Math. J. Okayama Univ. **47** (2005), 119–131

BELYI FUNCTION WHOSE GROTHENDIECK DESSIN IS A FLOWER TREE WITH TWO RAMIFICATION INDICES

Toru KOMATSU

Abstract. In this paper we present an explicit construction of Belyi functions whose dessins are flower trees (i.e., graphs of diameter 4) with two ramification indices. We also give a method for obtaining Belyi functions defined over the moduli fields of the dessins.

1. Introduction

For a compact connected Riemann surface R and a finite covering $\beta : R \to \mathbb{P}^1$ one calls β a Belyi function on R if β is unramified outside the three points $0, 1$ and $\infty \in \mathbb{P}^1$. Belyi [1] shows that for a complete nonsingular algebraic curve R defined over a field of characteristic zero, R can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a covering $R \to \mathbb{P}^1$ with three ramification points. There are various studies on properties of Belyi functions (cf. [2], [11], [13], [14], ...). The main result in this paper is a unified method for constructing Belyi functions of a certain family. In the following we assume $R = \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$ and denote by $\mathcal{B}$ the set of Belyi functions $\beta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. One usually identifies $\mathbb{P}^1_{\mathbb{C}}$ with $\mathbb{C} \cup \{\infty\}$. Let $[0, 1] \subset \mathbb{P}^1_{\mathbb{C}}$ be the segment on the real line with end points 0 and 1 not through ∞, that is, $[0, 1] = \{z \in \mathbb{R} | 0 \leq z \leq 1\}$. We denote by D_β the inverse image $\beta^{-1}([0, 1])$ of $[0, 1]$ for $\beta \in \mathcal{B}$, and call D_β a dessin due to Grothendieck. Here β is a polynomial in $\mathbb{C}[X]$ if and only if D_β is a graph of tree type, i.e., a graph with no cycles. It is easily seen that D_β is a connected graph. Let A_0 and A_1 be the sets of points whose images by β are 0 and 1, respectively. Then $A_0 \amalg A_1$ coincides with the set V of vertices of the graph D_β. On the graph D_β one draws $\bullet$ and $\times$ at points of A_0 and A_1, respectively. Then D_β is a bipartite connected graph with two partitions A_0 and A_1 of V. Let $\mathcal{G}$ be the set of bipartite connected graphs on $\mathbb{P}^1_{\mathbb{C}}$ with finite edges. We define an equivalence relation in $\mathcal{G}$ such that $g_1 \sim g_2$ if g_1 is equivalent to g_2 as bipartite graphs on $\mathbb{P}^1_{\mathbb{C}}$. On the other hand, we denote $\beta_1 \sim \beta_2$ for $\beta_1, \beta_2 \in \mathcal{B}$ if there exists $\rho \in \mathrm{PSL}_2(\mathbb{C})$ such that $\beta_2 = \beta_1 \circ \rho$. It is an equivalence relation in $\mathcal{B}$. The following is known as Grothendieck's correspondence (cf. [12]). In fact, it follows from Riemann existence theorem and Weil descent theorem.

Mathematics Subject Classification. Primary 14H55; Secondary 14H30.

Key words and phrases. Belyi function, Grothendieck dessin.

The author is supported by the 21st Century COE Program "Development of Dynamic Mathematics with High Functionality".

Proposition 1.1. *There exists a one-to-one correspondence between two quotient sets* $\mathcal{B}/\sim$ *and* $\mathcal{G}/\sim$ *such that* $\beta \mapsto D_\beta$. *Moreover, we have* $\mathcal{B}/\sim\; = \mathcal{B}_{\overline{\mathbb{Q}}}/\sim$ *where* $\mathcal{B}_{\overline{\mathbb{Q}}} \subset \mathcal{B}$ *is the set of Belyi functions defined over* $\overline{\mathbb{Q}}$. *In particular, every graph* $g \in \mathcal{G}$ *can be realized as the dessin* D_β *of a Belyi function* β *defined over* $\overline{\mathbb{Q}}$.

In this paper we study an explicit construction of Belyi functions whose dessins are graphs in a family of plane trees. For every case we construct a Belyi function over a number field whose degree is as small as possible, so called the moduli field.

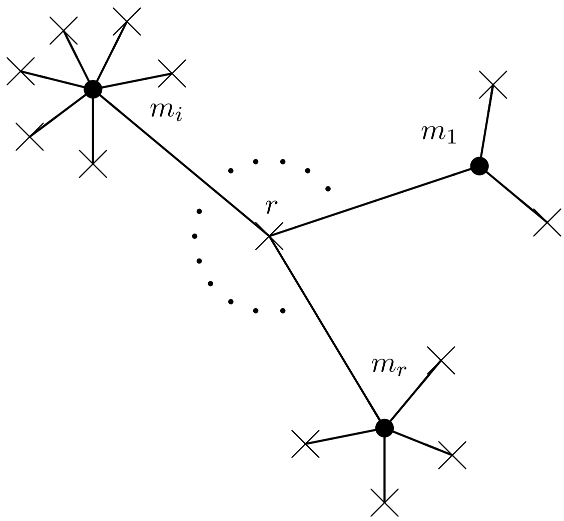

Figure 1.2 (flower tree with ramification $\langle m_1, m_2, \ldots, m_r\rangle$)

Let us call a tree of diameter 4 a flower tree. The flower tree in the Figure 1.2 above is called a flower tree with ramification $\langle m_1, m_2, \ldots, m_r\rangle$; or a flower tree of type $(m_1, m_2, \ldots, m_r)$ (cf. [20]). Here $\langle m_1, m_2, \ldots, m_r\rangle$ is considered to be a multi-set, that is, a set of numbers up to ordering. We denote $\langle m_1, m_2, \ldots, m_r\rangle$ by $\langle m_1, \ldots, m_i\rangle + \langle m_{i+1}, \ldots, m_r\rangle$. For example, we have $2\langle 4\rangle + 3\langle 5\rangle = \langle 4, 4, 5, 5, 5\rangle$. In the Figure 1.2 each point $\bullet$ has m_i edges, respectively. Here the edge connecting to the center point $\times$ is also counted for m_i. The center point $\times$ has r edges. Schneps [12], Shabat-Zvonkin [17] and Zapponi [20] study many properties of flower trees. The Belyi functions for flower trees with ramification $\langle 2, 3, 4, 5, 6\rangle$ are computed in [12]. Shabat-Zvonkin [17] calculate the Belyi functions for flower trees with the following ramifications:

(f.1) $i_1\langle m\rangle + i_2\langle n\rangle$,

(f.2) $j_1\langle m\rangle + j_2\langle n\rangle + j_3\langle p\rangle$, for $(i_1, i_2) = (2,3), (2,5)$ and $(j_1, j_2, j_3) = (1,1,1), (2,1,1), (3,1,1)$ where m, n and p are distinct positive integers. The Belyi function for a flower tree over a finite field and over a complete field are also studied (cf. [18],[19]). The main result in this paper is to present a complete solution for the case (f.1).

