Math. J. Okayama Univ. **47** (2005), 99–117

SYMPLECTIC INVARIANTS ARISING FROM A GRASSMANN QUOTIENT AND TRIVALENT GRAPHS

Hiroki AKAZAWA

Abstract. In this paper, we study the $\mathfrak{sp}$-invariant graded algebra arising in a specific quotient of a Grassmann algebra, and identify it with an algebra generated by trivalent graphs.

Contents

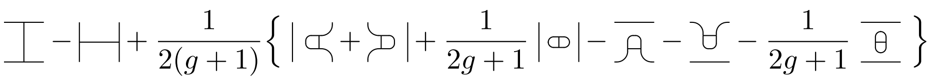
1. Introduction

Let $\mathfrak{sp} := \mathfrak{sp}_{2g}(\mathbb{C})$ be the Lie algebra of symplectic matrices of degree $2g$ over the complex number field $\mathbb{C}$. Let H be the fundamental representation of $\mathfrak{sp}_{2g}(\mathbb{C})$, $\bigwedge^k$ the k-th exterior functor, and U the irreducible $\mathfrak{sp}$-module isomorphic to $\bigwedge^3 H/H$. Suppose g is large enough. Then $\bigwedge^2 U$ as well as $\bigwedge^2(\bigwedge^3 H)$ contains a unique irreducible $\mathfrak{sp}$-component $[2^2]_{\mathfrak{sp}}$, where $[\lambda]_{\mathfrak{sp}}$ denotes the irreducible $\mathfrak{sp}$-module corresponding to a partition λ. We consider

Mathematics Subject Classification. Primary 20G05; Secondary 15A75,57M15.

Key words and phrases. invariant theory, symplectic Lie algebra, Grassmann algebra, trivalent graph.

the quotient space of the Grassmann algebra $\bigwedge U$ (or $\bigwedge(\bigwedge^3 H)$) divided by the ideal $([2^2]_{\mathfrak{sp}})$, where $([2^2]_{\mathfrak{sp}})$ is the ideal generated by $[2^2]_{\mathfrak{sp}} \subset \bigwedge^2 U$ (or $\bigwedge(\bigwedge^3 H)$). The algebras $\bigwedge U/([2^2]_{\mathfrak{sp}})$ and $\bigwedge(\bigwedge^3 H)/([2^2]_{\mathfrak{sp}})$ are important in the study of the mapping class groups of surfaces. In particular, using a classical theorem of H. Weyl in the invariant theory, S. Morita ([6]) gave an interpretation of the $\mathfrak{sp}$-invariant space $(\bigwedge U)^{\mathfrak{sp}}$ (or $(\bigwedge(\bigwedge^3 H))^{\mathfrak{sp}}$) by the algebra $\mathcal{C}(\phi)$, where $\mathcal{C}(\phi)$ denotes the commutative graded algebra freely generated by the connected trivalent graphs, i.e., by the connected graphs where each vertex meets exactly three ends of edges. Furthermore, recent studies by Garoufalidis-Nakamura ([2],[3]), Kawazumi-Morita ([4],[5]) gave relations of trivalent graphs in the symplectic invariant ideals of Grassmann algebras. Let IH_0^{bis} be the ideal of $\mathcal{C}(\phi)$ generated by the graph invariants of type:



Here, the symbols indicate graphs differing from each other only in parts where certain 4 distinct edges are connected as illustrated. Let (loop) denote the ideal generated by graphs containing loops, where a loop is an edge which begins and ends at the same vertex. Then the following theorem is shown by S. Garoufalidis and H. Nakamura [2], [3]:

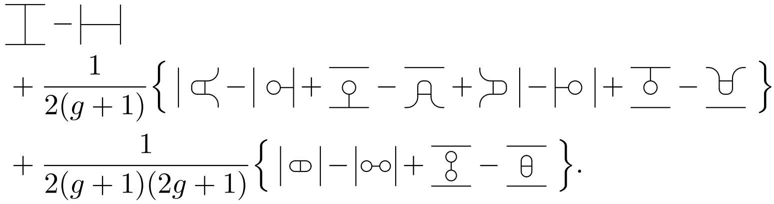
Theorem 1.1 ([2],[3], cf. [5, Remark 11.2]). *There exists a stable isomorphism of graded algebras*

$$\mathcal{C}(\phi)/(IH_0^{bis} + \text{loop}) \to \left(\bigwedge U/([2^2]_{\mathfrak{sp}})\right)^{\mathfrak{sp}}$$

which multiplies degrees by 2. *It gives also an isomorphism in the range of* $g \geq 3m$.

Here, ‘stable’ means that the homogeneous subspace of degree m in the left side is isomorphic to the homogeneous subspace of degree $2m$ in the right side if $g \geq 3m$.

Next, for the Grassmann algebra $\bigwedge(\bigwedge^3 H)$ and its ideal $([2^2]_{\mathfrak{sp}})$ generated by $[2^2]_{\mathfrak{sp}} \subset \bigwedge^2(\bigwedge^3 H)$, we introduce the ideal IH^* of $\mathcal{C}(\phi)$, which is generated by the graph invariants of type:

(1.1) 

Observe that killing the graph invariants with loops in (1.1) yields the graph invariants generating the ideal IH_0^{bis}, viz., $IH^* + \text{loop} = IH_0^{bis} + \text{loop} \subset \mathcal{C}(\phi)$. In this paper, we shall closely study the graph invariants generating the ideal IH^* in $\mathcal{C}(\phi)$. In particular, we obtain the following:

Theorem 1.2. *There exists a stable isomorphism of graded algebras*

$$\mathcal{C}(\phi)/IH^* \to \left(\bigwedge(\bigwedge^3 H)/([2^2]_{\mathfrak{sp}})\right)^{\mathfrak{sp}}$$

which multiplies degrees by 2*. It gives also an isomorphism in the range of* $g \geq 3m$.

The proof of Theorem 1.2 will be given in §4. By the above remark, Theorem 1.1 immediately follows from Theorem 1.2.

2. Preparation

In this section, we briefly review the representation theory of a semisimple Lie algebra and its Casimir operator. Especially, we describe about that in the case of $\mathfrak{sp}_{2g}(\mathbb{C})$.

