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A LOWER BOUND FOR THE LS CATEGORY OF A FORMAL ELLIPTIC SPACE

Dedicated to the memory of Professor Akie TAMAMURA

Yasusuke KOTANI and Toshihiro YAMAGUCHI

Abstract. We give a lower bound for the LS category of a formal elliptic space in terms of its rational cohomology.

1. Introduction

The Lusternik-Schnirelmann (LS) category, $\operatorname{cat} X$, for a space X is the least integer m such that X can be covered by $m+1$ open sets, each contractible in X. The rational LS category, $\operatorname{cat}_0(X)$, is the least integer n such that $X \simeq_0 Y$ and $\operatorname{cat} Y = n$. A simply connected CW complex X is called (rationally) elliptic if $\dim H^*(X;\mathbb{Q}) < \infty$ and $\dim \pi_*(X)\otimes\mathbb{Q} < \infty$. An elliptic space X has a positive Euler characteristic, i.e., $\sum_i (-1)^i \dim H^i(X;\mathbb{Q}) > 0$, if and only if it satisfies $\chi_\pi(X) = \sum_i (-1)^i \dim \pi_i(X)\otimes\mathbb{Q} = 0$ [6]. Then it is called often an "F_0-space". For example, a homogeneous space G/H, where G is a compact, connected Lie group and H is a closed subgroup of maximal rank, is an F_0-space. The rational cup length of a space Z, $\operatorname{cup}_0(Z)$, is the greatest integer n such that the n-product $H^+(Z;\mathbb{Q})\cdots H^+(Z;\mathbb{Q}) \neq 0$. Also the rational Toomer invariant of Z, $e_0(Z)$, is given by using the Sullivan minimal model [4] $\mathcal{M}(Z) = (\wedge V, d)$ as $\sup\{n \mid \text{there is an element } \alpha \in \wedge^{\geq n} V$ such that $[\alpha] \neq 0$ in $H^*(Z;\mathbb{Q})\}$. We remark that

$$(*) \qquad \operatorname{cup}_0(Y) \leq e_0(Y) = \operatorname{cat}_0(Y) \leq \operatorname{cat} Y$$

for an elliptic space Y (see [3]).

There is a problem [9]: *If Y is elliptic, then* $\dim H^*(Y;\mathbb{Q}) \leq 2^{\operatorname{cat} Y}$?. It is true if $\dim \pi_{\text{even}}(Y)\otimes\mathbb{Q} \leq 1$ [9].

Lemma. *For an F_0-space X,* $\dim H^*(X;\mathbb{Q}) \leq 2^{\operatorname{cup}_0(X)}$.

Especially, if X has the homotopy type of the r-product of even dimensional spheres, then $\dim H^*(X;\mathbb{Q}) = 2^r = 2^{\operatorname{cup}_0 X} = 2^{\operatorname{cat} X}$.

If an elliptic space Y is formal [4, p.156], roughly speaking, if the Sullivan minimal model [4] is a formal consequence of its rational cohomology, it has the rational homotopy type of the total space of a fibration $X \to E \to S$ in which X is an F_0-space and S is the product of odd dimensional spheres or the one point space (see [2]). Then we have

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Theorem. *If an elliptic space Y is formal, then* $\dim H^*(Y;\mathbb{Q}) \le 2^{\mathrm{cup}_0(Y)}$.

From the above remark (∗), we deduce a partial answer for our problem.

Corollary. *If an elliptic space Y is formal, then* $\dim H^*(Y;\mathbb{Q}) \le 2^{\mathrm{cat}\, Y}$.

In the next section, we give the proofs. In the third section, we compare our results with the *toral rank conjecture* [4, p.516] of Halperin for certain spaces.

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2. Proofs

Let X be an F_0-space. Then there is an isomorphism as algebras

$$H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1,\dots,x_p]/(f_1,\dots,f_p)$$

and there is an equation

$$\dim H^*(X;\mathbb{Q}) = |f_1|\cdots|f_p|/|x_1|\cdots|x_p|$$

where $\{|x_i|\}_i$ are all even and $f_1,\dots,f_p$ is a regular sequence [6]. Here $|*|$ is the degree of $*$ as an element in a graded algebra and f_i are homogeneous polynomials with no linear terms.