Let k, l, m and $n \in \mathbb{Z}$ be positive integers with $m \neq n$. Let S be the set $\{s_i | i = 1, 2, \ldots, k\}$ of k variables s_i. We may assume $s_0 = 1$ for convenience' sake. Let K be the field $\mathbb{Q}(S)$ adjoining to $\mathbb{Q}$ all of the elements in S, and $\mathcal{O}$ the ring of polynomials in K with $\mathbb{Q}$ coefficients, that is, $\mathcal{O} = \mathbb{Q}[S]$. For an $\mathfrak{s} = (s_1, s_2, \ldots, s_k) \in \mathcal{O}^k$ let $f(\mathfrak{s})(X)$ be a polynomial in $\mathcal{O}[X]$ such that

$$f(\mathfrak{s})(X) = \sum_{i=0}^{k} s_i X^i$$

where $s_0 = 1$. Then for each rational number $q \in \mathbb{Q}$ there exists a unique power series $g(q, \mathfrak{s})(X) \in K[[X]]$ such that

$$g(q, \mathfrak{s})(X) = f(\mathfrak{s})(X)^q$$

with the branch condition $g(q, \mathfrak{s}) \equiv 1 \pmod{XK[[X]]}$. For every non-negative integer $j \in \mathbb{Z}$, $j \geq 0$ let $c_j(q, \mathfrak{s}) \in K$ denote the coefficient of X^j in $g(q, \mathfrak{s})$, i.e.

$$g(q, \mathfrak{s})(X) = \sum_{j=0}^{\infty} c_j(q, \mathfrak{s}) X^j.$$

Here $c_j(q, \mathfrak{s}) \in \mathcal{O}$ holds for every $j \geq 0$ (cf. Lemma 2.1). We define a polynomial $\beta_{k,l}(m, n; \mathfrak{s})(X) \in \mathcal{O}[X]$ by

$$\beta_{k,l}(m, n; \mathfrak{s})(X) = \left(\sum_{i=0}^{k} s_i X^i \right)^m \left(\sum_{j=0}^{l} c_j(-m/n, \mathfrak{s}) X^j \right)^n.$$

Let us denote $\mathbb{C}^{k*} = \mathbb{C}^k - \{(0, 0, \ldots, 0)\}$ and

$$\begin{aligned} \mathcal{T} &= \mathcal{T}(k, l, m, n) \\ &= \{\mathfrak{t} \in \mathbb{C}^{k*} | c_j(-m/n, \mathfrak{t}) = 0 \text{ for every } j \in \mathbb{Z} \text{ with } l < j < k + l\}. \end{aligned}$$

Theorem 1.3. *For each* $\mathfrak{t} \in \mathcal{T}$, $\beta_{k,l}(m, n; \mathfrak{t})(X) \in \mathbb{C}[X]$ *is a Belyi function whose dessin is a flower tree with ramification* $k\langle m \rangle + l\langle n \rangle$.

For $\mathfrak{t} \in \mathcal{T}$ let $D_\mathfrak{t}$ denote the dessin which is obtained from $\beta_{k,l}(m, n; \mathfrak{t})$. Let $\mathcal{F} = \mathcal{F}(k, l, m, n)$ be the set of flower trees with ramification $k\langle m \rangle + l\langle n \rangle$ up to the graph equivalence $\sim$. Proposition 1.1 implies that for a graph $D \in \mathcal{F}$ there exists a Belyi function β over $\overline{\mathbb{Q}}$ corresponding to D. The action on the graph D of an element σ in the absolute Galois group $\Gamma_\mathbb{Q}$ of $\mathbb{Q}$ is defined via that on the coefficients of β, that is, $D^\sigma = D_{\beta^\sigma}$. Let Γ_D be the subgroup of $\Gamma_{\mathbb{Q}}$ such that $\Gamma_D = \{\sigma \in \Gamma_{\mathbb{Q}} | D^\sigma \sim D\}$. We denote the fixed field $\overline{\mathbb{Q}}^{\Gamma_D}$ by $\mathcal{M}(D)$ and call it the moduli field of D.

Theorem 1.4. *There exists a finite subest $\mathcal{T}_1$ of $\mathcal{T}$ satisfying the following two properties (i) and (ii) :*
(i) the map $\mathcal{T}_1 \to \mathcal{F}$, $\mathfrak{t} \mapsto D_{\mathfrak{t}}$ gives a bijection,
(ii) for each $\mathfrak{t} \in \mathcal{T}_1$, the Belyi function $\beta_{k,l}(m,n;\mathfrak{t})(X)$ is defined over $\mathcal{M}(D_{\mathfrak{t}})$.

Remark. See §2 for the explicit definition of the $\mathcal{T}_1$. The definition field of any Belyi function realizing a dessin D is an extension of the moduli field $\mathcal{M}(D)$.

Remark. Main construction in this paper generalizes our construction in [8] and contains Examples 5.2 and 5.3 in [17] as special cases.

Remark. In Theorems 1.3 and 1.4 we may take $l = 0$, which yields Belyi functions for the case with ramification $k\langle m\rangle$ (see Proposition 3.5).

2. Construction of Belyi functions

Let k, l, m and $n \in \mathbb{Z}$ be positive integers with $m \neq n$. We first show that $c_j(q, \mathfrak{s}) \in \mathcal{O}$ holds. One can calculate $c_j(q,\mathfrak{s}) \in K$ explicitly as follows. The branch condition implies $c_0(q,\mathfrak{s}) = 1$. For a positive integer $j \in \mathbb{Z}$, $j \geq 1$ we define

$$\mathcal{R}_j = \{(r_1, r_2, \ldots, r_k) \in \mathbb{Z}^k | r_i \geq 0 \text{ and } \sum_{i=1}^{k} r_i i = j\}.$$

For $\mathfrak{r} = (r_1, r_2, \ldots, r_k) \in \mathcal{R}_j$ let $\mathfrak{s}^{\mathfrak{r}}$ denote $\prod_{i=1}^{k} s_i^{r_i}$, and $M(q,\mathfrak{r})$ the multinomial coefficient $P(q, \sum_{i=1}^{k} r_i)/\prod_{i=1}^{k}(r_i!)$ where $P(q,r) = q(q-1)\cdots(q-r+1)$.