2.1. **Review of the representation theory of a semisimple Lie algebra.** Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. We take a maximal subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acting diagonally on $\mathfrak{g}$ by the adjoint representation

$$\mathrm{ad} : H \mapsto \mathrm{ad}(H)(X) := [H, X].$$

Such a subalgebra exists and is unique up to inner automorphisms of $\mathfrak{g}$. We fix one such $\mathfrak{h}$ and call it the Cartan subalgebra of $\mathfrak{g}$. Then we find that $\mathfrak{h}$ acts diagonally on any representation V of $\mathfrak{g}$ and that V will admit a direct sum decomposition

$$V = \bigoplus V_\alpha,$$

where the direct sum runs over a finite set of $\alpha \in \mathfrak{h}^*$ (linear characters of $\mathfrak{h}$). Here, $\mathfrak{h}$ acts on each V_α by multiplication by the eigenvalue α, i. e., for any $H \in \mathfrak{h}$ and $v \in V_\alpha$ we have

$$H(v) = \alpha(H)v.$$

These eigencharacters $\alpha \in \mathfrak{h}^*$ are called the weights of V and the V_α themselves are called weight spaces. Especially, for the adjoint representation we have a direct sum decomposition, called the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha. \tag{2.1}$$

The weights for the adjoint representation are called the roots of the Lie algebra and the corresponding subspaces $\mathfrak{g}_\alpha$ are called root spaces. The set of all roots is usually denoted $R \subset \mathfrak{h}^*$. For any $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ we will have

$$\mathrm{ad}(H)(X) = \alpha(H)X.$$

Then each root space $\mathfrak{g}_\alpha$ is one dimensional. We can choose a lexicographic ordering in $\mathfrak{h}^*$, and this allows that the roots can be divided into positive and negative ones. And the maximal weight of a representation V is called the highest weight of V.

The Killing form is a bilinear form on the Lie algebra $\mathfrak{g}$, and is defined by

$$B(X, Y) := \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

The Killing form B is nondegenerate on the semisimple Lie algebra $\mathfrak{g}$. We find that the restriction of B to a Cartan subalgebra $\mathfrak{h}$ is also nondegenerate and that

$$B(H, H') = \sum_{\alpha \in R} \alpha(H)\alpha(H') \tag{2.2}$$

for any $H, H' \in \mathfrak{h}$. By the nondegeneracy, any $\alpha \in \mathfrak{h}^*$ has a unique $T_\alpha \in \mathfrak{h}$ such that

$$B(T_\alpha, H) = \alpha(H)$$

for all $H \in \mathfrak{h}$. The map $\alpha \mapsto T_\alpha$ gives a linear isomorphism from $\mathfrak{h}^*$ to $\mathfrak{h}$.

For any $X_\alpha \in \mathfrak{g}_\alpha$ we can take some $Y_\alpha \in \mathfrak{g}_{-\alpha}$ so that $H_\alpha := [X_\alpha, Y_\alpha]$ satisfies $\alpha(H_\alpha) = 2$. Then $H_\alpha \in \mathfrak{h}$, and for any $H \in \mathfrak{h}$, we have

$$\begin{aligned} B(H_\alpha, H) &= B(H, H_\alpha) = B(H, [X_\alpha, Y_\alpha]) \\ &= B([H, X_\alpha], Y_\alpha) = \alpha(H)B(X_\alpha, Y_\alpha) \end{aligned}$$

by Jacobi's identity. Especially, we have $B(H_\alpha, H_\alpha) = 2B(X_\alpha, Y_\alpha)$. Hence, we find that

$$T_\alpha = \frac{H_\alpha}{B(X_\alpha, Y_\alpha)} = \frac{2H_\alpha}{B(H_\alpha, H_\alpha)}.$$

We denote the Killing form on $\mathfrak{h}^*$ by $(\alpha, \beta) := B(T_\alpha, T_\beta)$.

Let $U_1, ..., U_n$ be a basis for the semisimple Lie algebra $\mathfrak{g}$, and $U_1^*, ..., U_n^*$ be the dual basis with respect to the Killing form on $\mathfrak{g}$. Then the Casimir element of $\mathfrak{g}$ is given by

$$C = U_1 \cdot U_1^* + \cdots + U_n \cdot U_n^*.$$

Note that C is an element of the universal enveloping algebra $U_\mathfrak{g}$ of $\mathfrak{g}$ and independent of the choice of a basis. We shall take a basis $H_1, ..., H_g$ of $\mathfrak{h}$ and nonzero $X_\alpha \in \mathfrak{g}_\alpha$ for each root $\alpha \in R$ so that $\{H_i, X_\alpha; 1 \leqq i \leqq g, \alpha \in R\}$ forms a basis of $\mathfrak{g}$ by (2.1). Taking this $\{H_i, X_\alpha\}$ as the above $\{U_1, ..., U_n\}$, we obtain

$$C = \sum_{i=1}^{g} H_i \cdot H_i^* + \sum_{\alpha \in R} X_\alpha \cdot X_\alpha^*.$$

Moreover, by the nondegeneracy of B,

$$H_i^* = \frac{H_i}{B(H_i, H_i)}, \quad X_\alpha^* = \frac{2Y_\alpha}{B(H_\alpha, H_\alpha)}.$$

In general, the Casimir operator C acts on the irreducible representation with the highest weight λ by multiplication by the scalar

$$(\lambda+\rho, \lambda+\rho) - (\rho, \rho) = (\lambda, \lambda) + (2\lambda, \rho).$$

Here, ρ denotes the half sum of the positive roots.

2.2. **In the case of $\mathfrak{sp}_{2g}(\mathbb{C})$.** The Lie algebra $\mathfrak{sp}_{2g}(\mathbb{C})$ is defined by the space of the $2g \times 2g$ matrices X satisfying the relation

$$^t XJ + JX = O.$$

Here, $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ and I_g is the $g \times g$ identity matrix.

