P}^2) = \{(\eta,\nu)\kappa\}\oplus 2\pi^s_{21}(\mathbb{C}\mathrm{P}^2),$$
$$0 \in \langle(\eta,\nu),\varepsilon,\langle\nu,\eta,2\iota\rangle\rangle \mod (\eta,\nu)(G_{14};2) = \{(\eta,\nu)\kappa\}.$$

Therefore we have

$$0 \textit{ or } (\eta,\nu)\kappa \in \langle(\eta,\eta\kappa),\eta,2\iota\rangle \mod 2\pi^s_{21}(\mathbb{C}\mathrm{P}^2).$$

Because order of $(\eta,\nu)\kappa$ is 2, we obtain that

$$0 \in 2\langle(\eta,\eta\kappa),\eta,2\iota\rangle \mod 4\pi^s_{21}(\mathbb{C}\mathrm{P}^2).$$

(5) We apply (3.7) of [10] to the case $(\alpha,\beta,\gamma,\delta,\varepsilon) = (i,\eta,\eta^2\rho,\eta,2\iota)$ with the facts : $(\eta,\eta^2\rho) \in \langle i,\eta,\eta^2\rho\rangle \mod iG_{19}$, $0 \in \langle\eta,\eta^2\rho,\eta\rangle$ by Lemma 5.5 of [10] and $\overline{\zeta} \in \langle\eta^2\rho,\eta,2\iota\rangle$ by Theorem D of [6]. Then we have

$$\langle\langle i,\eta,\eta^2\rho\rangle,\eta,2\iota\rangle = \langle(\eta,\eta^2\rho),\eta,2\iota\rangle,$$
$$0 \in \langle i,\langle\eta,\eta^2\rho,\eta\rangle,2\iota\rangle \mod iG_{21}+2\pi^s_{23}(\mathbb{C}\mathrm{P}^2) = \{2(\eta,\overline{\zeta})\}\oplus\{i\sigma^3\}$$

and

$$\langle i,\eta,\langle\eta^2\rho,\eta,2\iota\rangle\rangle = \langle i,\eta,\overline{\zeta}\rangle \ni (\eta,\overline{\zeta}) \mod iG_{21}+2\pi^s_{23}(\mathbb{C}\mathrm{P}^2).$$

Therefore we obtain

$$(\eta,\overline{\zeta})+\ell i\sigma^3 \in \langle(\eta,\eta^2\rho),\eta,2\iota\rangle \mod \pi^s_{23}(\mathbb{C}\mathrm{P}^2) = \{2(\eta,\overline{\zeta})\}.$$

(6) We apply (3.7) of [10] to the case $(\alpha,\beta,\gamma,\delta,\varepsilon) = (i,\eta,\nu^*,8\iota,\eta)$ with the facts : $(\eta,\nu^*) \in \langle i,\eta,\nu^*\rangle$, $0 \in \langle\eta,\nu^*,8\iota\rangle$ by Lemma 2.1 (1) and $\langle\nu^*,8\iota,\eta\rangle \supset \langle 4\nu^*,2\iota,\eta\rangle = \langle\eta^2\eta^*,2\iota,\eta\rangle \supset \eta\eta^*\langle\eta,2\iota,\eta\rangle = 0 \mod \nu^*(G_2;2)+\eta(G_{19};2) = 0$. Then we have

$$\langle\langle i,\eta,\nu^*\rangle,8\iota,\eta\rangle = \langle(\eta,\nu^*),8\iota,\eta\rangle,$$
$$\langle i,\langle\eta,\nu^*,8\iota\rangle,\eta\rangle \ni 0 \mod iG_{22}+\pi^s_{23}(\mathbb{C}\mathrm{P}^2)\circ\eta = \{i\nu\overline{\sigma}\}$$

and $\langle i, \eta, \langle \nu^*, 8\iota, \eta\rangle\rangle \ni 0 \mod iG_{22} = \{i\nu\overline{\sigma}\}$. Therefore we obtain $0 \in \langle(\eta, \nu^*), 8\iota, \eta\rangle \mod \{i\nu\overline{\sigma}\}$. $\square$

Finally, we can determine the set $2m\,\mathrm{Ext}(\alpha, \eta)$ for $\alpha \in \pi_k^s(\mathbb{C}\mathrm{P}^2; 2) \cap \ker \eta^*$.