Lemma 2.1. *For a rational number $q \in \mathbb{Q}$ we have*

$$c_j(q,\mathfrak{s}) = \sum_{\mathfrak{r}\in\mathcal{R}_j} M(q,\mathfrak{r})\mathfrak{s}^{\mathfrak{r}}.$$

In particular, $c_j(q,\mathfrak{s}) \in \mathcal{O}$ holds for every $j \in \mathbb{Z}$ with $j \geq 0$.

Proof. One can check that two power series $\sum_{j=0}^{\infty}\sum_{\mathfrak{r}\in\mathcal{R}_j} M(q,\mathfrak{r})\mathfrak{s}^{\mathfrak{r}}X^j$ and $g(q,\mathfrak{s})(X)$ satisfy a partial differential equation $f(\mathfrak{s})\partial Y/\partial X - qY\partial f(\mathfrak{s})/\partial X = 0$. It follows from $g(q,\mathfrak{s})(X) \equiv \sum_{\mathfrak{r}\in\mathcal{R}_0} M(q,\mathfrak{r})\mathfrak{s}^{\mathfrak{r}} \equiv 1 \pmod{XK[[X]]}$ that $g(q,\mathfrak{s})(X) = \sum_{j=0}^{\infty}\sum_{\mathfrak{r}\in\mathcal{R}_j} M(q,\mathfrak{r})\mathfrak{s}^{\mathfrak{r}}X^j$. □

Let us fix a $\mathfrak{t} \in \mathcal{T} = \mathcal{T}(k,l,m,n)$ and denote $\beta_{k,l}(m,n;\mathfrak{t})$ simply by β. Note that the map $\beta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is non-constant for $\mathfrak{t} \in \mathbb{C}^{k*}$. Let e_x be the ramification index of β at $x \in \mathbb{P}^1_{\mathbb{C}}$. Let A_z be the set $\beta^{-1}(z) = \{x \in$

$\mathbb{P}^1_{\mathbb{C}}|\beta(x) = z\}$ for $z = 0, 1$ and ∞, and put $A = A_0 \amalg A_1 \amalg A_\infty$. We will calculate the indices e_a for all $a \in A$.

Lemma 2.2. *We have* $0 \in A_1$ *and* $e_0 \geq k + l$. *In particular,* $\sharp A_1 \leq \deg\beta - (k + l) + 1$.

Proof. It follows from the definitions of $\beta(X)$ and $\mathfrak{t} \in \mathcal{T}$ that

$$\begin{aligned}\beta(X) &= \left(\sum_{i=0}^{k} c_i(\mathfrak{t})X^i\right)^m \left(\sum_{j=0}^{l} c_j(-m/n, \mathfrak{t})X^j\right)^n \\ &= \left(\sum_{i=0}^{k} c_i(\mathfrak{t})X^i\right)^m \left(\sum_{j=0}^{k+l-1} c_j(-m/n, \mathfrak{t})X^j\right)^n.\end{aligned}$$

This implies

$$\begin{aligned}\beta(X) &\equiv \left(\sum_{i=0}^{k} c_i(\mathfrak{t})X^i\right)^m \left(\sum_{j=0}^{\infty} c_j(-m/n, \mathfrak{t})X^j\right)^n \pmod{X^{k+l}\mathbb{C}[[X]]} \\ &= 1.\end{aligned}$$

Since $\beta(X) \in \mathbb{C}[X]$, we have $\beta(X) - 1 \in X^{k+l}\mathbb{C}[X]$ and $e_0 \geq k + l$. □

Proof of Theorem 1.3. Lemma 2.2 implies that

$$\begin{aligned}\sharp A_0 + \sharp A_1 + \sharp A_\infty &\leq k + l + \deg\beta - (k + l) + 1 + 1 \\ &= \deg\beta + 2.\end{aligned} \tag{1}$$

Let us consider the following conditions.
(c.0) $(\sum_{i=0}^{k} c_i(\mathfrak{t})X^i)(\sum_{j=0}^{l} c_j(-m/n, \mathfrak{t})X^j) = 0$ has $k + l$ distinct roots in $\mathbb{C}$,
(c.1) $(\beta(X) - 1)/X^{k+l-1} = 0$ has $\deg\beta - (k + l) + 1$ distinct roots in $\mathbb{C}$.
Then both (c.0) and (c.1) are satisfied if and only if the equality in (1) holds. It is clear that

$$\sum_{x \in \mathbb{P}^1_{\mathbb{C}} - A} (e_x - 1) \geq 0. \tag{2}$$

By using (1) and (2) we have

$$\begin{aligned}\sum_{x \in \mathbb{P}^1_{\mathbb{C}}} (e_x - 1) &= \sum_{x \in A} (e_x - 1) + \sum_{x \in \mathbb{P}^1_{\mathbb{C}} - A} (e_x - 1) \\ &= 3\deg\beta - (\sharp A_0 + \sharp A_1 + \sharp A_\infty) + \sum_{x \in \mathbb{P}^1_{\mathbb{C}} - A} (e_x - 1) \\ &\geq 3\deg\beta - (\deg\beta + 2) \\ &= 2\deg\beta - 2.\end{aligned}$$

On the other hand, Riemann-Hurwitz formula shows $\sum_{x \in \mathbb{P}^1_{\mathbb{C}}}(e_x - 1) = 2\deg\beta - 2$ since β is a non-constant separable map from $\mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{P}^1_{\mathbb{C}}$. This means that the inequalities in (1) and (2) are, in fact, equalities. The equality $\sum_{x \in \mathbb{P}^1_{\mathbb{C}} - A}(e_x - 1) = 0$ verifies that $\beta(X)$ is a Belyi function.

By the above argument we see that both (c.0) and (c.1) hold. It follows from (c.0) that $c_k(\mathfrak{t})$ and $c_l(-m/n, \mathfrak{t})$ are non-zero. Thus we have $\deg\beta = km + ln$. Let $A_{0,1}$ and $A_{0,2}$ be subsets of A_0 such that $A_{0,1} =$

$\{x \in \mathbb{C} | \sum_{i=0}^{k} c_i(\mathfrak{t})x^i = 0\}$ and $A_{0,2} = \{x \in \mathbb{C} | \sum_{j=0}^{l} c_j(-m/n, \mathfrak{t})x^j = 0\}$, respectively. Then $A_0 = A_{0,1} \amalg A_{0,2}$. By (c.0) we have $e_a = v_i$ for every $a \in A_{0,i}$ where $v_1 = m$ and $v_2 = n$. The condition (c.1) means that $e_0 = k + l$ and $e_a = 1$ for each $a \in A_1 - \{0\}$. It is clear that $A_\infty = \{\infty\}$ and $e_\infty = \deg\beta = km + ln$. We now have a complete list of the ramification indices e_a for all $a \in A$. This data concludes that the dessin of $\beta(X)$ is a flower tree with ramification $k\langle m\rangle + l\langle n\rangle$. □

In the above proof, we have shown the following lemma which will be used later.