As a Cartan subalgebra of $\mathfrak{sp}_{2g}(\mathbb{C})$, we can take the following:

$$\mathfrak{h} = \left\{ H = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_g & & & \\ & & & -a_1 & & \\ & 0 & & & \ddots & \\ & & & & & -a_g \end{pmatrix} ;\ a_i \in \mathbb{C} \right\} \subset \mathfrak{sp}_{2g}(\mathbb{C}).$$

For convenience, the $(g+i)$-th row or the $(g+j)$-th column $(1 \leqq i, j \leqq g)$ of any $2g \times 2g$ matrix will be called the $(-i)$-th row or the $(-j)$-th column respectively. Hence, $\mathfrak{h}$ is spanned by the g matrices $H_i := E_{i,i} - E_{-i,-i}$ $(1 \leqq i \leqq g)$ where $E_{i,j}$ is the matrix whose entries are 0 except 1 at the (i,j)-th entry. We correspondingly take L_j $(1 \leqq j \leqq g)$ with $L_j(H_i) = \delta_{i,j}$ as basis of the dual vector space $\mathfrak{h}^*$. Furthermore, as a basis of $\mathfrak{sp}_{2g}(\mathbb{C})$ we can take the followings:

$$\begin{array}{lll} H_i & = E_{i,i} - E_{-i,-i} & (1 \leqq i \leqq g), \\ X_{i,j} & = E_{i,j} - E_{-j,-i} & (1 \leqq i, j \leqq g,\ i \neq j), \\ Y_{i,j} & = E_{i,-j} + E_{j,-i} & (1 \leqq i < j \leqq g), \\ Z_{i,j} & = E_{-i,j} + E_{-j,i} & (1 \leqq i < j \leqq g), \\ U_i & = E_{i,-i} & (1 \leqq i \leqq g), \\ V_i & = E_{-i,i} & (1 \leqq i \leqq g). \end{array}$$

For any $H = \sum_i a_i H_i \in \mathfrak{h}$, we find that

$$\begin{array}{lll}
\mathrm{ad}(H)(X_{i,j}) & = (a_i - a_j)X_{i,j} & = (L_i - L_j)(H)X_{i,j}, \\
\mathrm{ad}(H)(Y_{i,j}) & = (a_i + a_j)Y_{i,j} & = (L_i + L_j)(H)Y_{i,j}, \\
\mathrm{ad}(H)(Z_{i,j}) & = (-a_i - a_j)Z_{i,j} & = (-L_i - L_j)(H)Z_{i,j}, \\
\mathrm{ad}(H)(U_i) & = 2a_i U_i & = 2L_i(H)U_i, \\
\mathrm{ad}(H)(V_i) & = -2a_i V_i & = -2L_i(H)V_i.
\end{array}$$

Hence, the set of all roots for $\mathfrak{sp}_{2g}(\mathbb{C})$ is

$$R = \{\pm L_i \pm L_j;\ 1 \leqq i, j \leqq g\} \subset \mathfrak{h}^*.$$

The set of all positive roots is

$$R^+ = \{L_i + L_j\}_{i \leqq j} \cup \{L_i - L_j\}_{i<j}.$$

For each root $\alpha \in R$, we find that

$$\begin{array}{llll}
\alpha = L_i - L_j & \Rightarrow\ X_\alpha = X_{i,j}, & Y_\alpha = X_{i,j}, & H_\alpha = H_i - H_j, \\
\alpha = L_i + L_j & \Rightarrow\ X_\alpha = Y_{i,j}, & Y_\alpha = Z_{i,j}, & H_\alpha = H_i + H_j, \\
\alpha = -L_i - L_j & \Rightarrow\ X_\alpha = Z_{i,j}, & Y_\alpha = Y_{i,j}, & H_\alpha = -H_i - H_j, \\
\alpha = 2L_i & \Rightarrow\ X_\alpha = U_i, & Y_\alpha = V_i, & H_\alpha = H_i, \\
\alpha = -2L_i & \Rightarrow\ X_\alpha = V_i, & Y_\alpha = U_i, & H_\alpha = -H_i.
\end{array}$$

Next, we will compute the Killing form for $\mathfrak{sp}_{2g}(\mathbb{C})$. From (2.2), we have

$$B(H, H') = (4g+4)(\sum a_i b_i)$$

for any $H = \sum a_i H_i$ and $H' = \sum b_i H_i \in \mathfrak{h}$. Hence, we can compute the Casimir element of $\mathfrak{sp}_{2g}(\mathbb{C})$ as

$$\begin{aligned}
C &= \frac{1}{4g+4}\Big\{ \sum_{i=1}^{g} H_i \cdot H_i + \sum_{1 \leqq i,j \leqq g,\ i \neq j} X_{i,j} \cdot X_{j,i} \\
&\qquad + \sum_{1 \leqq i<j \leqq g} \big(Y_{i,j} \cdot Z_{j,i} + Z_{i,j} \cdot Y_{j,i}\big) + 2\sum_{i=1}^{g} \big(U_i \cdot V_i + V_i \cdot U_i\big)\Big\} \\
&= \frac{1}{4g+4}\Big\{ \sum_{(i)} \big(E_{i,i} \cdot E_{i,i} - E_{i,i} \cdot E_{-i,-i}\big) \\
&\qquad + \sum_{(i,j), ij>0, i \neq j} \big(E_{i,j} \cdot E_{j,i} - E_{i,j} \cdot E_{-i,-j}\big) \\
&\qquad + \sum_{(i,j), ij<0, i \neq -j} \big(E_{i,j} \cdot E_{j,i} + E_{i,j} \cdot E_{-i,-j}\big) \\
&\qquad + \sum_{(i)} \big(E_{i,-i} \cdot E_{-i,i} + E_{i,-i} \cdot E_{-i,i}\big)\Big\}.
\end{aligned}$$

Here, $(i_1, ..., i_n)$ indicates the summation over the set

$$\{(i_1, ..., i_n); -g \leqq i_1, ..., i_n \leqq g,\ i_1 \cdots i_n \neq 0\},$$

and henceforce, we conclude the following proposition.

Proposition 2.1. *The Casimir element of* $\mathfrak{sp}_{2g}(\mathbb{C})$ *is given by*

$$C = \frac{1}{4g+4} \sum_{(i,j)} \big(E_{i,j} \cdot E_{j,i} - \epsilon(ij) E_{i,j} \cdot E_{-i,-j}\big) \in U_{\mathfrak{sp}_{2g}(\mathbb{C})}.$$

Here, ϵ *means the signum function.*

3. $\mathfrak{sp}$-invariant algebras and trivalent graphs

As the fundamental representation of $\mathfrak{sp}_{2g}(\mathbb{C})$, we have the $2g$-dimensional vector space

$$H = \mathbb{C}^{2g} = \sum_{(i)} \mathbb{C}x_i.$$

In this section, we summarize the correspondence between the $\mathfrak{sp}$-invariant of the exterior algebra $\bigwedge(\bigwedge^3 H)$ and the commutative algebra generated by trivalent graphs.