Lemma 4.6. *The set* $2m\,\mathrm{Ext}(\alpha, \eta)$ *contains the corresponding element in the following table.*

α	$2(\eta,\nu)$	$i\nu^2$	$i\sigma$	$(\eta, 2\iota)\sigma$	$(\eta, \eta\varepsilon)$	$(\eta,\nu)\nu^2$	(η,ζ)	$i\kappa$
$2m$	4	2	16	16	8	2	8	2
representative of $2m\,\mathrm{Ext}(\alpha, \eta)$	0	0	0	$4(\eta, \eta\varepsilon)\pi$	0	0	$4i\rho\pi$	0

$i\sigma^2$	$i\rho$	(η, σ^2)	$(\eta, 2\iota)\rho$	$(\eta, \eta\kappa)$	$i\nu^*$	$(\eta, \eta^2\rho)$	$i\overline{\sigma}$	$i\overline{\kappa}$	$2(\eta, \nu^*)$
2	32	4	32	4	4	8	2	8	4
0	0	$2i\nu^*\pi$	$4(\eta, \eta^2\rho)\pi$	0	0	0	0	0	0

Proof. We shall prove this Lemma for the case of $\alpha = 2(\eta,\nu), i\kappa$ and $(\eta, \eta^2\rho)$. All of the remainders can be computed by trying the similar argument.

- $0 \in 4\,\mathrm{Ext}(2(\eta,\nu), \eta)$.
  We can choose $(\eta,\nu)[2\iota, \eta] - 2i\sigma\pi$ as an element of $\mathrm{Ext}(2(\eta,\nu),\eta)$. By Lemma 5.13 of [10], $4(\eta,\nu)[2\iota,\eta] = (\eta, 2\iota)2\nu[2\iota,\eta] \in (\eta, 2\iota)\langle\eta, 2\iota, \eta\rangle [2\iota, \eta] \subset \langle 2i\nu, 2\iota, 2\nu\pi\rangle \subset \langle i\nu, 8\iota, \nu\pi\rangle \supset i\langle\nu, 8\iota, \nu\rangle\pi \ni 8i\sigma\pi$. Here, the Toda bracket $\langle i\nu, 8\iota, \nu\pi\rangle$ is a coset of trivial subgroup. Thus we obtain a relation $4(\eta,\nu)[2\iota,\eta] = 8i\sigma\pi$. Therefore, we have that the order of $(\eta,\nu)[2\iota,\eta] - 2i\sigma\pi$ is 4.
- $0 \in 2\,\mathrm{Ext}(i\kappa, \eta)$.
  By Lemma 15.2 of [4], we have the relation $\kappa[2\iota, \eta] = 0$. Therefore,

$$
\begin{aligned}
2\,\mathrm{Ext}(i\kappa, \eta) &= \mathrm{Ext}(i\kappa,\eta) \circ 2(1_{\mathbb{C}\mathrm{P}^2}) \\
&= \mathrm{Ext}(i\kappa,\eta) \circ (i[2\iota,\eta] + (\eta, 2\iota)\pi) \quad \text{by (8.4) of [1]} \\
&= i\kappa[2\iota,\eta] + \langle i\kappa, \eta, 2\iota\rangle\pi \\
&= \langle i\kappa, \eta, 2\iota\rangle\pi \\
&\ni 0 \qquad \text{by (2) of Corollary 4.4.}
\end{aligned}
$$

- $0 \in 8\,\mathrm{Ext}((\eta, \eta^2\rho), \eta)$.
  For an element $\overline{\zeta}[2\iota,\eta]$ in $\{\Sigma^{17}\mathbb{C}\mathrm{P}^2, S^0\}$, we have $i^*(\overline{\zeta}[2\iota,\eta]) = 2\overline{\zeta} = i^*(2[\overline{\zeta}, \eta])$. By exactness of the sequence

$$0 \longrightarrow G_{21}/G_{20} \circ \eta \xrightarrow{\pi^*} \{\Sigma^{17}\mathbb{C}\mathrm{P}^2, S^0\} \xrightarrow{i^*} G_{19} \cap \ker \eta^* \longrightarrow 0,$$

we have

$$\overline{\zeta}[2\iota,\eta] \equiv 2[\overline{\zeta},\eta] \mod \{\sigma^3\pi\}.$$