Lemma 2.3. *For* $\mathfrak{t} \in \mathcal{T}$ *neither* $c_k(\mathfrak{t})$ *nor* $c_l(-m/n, \mathfrak{t})$ *vanishes.*

We define the action of $\alpha \in \mathbb{C}^\times$ on $\mathfrak{t} = (t_1, t_2, \ldots, t_k) \in \mathcal{T}$ by

$$\alpha\mathfrak{t} = (\alpha t_1, \ldots, \alpha^i t_i, \ldots, \alpha^k t_k).$$

In fact, one sees that $\alpha\mathfrak{t} \in \mathcal{T}$ since $c_j(-m/n, \alpha\mathfrak{t}) = \alpha^j c_j(-m/n, \mathfrak{t})$ holds for every positive integer $j \in \mathbb{Z}$. Let $\overline{\mathcal{T}}$ denote the quotient $\mathbb{C}^\times \backslash \mathcal{T}$ of $\mathcal{T}$ by the action of $\mathbb{C}^\times$. Now recall that $\mathcal{F} = \mathcal{F}(k, l, m, n)$ is the set of flower trees with ramification $k\langle m\rangle + l\langle n\rangle$ up to the graph equivalence.

Proposition 2.4. *There exists a one-to-one correspondence between* $\overline{\mathcal{T}}$ *and* $\mathcal{F}$ *by* $\mathfrak{t} \mapsto D_\mathfrak{t}$.

Proof. For $\mathfrak{t} \in \mathcal{T}$ and $\alpha \in \mathbb{C}^\times$ we have $\beta_{k,l}(m, n; \alpha\mathfrak{t})(X) = \beta_{k,l}(m, n; \mathfrak{t})(\alpha X)$. Thus $\beta_{k,l}(m, n; \alpha\mathfrak{t}) \sim \beta_{k,l}(m, n; \mathfrak{t})$. Proposition 1.1 shows that $D_{\alpha\mathfrak{t}} = D_\mathfrak{t}$ in $\mathcal{F}$. Thus the map $\varphi : \overline{\mathcal{T}} \to \mathcal{F}, \mathfrak{t} \mapsto D_\mathfrak{t}$ is well-defined. We first see that φ is injective. For $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}$ let us denote $\beta_{k,l}(m, n; \mathfrak{t}_i)$ simply by β_i, respectively. Now assume $D_{\mathfrak{t}_1} = D_{\mathfrak{t}_2}$ in $\mathcal{F}$. Then Proposition 1.1 implies that there exists an automorphism $\rho \in \mathrm{PSL}_2(\mathbb{C})$ of $\mathbb{P}^1_\mathbb{C}$ such that $\beta_2(X) = \beta_1(\rho(X))$. Here, $\beta_1^{-1}(\infty) = \beta_2^{-1}(\infty) = \{\infty\}$. This means that $\rho(X) = \alpha_1 X + \alpha_0$ with $\alpha_1 \in \mathbb{C}^\times$ and $\alpha_0 \in \mathbb{C}$. By the argument in the proof of Theorem 1.3, it satisfies that $\{x \in \mathbb{P}^1_\mathbb{C} | \beta_i(x) = 1 \text{ and } e_x = k + l\} = \{0\}$ for each $i = 1$ and 2. This implies $\alpha_0 = 0$. Thus we have $\beta_2(X) = \beta_1(\alpha_1 X)$ and $\beta_{k,l}(m, n; \mathfrak{t}_2)(X) = \beta_{k,l}(m, n; \mathfrak{t}_1)(\alpha_1 X) = \beta_{k,l}(m, n; \alpha_1\mathfrak{t}_1)(X)$. For $m \neq n$, one sees $\mathfrak{t}_2 = \alpha_1\mathfrak{t}_1$. Hence $\mathfrak{t}_1 = \mathfrak{t}_2$ holds in $\overline{\mathcal{T}}$.

We next show that φ is surjective. Let D be a graph in $\mathcal{F} = \mathcal{F}(k, l, m, n)$. Then there exists a Belyi function $\beta \in \mathcal{B}$ whose dessin is equivalent to D. Let $y \in \mathbb{P}^1_\mathbb{C}$ be a unique point such that $\beta(y) = 1$ and $e_y = k + l$. We denote $\beta(X + y)$ by $\beta_y(X)$. Note that $D_\beta \sim D_{\beta_y}$. Let $A_{0,1}$ (resp. $A_{0,2}$) be the sets of points $x \in \mathbb{C}$ such that $e_x = m$ (resp. $e_x = n$) and $\beta_y(x) = 0$. Then

$$\beta_y(X) = \gamma_1 \Big(\prod_{a \in A_{0,1}} (X - a) \Big)^m \Big(\prod_{a \in A_{0,2}} (X - a) \Big)^n$$

where γ_1 is the coefficient of the highest degree in $\beta_y(X)$. Since $\beta_y(0) = 1$, we have $a \neq 0$ for every $a \in A_0 = A_{0,1} \amalg A_{0,2}$. Thus there exists a constant $\gamma_2 \in \mathbb{C}^\times$ satisfying

$$\beta_y(X) = \gamma_2\Big(\prod_{a\in A_{0,1}} (1 - a^{-1}X)\Big)^m\Big(\prod_{a\in A_{0,2}} (1 - a^{-1}X)\Big)^n.$$

Indeed, $\gamma_2 = 1$ from $\beta_y(0) = 1$. Let t_i and u_j be complex numbers such that

$$\sum_{i=0}^{k} t_i X^i = \prod_{a\in A_{0,1}} (1 - a^{-1}X) \quad \text{and} \quad \sum_{j=0}^{l} u_j X^j = \prod_{a\in A_{0,2}} (1 - a^{-1}X).$$

Then we have

$$\beta_y(X) = \Big(\sum_{i=0}^{k} t_i X^i\Big)^m\Big(\sum_{j=0}^{l} u_j X^j\Big)^n.$$