3.1. **Isomorphisms of graded algebras.** At first, we consider tensors of H as the representation of $\mathfrak{sp}_{2g}(\mathbb{C})$. The third exterior power of H is decomposed as an $\mathfrak{sp}$-module as follows:

$$\bigwedge^3 H \cong H \oplus U \cong [1]_{\mathfrak{sp}} \oplus [1^3]_{\mathfrak{sp}}$$

for $g \geq 3$. Furthermore, the second exterior power of $\bigwedge^3 H$ and U are decomposed as $\mathfrak{sp}$-modules in the following way [6, Lemma 6.3]:

$$\bigwedge^2(\bigwedge^3 H) \cong [0]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [2^1 1^2]_{\mathfrak{sp}} \oplus [1^4]_{\mathfrak{sp}} \oplus \bigwedge^2 U,$$
$$\bigwedge^2 U \cong [0]_{\mathfrak{sp}} \oplus [2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [1^4]_{\mathfrak{sp}} \oplus [2^2 1^2]_{\mathfrak{sp}} \oplus [1^6]_{\mathfrak{sp}}.$$

In general, it is known that the $\mathfrak{sp}$-invariant space $(H^{\otimes 2n})^{\mathfrak{sp}}$ are generated by the basis corresponding to the graphs which are determined by ways of choosing n pairs from $2n$ vertices. For example, in the case of $n = 1$, the space $(H \otimes H)^{\mathfrak{sp}}$ is generated by the $\mathfrak{sp}$-invariant $\sum_{(i)} \epsilon(i)(x_i \otimes x_{-i})$. In the case of $n = 2$, the space $(H^{\otimes 4})^{\mathfrak{sp}}$ is generated by the $\mathfrak{sp}$-invariants $\sum_{(i,j)} \epsilon(ij)(x_i \otimes x_{-i} \otimes x_j \otimes x_{-j})$, $\sum_{(i,j)} \epsilon(ij)(x_i \otimes x_j \otimes x_{-i} \otimes x_{-j})$ and $\sum_{(i,j)} \epsilon(ij)(x_i \otimes x_j \otimes x_{-j} \otimes x_{-i})$. Next, we will introduce an $\mathfrak{sp}$-invariant $\alpha_\Gamma \in \bigwedge^{2m}(\bigwedge^3 H)$, which are corresponding to the trivalent graph Γ with $2m$ vertices. We call m the degree of the trivalent graph Γ. The quotient space $(\bigwedge^{2m}(\bigwedge^3 H))^{\mathfrak{sp}}$ in $(H^{\otimes 6m})^{\mathfrak{sp}}$ is generated by the α_Γ's. The map $\Gamma \to \alpha_\Gamma$ is given as follows. Let Γ be a trivalent graph with vertex set $\mathrm{Vert}(\Gamma)$ and edge set Edge(Γ) and let Flag(Γ) be the set of flags, where a flag is by definition a pair consisting of vertex and an incident half-edge. Then a total ordering τ of Γ consists of the following data:

- a linear ordering of vertices $\mathrm{Vert}(\Gamma) = \{v_1, ..., v_{2m}\}$,
- a linear ordering of $\mathrm{Flag}(v) = \{f_1(v), f_2(v), f_3(v)\}$ for each $v \in \mathrm{Vert}(\Gamma)$,
- an ordering of $\mathrm{Flag}(e) = \{f_+(e), f_-(e)\}$ for each $e \in \mathrm{Edge}(\Gamma)$.

Such a τ is called $\wedge$-admissible if it satisfies the condition:

$$\mathrm{sgn}\begin{pmatrix} f_1(v_1) & f_2(v_1) & f_3(v_1) & f_1(v_2) & \cdots & f_3(v_{2m}) \\ f_+(e_1) & f_-(e_1) & f_+(e_2) & f_-(e_2) & \cdots & f_-(e_{3m}) \end{pmatrix} = 1$$

for every linear ordering of edges $\mathrm{Edge}(\Gamma) = \{e_1, ..., e_{3m}\}$. We define $f_{3i+j} := f_j(v_{i+1})$ $(0 \leqq i < 2m, j = 1, 2, 3)$ and $OR := \{f_+(e)\}_{e \in \mathrm{Edge}(\Gamma)}$. For a trivalent graph Γ given a total ordering τ, put

$$I := \{\mathbf{i} = (i_1, ..., i_{6m});\ -g \leqq i_j \leqq g,\ i_k = -i_l \Leftrightarrow f_k, f_l \text{ is in the same edge}\},$$

and for any $\mathbf{i} \in I$, set

$$\epsilon(\mathbf{i}) := \prod_{f_k \in OR} \epsilon(i_k),$$
$$x_{\mathbf{i}} := x_{i_1} \otimes \cdots \otimes x_{i_{6m}}.$$

Then, we define an $\mathfrak{sp}$-invariant

$$\alpha_{(\Gamma,\tau)} := \sum_{\mathbf{i} \in I} \epsilon(\mathbf{i}) x_{\mathbf{i}} \in (H^{\otimes 6m})^{\mathfrak{sp}}.$$

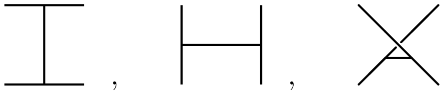
We find that the image of $\alpha_{(\Gamma,\tau)}$ via the standard projection $H^{\otimes 6m} \to \bigwedge^{2m}(\bigwedge^3 H)$ is independent of $\wedge$-admissible τ, which will be denoted by $\alpha_\Gamma \in (\bigwedge^{2m}(\bigwedge^3 H))^{\mathfrak{sp}}$. Since the kernel of $\bigwedge^3 H \to U$ equals to $H \wedge \sum_{(i)} \epsilon(i)(x_i \wedge x_{-i})$, the image of α_Γ in $\bigwedge^{2m} U$ vanishes when the graph Γ has a loop. Namely, $(\bigwedge^{2m} U)^{\mathfrak{sp}}$ is generated by the trivalent graphs with $2m$ verticies without loops [2]. Therefore, if g is large enough, we have stable isomorphisms of graded algebras

$$\mathcal{C}(\phi) \cong \bigoplus_{m \geq 0} (\bigwedge^{2m}(\bigwedge^3 H))^{\mathfrak{sp}} = \big(\bigwedge(\bigwedge^3 H)\big)^{\mathfrak{sp}}, \tag{3.1}$$

$$\mathcal{C}(\phi)/(\mathrm{loop}) \cong \bigoplus_{m \geq 0} (\bigwedge^{2m} U)^{\mathfrak{sp}} = \big(\bigwedge U\big)^{\mathfrak{sp}}. \tag{3.2}$$

Here, 'loop' denotes the ideal generated by the graphs containing loops.

3.2. **Isomorphisms of Grassmann quotients.** Let Γ be a trivalent graph and I, H, X be a graph as indicated in the following figures respectively.



The graphs I, H, X.