Proof of Lemma. Suppose that $|x_1| \le \cdots \le |x_p|$. Let Φ_i denote the set of all monomials occurring in f_i, i.e., $f_i = \sum_j c_{ij}\sigma_{ij}$ for some $c_{ij} \ne 0 \in \mathbb{Q}$ and $\sigma_{ij} \in \Phi_i$. Since $\dim \mathbb{Q}[x_1,\dots,x_p]/(f_1,\dots,f_p) < \infty$, sets $\Phi_1,\dots,\Phi_p$ must satisfy the "polynomial condition" P.C. [5, p.119] due to Friedlander and Halperin: for each s and for each set of s variables $x_{i_1},\dots,x_{i_s}$, there are at least s sets $\Phi_{j_1},\dots,\Phi_{j_s}$ containing a monomial in $\mathbb{Q}[x_{i_1},\dots,x_{i_s}]$ [5, Theorem 3]. By changing the indexes of Φ_i's, we can regard Φ_i as an element of $\mathbb{Q}[x_1,\dots,x_i]$ for any i. Thus we may assume that each f_i contains a term of the form $x_1^{k_{i1}}\cdots x_i^{k_{ii}}$ where $k_{i1},\dots,k_{ii}$ are non-negative integers with $k_{i1}+\cdots+k_{ii} \ge 2$. Then, for each $1 \le i \le p$, we have

$$\begin{aligned}|f_i|/|x_i| &= (k_{i1}|x_1| + \cdots + k_{ii}|x_i|)/|x_i| \\ &\le k_{i1} + \cdots + k_{ii} \\ &\le 2^{k_{i1}+\cdots+k_{ii}-1} \\ &\le 2^{\deg f_i - 1},\end{aligned}$$

where $\deg f_i$ is the degree of f_i as a polynomial. Thus

$$\dim H^*(X;\mathbb{Q}) = |f_1|\cdots|f_p|/|x_1|\cdots|x_p| \le 2^{\deg f_1+\cdots+\deg f_p - p}.$$

Since the Jacobian $\det(\partial f_i/\partial x_j)$ is the fundamental class of the Poincaré duality algebra [8, Proposition 3], we have $\deg f_1+\cdots+\deg f_p - p \le \mathrm{cup}_0(X)$. □

Recall the following Lemma in [9] for the proof of Theorem.

Lemma 2.1 ([9, Lemma 2.1]). *Let E be the total space of the rational fibration $F \to E \to S^{2n+1}$ with an elliptic space F satisfying* $\dim H^*(F;Q) \leq 2^{e_0(F)}$. *Then* $\dim H^*(E;Q) \leq 2^{e_0(E)}$.

Proof of Theorem. Since a formal elliptic space Y is hyperformal, there is an isomorphism as algebras

$$H^*(Y;\mathbb{Q}) \cong \mathbb{Q}[x_1,\ldots,x_p] \otimes \wedge(y_1,\ldots,y_q)/(h_1,\ldots,h_p) \qquad (p,q \geq 0),$$

where the elements h_i $(i = 1,\ldots,p)$ are written as $h_i = f_i + g_i$ with a regular sequence $f_1,\ldots,f_p$ in $\mathbb{Q}[x_1,\ldots,x_p]$ and elements g_i in the ideal generated by $y_1,\ldots,y_q$ [2, p.576–577]. We regard the algebra $H^*(Y;\mathbb{Q})$ as the exterior algebra $\wedge(y_1,\ldots,y_q)$ if $p = 0$. Thus Y has the rational homotopy type of the total space of a fibration

$$X \to E_q \to S^{|y_1|} \times \cdots \times S^{|y_q|},$$

where X is the F_0-space with $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1,\ldots,x_p]/(f_1,\ldots,f_p)$ and $S^{|y_i|}$ is the $|y_i|$-dimensional sphere. If $q > 0$, by using Lemma 2.1 with [4, p.388, Example 4] inductively for fibrations