Now put $\mathfrak{t} = (t_1, t_2, \ldots, t_k)$. One notes that $\mathfrak{t} \in \mathbb{C}^{k*}$ since $a^{-1} \neq 0$ for $a \in A_{0,1}$. It is easily seen that $\beta_y(X) \equiv 1 \pmod{X^{k+l}\mathbb{C}[X]}$ implies

$$c_j(-m/n, \mathfrak{t}) = \begin{cases} u_j & \text{if } 0 \leq j \leq l, \\ 0 & \text{if } l < j < k + l. \end{cases}$$

This shows that $\mathfrak{t} \in \mathcal{T}$ and $\beta_y(X) = \beta_{k,l}(m, n; \mathfrak{t})(X)$. Hence φ is surjective. □

We will find a suitable subset of $\mathcal{T}$ which is a complete system of representatives for $\overline{\mathcal{T}}$. Let us define the period $\mathrm{pd}(\mathfrak{t})$ of $\mathfrak{t} \in \mathcal{T}$ to be $\gcd\{1 \leq i \leq k | c_i(\mathfrak{t}) \neq 0\}$.

Lemma 2.5. *For every* $\mathfrak{t} \in \mathcal{T}$ *the period* $\mathrm{pd}(\mathfrak{t})$ *is a common divisor of* k *and* l.

Proof. By Lemma 2.3 we have $c_k(\mathfrak{t}) \neq 0$. Thus $\mathrm{pd}(\mathfrak{t})$ is a divisor of k. Let ζ be a primitive $\mathrm{pd}(\mathfrak{t})$-th root of unity. Then we have $\zeta\mathfrak{t} = \mathfrak{t}$. This implies that $c_j(-m/n, \mathfrak{t}) = c_j(-m/n, \zeta\mathfrak{t}) = \zeta^j c_j(-m/n, \mathfrak{t})$. Since $c_l(-m/n, \mathfrak{t}) \neq 0$, $\mathrm{pd}(\mathfrak{t})$ is a divisor of l. Hence $\mathrm{pd}(\mathfrak{t}) \mid \gcd(k, l)$ holds. □

For an element $\mathfrak{t} \in \mathcal{T}$ of period p we define the non-vanishing index set $I(\mathfrak{t})$ of $\mathfrak{t}$ by

$$\begin{aligned} I(\mathfrak{t}) &= \{i \in \mathbb{Z} | 1 \leq i \leq k,\ c_i(\mathfrak{t}) \neq 0\} \\ &= \{i_1 < i_2 < \cdots < i_\kappa\}. \end{aligned}$$

Then there exist non-negative integers $\lambda_j \in \mathbb{Z}$ such that $\lambda_1 i_1 - \sum_{j=2}^{\kappa} \lambda_j i_j = p$. The integers λ_j depending on $I(\mathfrak{t})$ can be determined uniquely in the following way. For an integer $j_1 \in \mathbb{Z}$ with $1 \leq j_1 \leq \kappa$ let μ_{j_1} denote $\gcd\{i_j | 1 \leq j \leq j_1 - 1\}$. For each integer j_1 decreasing from κ to 2, we define λ_{j_1} to be the smallest non-negative integer such that $p + \sum_{j=j_1}^{\kappa} \lambda_j i_j \equiv 0$

(mod μ_{j_1}) inductively. Then one puts $\lambda_1 = (p + \sum_{j=2}^{\kappa} \lambda_j i_j)/i_1$. We call such $(\lambda_1, \lambda_2, \dots, \lambda_\kappa)$ the minimization operator of $I(\mathfrak{t})$. Let us define the direction $\mathrm{dir}(\mathfrak{t}) \in \mathbb{C}^\times$ of $\mathfrak{t} \in \mathcal{T}$ by

$$\mathrm{dir}(\mathfrak{t}) = c_{i_1}(\mathfrak{t})^{\lambda_1} \prod_{j=2}^{\kappa} c_{i_j}(\mathfrak{t})^{-\lambda_j},$$

where $(\lambda_1, \lambda_2, \dots, \lambda_\kappa)$ is the minimization operator of $I(\mathfrak{t})$. Let α be a p-th root of $\mathrm{dir}(\mathfrak{t})$, that is, $\alpha^p = \mathrm{dir}(\mathfrak{t})$. We denote $\alpha^{-1}\mathfrak{t}$ by $\mathrm{nom}(\mathfrak{t})$, and call it the normalized element of $\mathfrak{t}$. Here $\zeta\alpha$ is also a p-th root of $\mathrm{dir}(\mathfrak{t})$ for a p-th root ζ of unity. Then $(\zeta\alpha)^{-1}\mathfrak{t} = \alpha^{-1}\zeta^{-1}\mathfrak{t} = \alpha^{-1}\mathfrak{t}$. Thus $\mathrm{nom}(\mathfrak{t}) \in \mathcal{T}$ is well-defined. Let us define $\mathcal{T}_1 = \{\mathfrak{t} \in \mathcal{T} | \mathrm{dir}(\mathfrak{t}) = 1\}$. Then the following lemma is easily seen.

Lemma 2.6. *There exists a bijective map from $\mathcal{T}$ to the direct product of two sets $\mathbb{C}^\times$ and $\mathcal{T}_1$ such that*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\sim} & \mathbb{C}^\times \times \mathcal{T}_1 \\ \mathfrak{t} & \mapsto & (\mathrm{dir}(\mathfrak{t}), \mathrm{nom}(\mathfrak{t})). \end{array}$$

In particular, every normalized element has direction 1.

Let ψ be the composite map of the canonical inclusion map $\mathcal{T}_1 \to \mathcal{T}$ and the projection $\mathcal{T} \to \overline{\mathcal{T}}$.

Lemma 2.7. *The map $\psi : \mathcal{T}_1 \to \overline{\mathcal{T}}$ is bijective, that is, $\mathcal{T}_1$ is a complete system of representatives for $\overline{\mathcal{T}}$.*

Proof. For every $\mathfrak{t} \in \mathcal{T}$ we have $\mathrm{nom}(\mathfrak{t}) \in \mathcal{T}_1$ and $\psi(\mathrm{nom}(\mathfrak{t})) = \mathfrak{t}$ in $\overline{\mathcal{T}}$, which means that ψ is surjective. Now assume $\psi(\mathfrak{t}_1) = \psi(\mathfrak{t}_2)$ for $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}_1$. Then there exists an $\alpha \in \mathbb{C}^\times$ such that $\mathfrak{t}_2 = \alpha\mathfrak{t}_1$. Here the period p of $\mathfrak{t}_1$ is equal to that of $\mathfrak{t}_2$. It follows from the definition that $\mathrm{dir}(\mathfrak{t}_2) = \mathrm{dir}(\alpha\mathfrak{t}_1) = \alpha^p \mathrm{dir}(\mathfrak{t}_1)$. The assumption $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}_1$ implies that $\alpha^p = 1$. Since $\mathrm{pd}(\mathfrak{t}_1) = p$, one sees $\alpha\mathfrak{t}_1 = \mathfrak{t}_1$. Hence we conclude $\mathfrak{t}_1 = \mathfrak{t}_2$, which shows the injectivity of ψ. □

Proposition 2.4 and Lemma 2.7 imply the first assertion of Theorem 1.4.