Given an embedding of graphs $I \hookrightarrow \Gamma$, let $\Gamma = \Gamma_I, \Gamma_H, \Gamma_X$ denote three trivalent graphs constructed by replacing I-part in Γ by the graph I, H, X respectively. For $t = t_1 \otimes t_2 \otimes t_3 \otimes t_4 \in H^{\otimes 4}$, we define $\mathfrak{sp}$-homomorphisms $f_I, f_H, f_X : H^{\otimes 4} \to \bigwedge^2(\bigwedge^3 H)$ as follows:

$$\begin{aligned}
f_I(t) &:= \sum_{(e)} \epsilon(e)(t_1 \wedge t_2 \wedge x_e) \wedge (t_3 \wedge t_4 \wedge x_{-e}), \\
f_H(t) &:= \sum_{(e)} \epsilon(e)(t_1 \wedge t_3 \wedge x_e) \wedge (t_4 \wedge t_2 \wedge x_{-e}), \\
f_X(t) &:= \sum_{(e)} \epsilon(e)(t_1 \wedge t_4 \wedge x_e) \wedge (t_2 \wedge t_3 \wedge x_{-e}).
\end{aligned}$$

And for any triple of scalars (a, b, c), we define

$$\begin{aligned}
f_{a,b,c} &:= af_I + bf_H + cf_X : H^{\otimes 4} \to \textstyle\bigwedge^2(\bigwedge^3 H), \\
I_{a,b,c} &:= (a\Gamma_I + b\Gamma_H + c\Gamma_X;\ I \hookrightarrow \Gamma) \subset \mathcal{C}(\phi).
\end{aligned}$$

Furthermore, we denote the composite of $f_{a,b,c}$ with the projection

$$\textstyle\bigwedge^2(\bigwedge^3 H) \to \bigwedge^2 U$$

by $\bar{f}_{a,b,c}$. Then,

Proposition 3.1 ([2, Proposition 2.1])**.** *The stable isomorphism of equations* (3.1), (3.2) *induces stable isomorphism of graded algebras*

$$\begin{aligned}
\mathcal{C}(\phi)/I_{a,b,c} &\cong \left(\textstyle\bigwedge(\bigwedge^3 H)/(\mathrm{Im} f_{a,b,c})\right)^{\mathfrak{sp}}, \\
\mathcal{C}(\phi)/(I_{a,b,c} + \mathrm{loop}) &\cong \left(\textstyle\bigwedge U/(\mathrm{Im} \bar{f}_{a,b,c})\right)^{\mathfrak{sp}}
\end{aligned}$$

which multiply degrees by 2.

Definition 1. We define

$$\begin{aligned}
f_{IH} &:= f_I - f_H = f_{1,-1,0} : H^{\otimes 4} \to \textstyle\bigwedge^2(\bigwedge^3 H), \\
IH &:= I_{1,-1,0} = (\Gamma_I - \Gamma_H;\ I \hookrightarrow \Gamma) \subset \mathcal{C}(\phi).
\end{aligned}$$

From [2, Corollary 2.2], $\mathrm{Im} f_{IH} \subset \bigwedge^2(\bigwedge^3 H)$ and $\mathrm{Im}\bar{f}_{IH} \subset \bigwedge^2 U$ are decomposed as an $\mathfrak{sp}$-module in the following way:

$$\mathrm{Im} f_{IH} \cong \mathrm{Im}\bar{f}_{IH} \cong [2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}.$$

Therefore, we have the stable isomorphisms:

$$\mathcal{C}(\phi)/IH \cong \big(\bigwedge(\bigwedge^3 H)/(\mathrm{Im} f_{IH})\big)^{\mathfrak{sp}},$$
$$\mathcal{C}(\phi)/(IH + \mathrm{loop}) \cong \big(\bigwedge U/(\mathrm{Im}\bar{f}_{IH})\big)^{\mathfrak{sp}}.$$

4. Proof of Theorem 1.2

In this section, using the Casimir operator for the representation of $\mathfrak{sp}_{2g}(\mathbb{C})$, we construct the projection from $\mathrm{Im} f_{IH}$ (or $\mathrm{Im}\bar{f}_{IH}$) to the irreducible component $[2^2]_{\mathfrak{sp}}$, and describe graph invariants generating $([2^2]_{\mathfrak{sp}})^{\mathfrak{sp}}$.

4.1. **Projection** $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}} : \mathrm{Im} f_{IH} \to [2^2]_{\mathfrak{sp}}$**.** As recalled in §2.1, the Casimir operator C is multiplication by the constant $(\lambda, \lambda)+(2\lambda, \rho)$ on the irreducible representation with the highest weight λ. Here, ρ denotes the half sum of positive roots and equals to $gL_1 + (g-1)L_2 + \cdots + L_g$ for $\mathfrak{sp}_{2g}(\mathbb{C})$. Hence, the eigenvalue of C equals to

$$\frac{2g+1}{g+1} \text{ on } [2^2]_{\mathfrak{sp}},$$
$$\frac{g}{g+1} \text{ on } [1^2]_{\mathfrak{sp}},$$
$$0 \text{ on } [0]_{\mathfrak{sp}}$$

respectively. Therefore, we compute the projection from $[2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}$ into the $[2^2]_{\mathfrak{sp}}$-part as

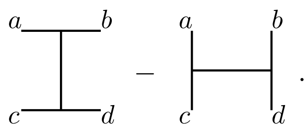
$$\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}} = \frac{g+1}{2g+1} C\big(C - \frac{g}{g+1}\big) = \frac{g+1}{2g+1} C^2 - \frac{g}{2g+1} C.$$

4.2. **Tensors in the ideal** $([2^2]_{\mathfrak{sp}})$**.** Recall from §3.2 that

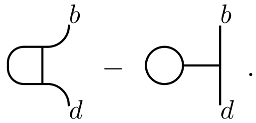
$$(\mathrm{Im} f_{IH})^{\mathfrak{sp}} = ([2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}})^{\mathfrak{sp}} \cong IH = (\Gamma_I - \Gamma_H;\ I \hookrightarrow \Gamma).$$

The generators $\Gamma_I - \Gamma_H$ for the ideal IH can be classified into the following three patterns:

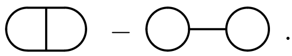
- IH_0-type, where indices a, b, c, d indicate 4 distinct edges as



• IH_1-type, where a and c indicate a same edge and b, d indicate distinct edges as



• IH_2-type, where a, c indicate a same edge and b, d also indicate another same edge as