By Proposition 4.1 (4) and the relation of Lemma 3.1, we have

$$4(\eta, \eta^2\rho)[2\iota, \eta] = 2i\bar{\zeta}[2\iota, \eta] \equiv 4i[\bar{\zeta}, \eta] = 4(\eta, \bar{\zeta})\pi \mod \{2i\sigma^3\pi\} = 0.$$

Therefore,

$$\begin{aligned} 8\,\mathrm{Ext}((\eta, \eta^2\rho), \eta) &= 4(\eta, \eta^2\rho)[2\iota, \eta] + 4\langle(\eta, \eta^2\rho), \eta, 2\iota\rangle\pi && \text{by (8.4) of [1]} \\ &\ni 4(\eta, \bar{\zeta})\pi + 4(\eta, \bar{\zeta})\pi && \text{by (5) of Lemma 4.5} \\ &= 0. \end{aligned}$$

□

proof of Theorem 1.1. Now we compute $G_{19}(\mathbb{C}\mathrm{P}^2)$. Other cases can be computed by trying the similar argument. Consider the short exact sequence

$$0 \longrightarrow \pi_{23}^s(\mathbb{C}\mathrm{P}^2)/\pi_{22}^s(\mathbb{C}\mathrm{P}^2) \circ \eta \xrightarrow{\pi^*} G_{19}(\mathbb{C}\mathrm{P}^2) \xrightarrow{i^*} \pi_{21}^s(\mathbb{C}\mathrm{P}^2) \cap \ker \eta^* \longrightarrow 0,$$

where $\pi_{23}^s(\mathbb{C}\mathrm{P}^2)/\pi_{22}^s(\mathbb{C}\mathrm{P}^2)\circ\eta = \{(\eta, \bar{\zeta})\} \oplus \{(\eta, \bar{\sigma})\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\pi_{21}^s(\mathbb{C}\mathrm{P}^2) \cap \ker \eta^* = \{(\eta, \eta^2\rho)\} \oplus \{i\bar{\sigma}\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2$. By the table of Lemma 4.6, we have

$$0 \in 8\,\mathrm{Ext}((\eta, \eta^2\rho), \eta) \quad \text{and} \quad 0 \in 2\,\mathrm{Ext}(i\bar{\sigma}, \eta).$$

These show that the above exact sequence splits. Thus, we have

$$\begin{aligned} G_{19}(\mathbb{C}\mathrm{P}^2) &= \{(\eta, \bar{\zeta})\pi\} \oplus \{[(\eta, \eta^2\rho), \eta]\} \oplus \{(\eta, \bar{\sigma})\pi\} \oplus \{i[\bar{\sigma}, \eta]\} \\ &\cong \mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

□

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References

[1] Araki, S., Toda, H.: Multiplicative structures in mod q cohomology theories I, and II, Osaka j. Math., **2**, 71-115 (1965) and **3**, 81-120 (1966)

[2] Ishikawa, N.: Multiplications in cohomology theories with coefficient maps, J. Math. Soc. Japan, **22**, 456-489 (1970)

[3] Mimura, M.: On the generalized Hopf homomorphism and the higher composition. Part U. $\pi_{n+i}(S^n)$ for $i = 21$ and 22, J. Math. Kyoto Univ., **4**, 301-326 (1965)

[4] Mimura, M., Toda, H.: The $(n + 20)$-th homotopy groups of n-spheres, J. Math. Kyoto Univ., **3**, 37-58 (1963)

[5] Mimura, M., Toda, H.: Homotopy groups of $SU(3)$, $SU(4)$ and $Sp(2)$, J. Math. Kyoto Univ., **3**, 217-250 (1963)

[6] Mukai, J.: Stable homotopy of some elementary complexes, Memoirs Faculty of Sci. Kyushu Univ., **20**, 266-282 (1966)

[7] Mukai, J.: On the stable homotopy of a Z_2-Moore space, Osaka J. Math., **6**, 63-91 (1969)

[8] Nishida, G.: The nilpotency of elements of the stable homotopy groups of spheres, J. Math. Soc. Japan, **25**, 707-732 (1973)
[9] Oda, N.: Unstable homotopy groups of spheres, Faculty of Science ,Fukuoka Univ., **44**, 49-152 (1979)
[10] Toda, H.: Composition methods in homotopy groups of spheres, Princeton University Press, New Jersey, 1962

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