Corollary 2.8. *There exists a one-to-one correspondence between $\mathcal{T}_1$ and $\mathcal{F}$ by $\mathfrak{t} \mapsto D_\beta$ where $\beta = \beta_{k,l}(m, n; \mathfrak{t})$.*

Remark. For a $\mathfrak{t} \in \mathcal{T}$ with $c_1(\mathfrak{t}) \neq 0$, the condition $\mathfrak{t} \in \mathcal{T}_1$ is equivalent to $c_1(\mathfrak{t}) = 1$.

Let $\mathcal{T}_{\overline{\mathbb{Q}}}$ be the algebraic subset of $\mathcal{T}$, i.e.,

$$\mathcal{T}_{\overline{\mathbb{Q}}} = \{\mathfrak{t} \in \mathcal{T} | c_i(\mathfrak{t}) \in \overline{\mathbb{Q}} \text{ for every } 0 \le i \le k\}.$$

Lemma 2.9. *We have $\mathcal{T}_1 \subset \mathcal{T}_{\overline{\mathbb{Q}}}$.*

Proof. It follows from Corollary 2.8 that $\sharp\mathcal{T}_1 = \sharp\mathcal{F} < \infty$. Let us fix an element $\mathfrak{t}_1 \in \mathcal{T}_1$ and put $\mathcal{T}_2 = \{\mathfrak{t} \in \mathcal{T}_1 | I(\mathfrak{t}) = I(\mathfrak{t}_1)\}$. For an integer $i \in \mathbb{Z}$ with $c_i(\mathfrak{t}) \neq 0$ we define a polynomial $f_i(\mathfrak{s}) \in \mathbb{C}[S]$ such that

$$f_i(\mathfrak{s}) = \prod_{\mathfrak{t}\in\mathcal{T}_2} (c_i(\mathfrak{s}) - c_i(\mathfrak{t})) \in \mathbb{C}[s_i].$$

Then $f_i(\mathfrak{t}) = 0$ holds for all $\mathfrak{t} \in \mathcal{T}_2$. Note that $\mathcal{T}_2$ is equal to the set of zeros of simultaneous equations

$$\begin{aligned} &c_i(-m/n, \mathfrak{s}) = 0 \text{ for } l < i < k + l, \\ &c_i(\mathfrak{s}) = 0 \text{ for } 1 \leq i \leq k \text{ and } i \notin I(\mathfrak{t}_1), \\ &c_{i_1}(\mathfrak{s})^{\lambda_1} - \textstyle\prod_{j=2}^{\kappa} c_{i_j}(\mathfrak{s})^{\lambda_j} = 0, \end{aligned}$$

where $I(\mathfrak{t}_1) = \{i_1 < i_2 < \ldots < i_\kappa\}$ is the non-vanishing index set of $\mathfrak{t}_1$ and $(\lambda_1, \lambda_2, \ldots, \lambda_\kappa)$ is the minimization operator of $I(\mathfrak{t}_1)$. Here the above simultaneous equations consist of polynomials in $\mathbb{Q}[S]$. Thus Hilbert zero point theorem implies that $f_i(\mathfrak{s})^r \in \mathbb{Q}[S]$ for some positive integer $r \in \mathbb{Z}$. Since $f_i(\mathfrak{s})^r \in \mathbb{Q}[S] \cap \mathbb{C}[s_i] = \mathbb{Q}[s_i]$, it holds that $c_i(\mathfrak{t}) \in \overline{\mathbb{Q}}$ for every $\mathfrak{t} \in \mathcal{T}_2$. Hence we have $\mathcal{T}_1 \subset \mathcal{T}_{\overline{\mathbb{Q}}}$. □

Let $\Gamma_{\mathbb{Q}}$ be the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$. The action of $\sigma \in \Gamma_{\mathbb{Q}}$ on $\mathfrak{t} = (t_1, t_2, \ldots, t_k) \in \mathcal{T}_{\overline{\mathbb{Q}}}$ is defined by $\sigma\mathfrak{t} = (\sigma t_1, \sigma t_2, \ldots, \sigma t_k)$. For a fixed $\mathfrak{t} = (t_1, t_2, \ldots, t_k) \in \mathcal{T}_{\overline{\mathbb{Q}}}$ let us denote by $\mathbb{Q}(\mathfrak{t})$ the field $\mathbb{Q}(t_1, t_2, \ldots, t_k)$, and by $\mathbb{Q}(\beta)$ the definition field of the polynomial $\beta(X) = \beta_{k,l}(m, n; \mathfrak{t})(X)$. The moduli field $\mathcal{M}(D)$ of the dessin $D = D_\beta$ is the fixed field $\overline{\mathbb{Q}}^{\Gamma_D}$ where $\Gamma_D = \{\sigma \in \Gamma_{\mathbb{Q}} | D^\sigma \sim D\}$. Then we have $\mathcal{M}(D) \subseteq \mathbb{Q}(\beta) \subseteq \mathbb{Q}(\mathfrak{t})$ in general.

Proposition 2.10. *If* $\mathfrak{t} \in \mathcal{T}_1$, *then* $\mathcal{M}(D) = \mathbb{Q}(\beta) = \mathbb{Q}(\mathfrak{t})$.