The $\mathfrak{sp}$-invariants corresponding to these types are respectively in the forms of

$$\sum_{(a,b,c,d,e,\ldots)} \epsilon(abcde\cdots)\big\{(x_a \wedge x_b \wedge x_e) \wedge (x_c \wedge x_d \wedge x_{-e}) - (x_a \wedge x_c \wedge x_e) \wedge (x_d \wedge x_b \wedge x_{-e})\big\} \wedge \cdots \wedge x_{-a} \wedge \cdots \wedge x_{-b} \wedge \cdots \wedge x_{-c} \wedge \cdots \wedge x_{-d} \wedge \cdots,$$

$$\sum_{(h,b,d,e,\ldots)} \epsilon(hbde\cdots)\big\{(x_h \wedge x_b \wedge x_e) \wedge (x_{-h} \wedge x_d \wedge x_{-e}) - (x_h \wedge x_{-h} \wedge x_e) \wedge (x_d \wedge x_b \wedge x_{-e})\big\} \wedge \cdots \wedge x_{-b} \wedge \cdots \wedge x_{-d} \wedge \cdots,$$

$$\sum_{(h,f,e,\ldots)} \epsilon(hfe\cdots)\big\{(x_h \wedge x_f \wedge x_e) \wedge (x_{-h} \wedge x_{-f} \wedge x_{-e}) - (x_h \wedge x_{-h} \wedge x_e) \wedge (x_{-f} \wedge x_f \wedge x_{-e})\big\} \wedge \cdots,$$

where we should understand that the total orderings of the corresponding graphs are given to be $\wedge$-admissible. In order to get the ideal $([2^2]_{\mathfrak{sp}})^{\mathfrak{sp}} \subset \mathcal{C}(\phi)$, we want to apply the projection $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}}$ to the first $\{*\}$-part of the above forms. More precisely, we argue as follows.

Let J be the ideal of $\bigwedge(\bigwedge^3 H)$ generated by $[2^2]_{\mathfrak{sp}} \subset \bigwedge^2(\bigwedge^3 H)$, i.e, generated by the image of $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}} \circ f_{IH}$ and J_{2m} its homogeneous part of degree $2m$. Then we will describe the $\mathfrak{sp}$-invariant part $J_{2m}^{\mathfrak{sp}}$. We can obtain J_{2m} as the image of an $\mathfrak{sp}$-homomorphism

$$H^{\otimes 6m} \to \textstyle\bigwedge^{2m}(\bigwedge^3 H) \tag{4.1}$$

which is defined as follows. We define the $\mathfrak{sp}$-homomorphism

$$H^{\otimes 6m} \to \textstyle\bigwedge^2(\bigwedge^3 H) \otimes \bigwedge^{2m-2}(\bigwedge^3 H)$$

in such a way that the domain components of the canonical contraction $H^{\otimes 2} \to \mathbb{C}$ (which maps $\sum_{(i)} \epsilon(i)(x_i \otimes x_{-i})$ to 1) share the third and sixth factors of $H^{\otimes 6m}$, and

$$\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}} \circ f_{IH} : H^{\otimes 4} \to \bigwedge^2(\bigwedge^3 H)$$

share the first, second, fourth and fifth factors of $H^{\otimes 6m}$, and the standard surjection $H^{\otimes 6m-6} \to \bigwedge^{2m-2}(\bigwedge^3 H)$ share the last $6m-6$ factors of $H^{\otimes 6m}$. The composite of this map with the obvious surjection

$$\bigwedge^2(\bigwedge^3 H) \otimes \bigwedge^{2m-2}(\bigwedge^3 H) \to \bigwedge^{2m}(\bigwedge^3 H)$$

yields the $\mathfrak{sp}$-homomorphism (4.1) and gives a surjection onto J_{2m}. Here, the semisimplicity of $\mathfrak{sp}$-representations implies that $J_{2m}^{\mathfrak{sp}}$ is generated by the images of $\mathfrak{sp}$-invariants $\alpha_{(\Gamma,\tau)} \in (H^{\otimes 6m})^{\mathfrak{sp}}$ via the $\mathfrak{sp}$-homomorphism (4.1), where (Γ, τ) runs over the trivalent graphs of degree m with $\wedge$-admissible total orderings such that $f_3 = f_+(e), f_6 = f_-(e)$ for some $e \in \mathrm{Edge}(\Gamma)$, cf. also [2, §3.1, p.396].

4.3. **Tensors in the image of** $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}} \circ f_{IH}$**.** In order to identify the image of $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}} \circ f_{IH}$ above, let us first compute the image of

$$\begin{aligned} &f_I(x_a \otimes x_b \otimes x_c \otimes x_d) \\ &\quad = \sum_{(e)} \epsilon(e)(x_a \wedge x_b \wedge x_e) \wedge (x_c \wedge x_d \wedge x_{-e}) \in \mathrm{Im} f_I \subset \bigwedge^2(\bigwedge^3 H) \end{aligned}$$

by the Casimir operator C. Since $f_I : H^{\otimes 4} \to \bigwedge^2(\bigwedge^3 H)$ commutes with the action of C, we may only take care of the actions on x_a, x_b, x_c, x_d. Using Proposition 2.1, we compute that

$$\Big(\sum_{(i,j)} E_{i,j} \cdot E_{j,i}\Big) x_a = 2g x_a, \quad \Big(\sum_{(i,j)} \epsilon(ij) E_{i,j} \cdot E_{-i,-j}\Big) x_a = -x_a,$$

and the operators act on x_b, x_c, x_d in similar ways. Furthermore, we compute that

$$\sum_{(i,j)} (E_{i,j} x_a \otimes E_{j,i} x_b) = \sum_{(i,j)} (E_{j,i} x_a \otimes E_{i,j} x_b) = x_b \otimes x_a,$$

$$\begin{aligned} &\sum_{(i,j)} \epsilon(ij)(E_{i,j} x_a \otimes E_{-i,-j} x_b) \\ &\qquad = \sum_{(i,j)} \epsilon(ij)(E_{-i,-j} x_a \otimes E_{i,j} x_b) = \epsilon(a)\delta_{a,-b} \sum_{(i)} \epsilon(i)(x_i \otimes x_{-i}), \end{aligned}$$

and the operators similarly act on the pairs of x_a and x_c, etc. We now introduce the following symbols:

Definition 2.