$$\begin{gathered} X \to E_1 \to S^{|y_1|} \\ \vdots \\ E_i \to E_{i+1} \to S^{|y_{i+1}|} \\ \vdots \\ E_{q-1} \to E_q \to S^{|y_q|}, \end{gathered}$$

we have $\dim H^*(E_i;\mathbb{Q}) \leq 2^{\mathrm{cup}_0(E_i)}$ for $i = 1,\ldots,q$. Here each E_i is an elliptic space with

$$H^*(E_i;\mathbb{Q}) \cong \mathbb{Q}[x_1,\ldots,x_p] \otimes \wedge(y_1,\ldots y_i)/(\overline{h}_1,\ldots,\overline{h}_p),$$

where $\overline{h}_j$ is the obvious projection of h_j. □

3. Toral rank vs LS category

Let X be a simply connected finite cell complex. Recall that the toral rank of a space X, $\mathrm{rk}(X)$, is the largest integer n such that an n-torus can act continuously on X with all its isotropy subgroups finite. In [7], S. Halperin conjectured that $2^{\mathrm{rk}(X)} \leq \dim H^*(X;\mathbb{Q})$, which gives an upper bound for toral rank. We compare the two bounds around formal elliptic spaces in the following table. Refer [4, Part II and Section 32] for the Sullivan minimal model theory.

X: elliptic		$2^{\mathrm{rk}(X)} \leq \dim H^*(X;\mathbb{Q})$	$\dim H^*(X;\mathbb{Q}) \leq 2^{\mathrm{cat}\,X}$
$\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}=0$		? (a)	yes ([9])
X: formal	$\chi_\pi(X)=0$	yes (since $\mathrm{rk}(X)=0$)	yes (Lemma)
	$\chi_\pi(X)<0$	yes (b)	yes (Corollary)
X: pure		yes ([7, Proposition 1.5])	? (c)

Here X is called pure [4, p.435] if the differential d satisfies $dV^{\mathrm{even}}=0$ and $dV^{\mathrm{odd}}\subset\wedge V^{\mathrm{even}}$ for the Sullivan minimal model $\mathcal{M}(X)=(\wedge V,d)$ of X, where $V=\oplus_{i>1}V^i$ with $\dim V^i=\dim\pi_i(X)\otimes\mathbb{Q}$. For example, a homogeneous space is pure. Note a pure space with $\dim V^{\mathrm{even}}=\dim V^{\mathrm{odd}}$ is an F_0-space.

(a) is "yes" if X is a space of two-stage Sullivan minimal model and coformal, i.e., V decomposes as $V\cong U\oplus W$ with $dU=0$, $dW\subset\wedge U$ and d is quadratic, respectively [1, Proposition 3.1].

(b) is obtained from $\mathrm{rk}(X)\leq-\chi_\pi(X)=-(\dim V^{\mathrm{even}}-\dim V^{\mathrm{odd}})=q$ [7, Theorem 1.1] and $\dim H^*(X;\mathbb{Q})\geq 2^q$ when $H^*(X;\mathbb{Q})$ is given by $\mathbb{Q}[x_1,\dots,x_p]\otimes\wedge(y_1,\dots,y_q)/(h_1,\dots,h_p)$, where $\mathcal{M}(X)=(\wedge(x_1,\dots,x_p,y_1,\dots,y_q,v_1,\dots,v_p),d)$ with $dx_i=dy_i=0,\ dv_i=h_i$.

(c) is "yes" if $\dim V^{\mathrm{even}}=1$, because then there is an isomorphism $H^*(X;\mathbb{Q})\cong\mathbb{Q}[x]/(x^n)\otimes\wedge(y_1,\dots,y_q)$.

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Yasusuke Kotani
Faculty of Science
Kochi University
Kochi 780-8520, Japan
e-mail address: kotani@math.kochi-u.ac.jp

Toshihiro Yamaguchi
Faculty of Education
Kochi University
Kochi 780-8520, Japan
e-mail address: tyamag@cc.kochi-u.ac.jp

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