Proof. Let us note that $\beta_{k,l}(m, n; \mathfrak{s})(X) \in \mathbb{Q}[S][X]$. Thus $\sigma\beta_{k,l}(m, n; \mathfrak{t})(X) = \beta_{k,l}(m, n; \sigma\mathfrak{t})(X)$ for $\sigma \in \Gamma_{\mathbb{Q}}$ and $\mathfrak{t} \in \mathcal{T}_1$. Let $\sigma \in \Gamma_{\mathbb{Q}}$ be an element in Γ_D, that is, $D_{\beta^\sigma} \sim D_\beta$. By the same argument as in the proof of Proposition 2.4 we have $\sigma\beta_{k,l}(m, n; \mathfrak{t})(X) = \beta_{k,l}(m, n; \mathfrak{t})(\alpha X)$ for an $\alpha \in \mathbb{C}^\times$. It is easy to see that $\sigma\mathfrak{t} = \alpha\mathfrak{t}$ since $m \neq n$. It follows from the definition that $\mathrm{dir}(\sigma\mathfrak{t}) = \sigma(\mathrm{dir}(\mathfrak{t})) = \sigma(1) = 1$ for $\mathfrak{t} \in \mathcal{T}_1$. On the other hand, we have $\mathrm{dir}(\alpha\mathfrak{t}) = \alpha^p \mathrm{dir}(\mathfrak{t}) = \alpha^p$ where $p = \mathrm{pd}(\mathfrak{t})$. This means that $\alpha^p = 1$ and $\sigma\mathfrak{t} = \alpha\mathfrak{t} = \mathfrak{t}$. Hence we have $\mathbb{Q}(\mathfrak{t}) \subseteq \mathcal{M}(D)$, which concludes $\mathcal{M}(D) = \mathbb{Q}(\beta) = \mathbb{Q}(\mathfrak{t})$. □

Proposition 2.10 verifies the second assertion of Theorem 1.4.

Remark. The notion of the normalized element $\mathfrak{t} \in \mathcal{T}_1$ is essentially similar to that of a normalized model in [20].

3. Some numerical examples

In this section we calculate some Belyi functions by using Theorem 1.4. Let us consider the case of the flower tree with ramification $\langle m\rangle + l\langle n\rangle$ where l, m and n are positive integers with $m \neq n$. Since the set $\{j \in \mathbb{Z} | l < j < l+1\}$ is empty, one sees $\mathcal{T}(1,l,m,n) = \{(t_1) | t_1 \in \mathbb{C}^{\times}\}$ and $\mathcal{T}(1,l,m,n)_1 = \{(1)\}$.

Proposition 3.1. *We have* $\mathcal{T}(1,l,m,n)_1 = \{(1)\}$ *and*

$$\beta_{1,l}(m,n;(1))(X) = (1+X)^m (\sum_{j=0}^{l} c_j(-m/n,(1))X^j)^n$$

where $c_j(-m/n,(1)) = (-m/n)(-m/n-1)\cdots(-m/n-j+1)/(j!)$ *for every* $j \in \mathbb{Z}$. *In particular, the Belyi function* $\beta_{1,l}(m,n;(1))(X)$ *is defined over* $\mathbb{Q}$.

We have the following proposition for the case of the flower trees with ramification $2\langle m\rangle + l\langle n\rangle$ where $m \neq n$.

Proposition 3.2. *If* l *is odd, then*

$$\mathcal{T}(2,l,m,n)_1 = \{(1,t_2) | c_{l+1}(-m/n,(1,t_2)) = 0\}.$$

When l *is even, we have*

$$\mathcal{T}(2,l,m,n)_1 = \{(1,t_2) | c_{l+1}(-m/n,(1,t_2)) = 0\} \cup \{(0,1)\}.$$

For each $(1,t_2) \in \mathcal{T}(2,l,m,n)_1$, *it holds that*

$$\beta_{2,l}(m,n;(1,t_2))(X) = (1+X+t_2X^2)^m (\sum_{j=0}^{l} c_j(-m/n,(1,t_2))X^j)^n$$

where

$$c_j(-m/n,(1,t_2)) = \sum_{i=0}^{\lfloor j/2 \rfloor} \frac{(-m/n)(-m/n-1)\cdots(-m/n-(j-i)+1)}{(j-2i)!i!} t_2^i.$$

In particular, $\beta_{2,l}(m,n;(1,t_2))(X)$ *is defined over the moduli field* $\mathbb{Q}(t_2)$. *For* $(0,1) \in \mathcal{T}(2,l,m,n)_1$ *it satisfies* $\beta_{2,l}(m,n;(0,1))(X) = \beta_{1,l/2}(m,n;(1))(X^2)$.

Proof. Lemma 2.5 implies that $\mathrm{pd}(\mathfrak{t}) = 1$ for every $\mathfrak{t} \in \mathcal{T}(2,l,m,n)$ if l is odd. This means $c_1(\mathfrak{t}) \neq 0$ and $\mathrm{nom}(\mathfrak{t}) = (1,t_2)$ for some $t_2 \in \mathbb{C}$. When l is even, we have $(0,t_2) \in \mathcal{T}(2,l,m,n)$ since $c_{l+1}((-1)\mathfrak{s}) = -c_{l+1}(\mathfrak{s})$. Note that $\mathrm{nom}((0,t_2)) = (0,1) \in \mathcal{T}(2,l,m,n)_1$. Thus Theorem 1.4 shows the assertion. □

For the flower trees with ramification $2\langle m\rangle + 3\langle n\rangle = \langle m, m, n, n, n\rangle$ we have $\mathcal{T}(2, 3, m, n)_1 = \{(1, t_2^+), (1, t_2^-)\}$ where

$$t_2^{\pm} = \frac{3(m/n+2) \pm \sqrt{3(m/n+2)(2m/n+3)}}{6},$$

respectively.

Corollary 3.3. *For every real quadratic field $\mathbb{Q}(\sqrt{d})$ there exist infinitely many flower tree dessins D with ramification $2\langle m\rangle + 3\langle n\rangle$ so that $\mathcal{M}(D) = \mathbb{Q}(\sqrt{d})$.*

Proof. Let $d \in \mathbb{Z}$ be a positive integer. Then there exist infinitely many rational numbers $r \in \mathbb{Q}$ such that $3/2 < r < 2$ and $r = du^2$ for some $u \in \mathbb{Q}$. For such an $r \in \mathbb{Q}$ not equal to $9/5$, let m and n be positive integers with $m/n = -3(r-2)/(2r-3)$. Then we have $\mathbb{Q}(t_2^+) = \mathbb{Q}(t_2^-) = \mathbb{Q}(\sqrt{d})$. □

For the flower trees with ramification $2\langle 4\rangle + 3\langle 1\rangle = \langle 1, 1, 1, 4, 4\rangle$, we have $\mathcal{T}(2, 3, 4, 1)_1 = \{(1, t_2^-), (1, t_2^+)\}$ where $t_2^{\pm} = (6 \pm \sqrt{22})/2$. The Belyi functions are equal to

$$\begin{aligned}\beta_{2,3}(4, 1; (1, t_2^{\pm}))(X) &= (1 + X + (6 \pm \sqrt{22})/2X^2)^4 \\ &\quad \times (1 - 4X - 2(1 \pm \sqrt{22})X^2 + 10(4 \pm \sqrt{22})X^3) \\ &\equiv 1 + 22(23 \pm 5\sqrt{22})X^5 \pmod{X^6\mathbb{Q}(\sqrt{22})[X]},\end{aligned}$$

respectively. One can check that the dessin of $\beta_{2,3}(4, 1; (1, t_2^+))(X)$ is the left graph in Figure 3.4 and that of $\beta_{2,3}(4, 1; (1, t_2^-))(X)$ is the right one. The two graphs in Figure 3.4 are conjugate of each other under a Galois action $\sigma \in \Gamma_{\mathbb{Q}}$ such that $\sigma(\sqrt{22}) = -\sqrt{22}$.