$$\langle a, b | c, d \rangle := \sum_{(e)} \epsilon(e)(x_a \wedge x_b \wedge x_e) \wedge (x_c \wedge x_d \wedge x_{-e}),$$

$$\begin{aligned}\langle \underline{h}, b| - \underline{h}, d\rangle &:= \sum_{(h,e)} \epsilon(he)(x_h \wedge x_b \wedge x_e) \wedge (x_{-h} \wedge x_d \wedge x_{-e}) \\ &= \sum_{(h)} \epsilon(h)\langle h, b| - h, d\rangle, \\ \langle \underline{h}, \underline{f}| - \underline{h}, -\underline{f}\rangle &:= \sum_{(h,f,e)} \epsilon(hfe)(x_h \wedge x_f \wedge x_e) \wedge (x_{-h} \wedge x_{-f} \wedge x_{-e}) \\ &= \sum_{(h,f)} \epsilon(hf)\langle h, f| - h, -f\rangle.\end{aligned}$$

It is easy to see

$$\langle a, b|c, d\rangle = -\langle b, a|c, d\rangle, \quad \langle \underline{h}, b| - \underline{h}, d\rangle = -\langle -\underline{h}, b|\underline{h}, d\rangle$$

and so on. Under these symbols, the above computations combined with Proposition 2.1 yield:

$$\begin{aligned}C\langle a, b|c, d\rangle =& \frac{1}{4g+4}\big\{8g\langle a, b|c, d\rangle + 4\langle a, b|c, d\rangle \\ &+ 2\langle b, a|c, d\rangle - 2\epsilon(a)\delta_{a,-b}\langle \underline{h}, -\underline{h}|c, d\rangle \\ &+ 2\langle c, b|a, d\rangle - 2\epsilon(a)\delta_{a,-c}\langle \underline{h}, b| - \underline{h}, d\rangle \\ &+ 2\langle b, d|c, a\rangle - 2\epsilon(a)\delta_{a,-d}\langle \underline{h}, b|c, -\underline{h}\rangle \\ &+ 2\langle a, c|b, d\rangle - 2\epsilon(b)\delta_{b,-c}\langle a, \underline{h}| - \underline{h}, d\rangle \\ &+ 2\langle a, d|c, b\rangle - 2\epsilon(b)\delta_{b,-d}\langle a, \underline{h}|c, -\underline{h}\rangle \\ &+ 2\langle a, b|d, c\rangle - 2\epsilon(c)\delta_{c,-d}\langle a, b|\underline{h}, -\underline{h}\rangle\big\} \\ =& \frac{1}{2g+2}\big\{4g\langle a, b|c, d\rangle + 2\langle c, b|a, d\rangle + 2\langle a, c|b, d\rangle \\ &- \epsilon(a)\delta_{a,-c}\langle \underline{h}, b| - \underline{h}, d\rangle - \epsilon(b)\delta_{b,-d}\langle a, \underline{h}|c, -\underline{h}\rangle \\ &- \epsilon(a)\delta_{a,-d}\langle \underline{h}, b|c, -\underline{h}\rangle - \epsilon(b)\delta_{b,-c}\langle a, \underline{h}| - \underline{h}, d\rangle \\ &- \epsilon(a)\delta_{a,-b}\langle \underline{h}, -\underline{h}|c, d\rangle - \epsilon(c)\delta_{c,-d}\langle a, b|\underline{h}, -\underline{h}\rangle\big\}.\end{aligned}$$

Furthermore, we compute

$$\begin{aligned}&C\langle \underline{h}, b| - \underline{h}, d\rangle \\ &= \sum_{(h)} \epsilon(h)\frac{1}{2g+2}\big\{4g\langle h, b| - h, d\rangle + 2\langle -h, b|h, d\rangle + 2\langle h, -h|b, d\rangle \\ &\quad - \epsilon(h)\delta_{h,h}\langle \underline{f}, b| - \underline{f}, d\rangle - \epsilon(b)\delta_{b,-d}\langle h, \underline{f}| - h, -\underline{f}\rangle \\ &\quad - \epsilon(h)\delta_{h,-d}\langle \underline{f}, b| - h, -\underline{f}\rangle - \epsilon(b)\delta_{b,h}\langle h, \underline{f}| - \underline{f}, d\rangle \\ &\quad - \epsilon(h)\delta_{h,-b}\langle \underline{f}, -\underline{f}| - h, d\rangle - \epsilon(-h)\delta_{-h,-d}\langle h, b|\underline{f}, -\underline{f}\rangle\big\} \\ &= \frac{1}{2g+2}\big\{4g\langle \underline{h}, b| - \underline{h}, d\rangle + 2\langle -\underline{h}, b|\underline{h}, d\rangle + 2\langle \underline{h}, -\underline{h}|b, d\rangle\end{aligned}$$

$$- 2g\langle \underline{f}, b| - \underline{f}, d\rangle - \epsilon(b)\delta_{b,-d}\langle \underline{h}, \underline{f}| - \underline{h}, -\underline{f}\rangle$$
$$- \langle \underline{f}, b|d, -\underline{f}\rangle - \langle b, \underline{f}| - \underline{f}, d\rangle - \langle \underline{f}, -\underline{f}|b, d\rangle + \langle d, b|\underline{f}, -\underline{f}\rangle\big\}$$
$$= \frac{1}{2g+2}\big\{2g\langle \underline{h}, b| - \underline{h}, d\rangle - \epsilon(b)\delta_{b,-d}\langle \underline{h}, \underline{f}| - \underline{h}, -\underline{f}\rangle\big\},$$

and thus

$$C\langle \underline{h}, \underline{f}| - \underline{h}, -\underline{f}\rangle$$
$$= \sum_{(f)} \epsilon(f)\frac{1}{2g+2}\big\{2g\langle \underline{h}, f| - \underline{h}, -f\rangle - \epsilon(f)\delta_{f,f}\langle \underline{h}, \underline{i}| - \underline{h}, -\underline{i}\rangle\big\} = 0.$$

Since

$$f_{IH}(x_a \otimes x_b \otimes x_c \otimes x_d) = f_I(x_a \otimes x_b \otimes x_c \otimes x_d) - f_H(x_a \otimes x_b \otimes x_c \otimes x_d)$$
$$= \langle a, b|c, d\rangle - \langle a, c|d, b\rangle,$$

according to the above calculations, we conclude the action of C on the general element $f_{IH}(x_a \otimes x_b \otimes x_c \otimes x_d)$ of $\mathrm{Im} f_{IH} \subset \bigwedge^2(\bigwedge^3 H)$ as in the following lemma:

Lemma 4.1.