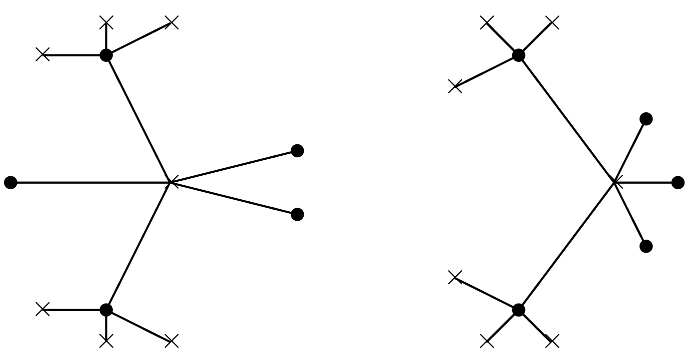

Figure 3.4 (two flower trees with ramification $2\langle 4\rangle + 3\langle 1\rangle$)

As a special case we can obtain the flower tree with one ramification index, that is, the flower tree with ramification $k\langle m\rangle$ where $k, m \in \mathbb{Z}$. In Theorems 1.3 and 1.4 let us take $l = 0$ and $n \geq 1$. Then $\mathfrak{t} = (0, 0, \ldots, 0, 1) \in \mathbb{C}^{k*}$

satisfies $c_j(-m/n, \mathfrak{t}) = 0$ for every $0 < j < k$. Here $\beta_{k,0}(m, n; \mathfrak{t}) = (1+X^k)^m$ is a Belyi function whose dessin is the flower tree with ramification $k\langle m\rangle$. It is clear that there exists only one flower tree with ramification $k\langle m\rangle$.

Proposition 3.5. *The Belyi function for the flower tree with ramification $k\langle m\rangle$ is equal to $(1 + X^k)^m$, which is defined over $\mathbb{Q}$.*

Acknowledgement. The author is grateful to the referee for many helpful comments and careful reading of the manuscript.

References

[1] G. V. Belyi, *Galois extensions of a maximal cyclotomic field (Russian)*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 2, 267–276, 479.

[2] P. L. Bowers, K. Stephenson, Uniformizing dessins and Belyi maps via circle packing, Mem. Amer. Math. Soc. **805** (2004).

[3] J.–M. Couveignes, *Calcul et rationalite de fonctions de Belyi en genre* 0 (*French*), Ann. Inst. Fourier (Grenoble) **44** (1994), no. 1, 1–38.

[4] M. D. Fried, Y. Ihara (ed.), Arithmetic fundamental groups and noncommutative algebra, Proc. Sympos. Pure Math., **70**, Amer. Math. Soc., Providence, RI, 2002.

[5] A. Grothendieck, *Esquisse d'un programme*, in [13], 5–48 (an English trans. on pp. 243–283).

[6] Y. Y. Kochetkov, *Trees of diameter 4*, in [10], 447–453.

[7] T. Komatsu, *Geometric balance of cuspidal points realizing dessin d'enfants on the Riemann sphere*, Math. Ann. **320** (2001), no.3, 417–429.

[8] T. Komatsu, *On flower-tree dessins and their Belyi functions* (*Japanese*), Comm. in arith. fund. groups (Kyoto, 1999/2001), Surikaisekikenkyusho Kokyuroku **1267** (2002), 26–47.

[9] E. M. Kreines, *Belyi functions related to plane graphs: multiplicities and parasitic solutions*, in [10], 468–475.

[10] D. Krob, A. A. Mikhalev, A. V. Mikhalev (ed.), Formal power series and algebraic combinatorics, Proceedings of the 12th International Conference (FPSAC'00), Springer-Verlag, Berlin, 2000.

[11] L. Schneps (ed.), The Grothendieck theory of dessins d'enfants, London Math. Soc. Lecture Note Ser. **200**, Cambridge Univ. Press, Cambridge, 1994.

[12] L. Schneps, *Dessins d'enfants on the Riemann sphere*, in [11], 47–77.

[13] L. Schneps, P. Lochak (ed.), Geometric Galois actions 1, Around Grothendieck's "Esquisse d'un programme", London Math. Soc. Lecture Note Ser. **242**, Cambridge Univ. Press, Cambridge, 1997.

[14] L. Schneps, P. Lochak (ed.), Geometric Galois actions 2, The inverse Galois problem, moduli spaces and mapping class groups, London Math. Soc. Lecture Note Ser. **243**, Cambridge Univ. Press, Cambridge, 1997.

[15] G. Shabat, *On a class of families of Belyi functions*, in [10], 575–580.

[16] G. B. Shabat, V. A. Voevodsky, *Drawing curves over number fields*, The Grothendieck Festschrift, Vol. III, 199–227, Progr. Math., **88**, Birkhauser Boston, Boston, MA, 1990.

[17] G. Shabat, A. Zvonkin, *Plane trees and algebraic numbers*, Jerusalem combinatorics '93, 233–275, Contemp. Math. **178**, Amer. Math. Soc., Providence, RI, 1994.

[18] L. Zapponi, *The arithmetic of prime degree trees*, Int. Math. Res. Not. 2002, no. 4, 211–219.

[19] L. Zapponi, *Specialization of polynomial covers of prime degree*, Pacific J. Math. **214** (2004), no. 1, 161–183.
[20] L. Zapponi, *Galois action on diameter four trees*, arXiv: math.AG/0108031 preprint.

Toru Komatsu
Department of Mathematics
Tokyo Metropolitan University
1–1 Minami-Ohsawa Hachioji-shi
Tokyo, 192–0397 Japan
e-mail address: trkomatu@comp.metro-u.ac.jp

Current address:
Faculty of Mathematics
Kyushu University
6-10-1 Hakozaki Higashiku
Fukuoka, 812-8581 Japan
e-mail address: trkomatu@math.kyushu-u.ac.jp

(*Received April 6, 2005*)
(*Revised July 25, 2005*)