(i) $$C\big(\langle a, b|c, d\rangle - \langle a, c|d, b\rangle\big)$$
$$= \frac{2g+1}{g+1}\big\{\langle a, b|c, d\rangle - \langle a, c|d, b\rangle\big\}$$
$$- \frac{1}{2(g+1)}\Big\{ \begin{array}{l} \epsilon(a)\delta_{a,-c}\big(\langle \underline{h}, b| - \underline{h}, d\rangle - \langle \underline{h}, -\underline{h}|d, b\rangle\big) \\ + \epsilon(a)\delta_{a,-b}\big(\langle \underline{h}, -\underline{h}|c, d\rangle - \langle \underline{h}, c|d, -\underline{h}\rangle\big) \\ + \epsilon(b)\delta_{b,-d}\big(\langle a, \underline{h}|c, -\underline{h}\rangle - \langle a, c| - \underline{h}, \underline{h}\rangle\big) \\ + \epsilon(c)\delta_{c,-d}\big(\langle a, b|\underline{h}, -\underline{h}\rangle - \langle a, \underline{h}| - \underline{h}, b\rangle\big) \end{array} \Big\}.$$

(ii) $$C\big(\langle \underline{h}, b| - \underline{h}, d\rangle - \langle \underline{h}, -\underline{h}|d, b\rangle\big)$$
$$= \frac{g}{g+1}\big\{\langle \underline{h}, b| - \underline{h}, d\rangle - \langle \underline{h}, -\underline{h}|d, b\rangle\big\}$$
$$- \frac{1}{2(g+1)}\big\{\epsilon(b)\delta_{b,-d}\big(\langle \underline{h}, \underline{f}| - \underline{h}, -\underline{f}\rangle - \langle \underline{h}, -\underline{h}| - \underline{f}, \underline{f}\rangle\big)\big\}.$$

(iii) $$C\big(\langle \underline{h}, \underline{f}| - \underline{h}, -\underline{f}\rangle - \langle \underline{h}, -\underline{h}| - \underline{f}, \underline{f}\rangle\big) = 0.$$

Letting C act on the above results, and applying Lemma 4.1 again, we obtain the action of C^2 as follows:

Lemma 4.2.

(i) $$C^2\big(\langle a, b|c, d\rangle - \langle a, c|d, b\rangle\big)$$

$$= \frac{(2g+1)^2}{(g+1)^2}\big\{\langle a,b|c,d\rangle - \langle a,c|d,b\rangle\big\}$$
$$- \frac{3g+1}{2(g+1)^2}\Big\{ \epsilon(a)\delta_{a,-c}\big(\langle \underline{h},b| -\underline{h},d\rangle - \langle \underline{h},-\underline{h}|d,b\rangle\big) + \epsilon(a)\delta_{a,-b}\big(\langle \underline{h},-\underline{h}|c,d\rangle - \langle \underline{h},c|d,-\underline{h}\rangle\big) + \epsilon(b)\delta_{b,-d}\big(\langle a,\underline{h}|c,-\underline{h}\rangle - \langle a,c|-\underline{h},\underline{h}\rangle\big) + \epsilon(c)\delta_{c,-d}\big(\langle a,b|\underline{h},-\underline{h}\rangle - \langle a,\underline{h}|-\underline{h},b\rangle\big)\Big\}$$
$$+ \frac{1}{2(g+1)^2}\Big\{ \epsilon(ab)\delta_{a,-c}\delta_{b,-d}\big(\langle \underline{h},\underline{f}|-\underline{h},-\underline{f}\rangle - \langle \underline{h},-\underline{h}|-\underline{f},\underline{f}\rangle\big) + \epsilon(ac)\delta_{a,-b}\delta_{c,-d}\big(\langle \underline{h},-\underline{h}|\underline{f},-\underline{f}\rangle - \langle \underline{h},\underline{f}|-\underline{f},-\underline{h}\rangle\big)\Big\}.$$

(ii) $C^2\big(\langle \underline{h},b|-\underline{h},d\rangle - \langle \underline{h},-\underline{h}|d,b\rangle\big)$
$$= \frac{g^2}{(g+1)^2}\big\{\langle \underline{h},b|-\underline{h},d\rangle - \langle \underline{h},-\underline{h}|d,b\rangle\big\} - \frac{g}{2(g+1)^2}\big\{\epsilon(b)\delta_{b,-d}\big(\langle \underline{h},\underline{f}|-\underline{h},-\underline{f}\rangle - \langle \underline{h},-\underline{h}|-\underline{f},\underline{f}\rangle\big)\big\}.$$

(iii) $C^2\big(\langle \underline{h},\underline{f}|-\underline{h},-\underline{f}\rangle - \langle \underline{h},-\underline{h}|-\underline{f},\underline{f}\rangle\big) = 0.$

According to the above lemmas, we conclude the action of

$$\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}} = \frac{g+1}{2g+1}C^2 - \frac{g}{2g+1}C$$

on the general element of $\mathrm{Im} f_{IH} \subset \bigwedge^2(\bigwedge^3 H)$ as follows:

Lemma 4.3.

(i) $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}}\big(\langle a,b|c,d\rangle - \langle a,c|d,b\rangle\big)$
$$= \langle a,b|c,d\rangle - \langle a,c|d,b\rangle$$
$$- \frac{1}{2(g+1)}\Big\{ \epsilon(a)\delta_{a,-c}\big(\langle \underline{h},b| -\underline{h},d\rangle - \langle \underline{h},-\underline{h}|d,b\rangle\big) + \epsilon(a)\delta_{a,-b}\big(\langle \underline{h},-\underline{h}|c,d\rangle - \langle \underline{h},c|d,-\underline{h}\rangle\big) + \epsilon(b)\delta_{b,-d}\big(\langle a,\underline{h}|c,-\underline{h}\rangle - \langle a,c|-\underline{h},\underline{h}\rangle\big) + \epsilon(c)\delta_{c,-d}\big(\langle a,b|\underline{h},-\underline{h}\rangle - \langle a,\underline{h}|-\underline{h},b\rangle\big)\Big\}$$
$$+ \frac{1}{2(g+1)(2g+1)}\Big\{ \epsilon(ab)\delta_{a,-c}\delta_{b,-d}\big(\langle \underline{h},\underline{f}|-\underline{h},-\underline{f}\rangle - \langle \underline{h},-\underline{h}|-\underline{f},\underline{f}\rangle\big) + \epsilon(ac)\delta_{a,-b}\delta_{c,-d}\big(\langle \underline{h},-\underline{h}|\underline{f},-\underline{f}\rangle - \langle \underline{h},\underline{f}|-\underline{f},-\underline{h}\rangle\big)\Big\}.$$

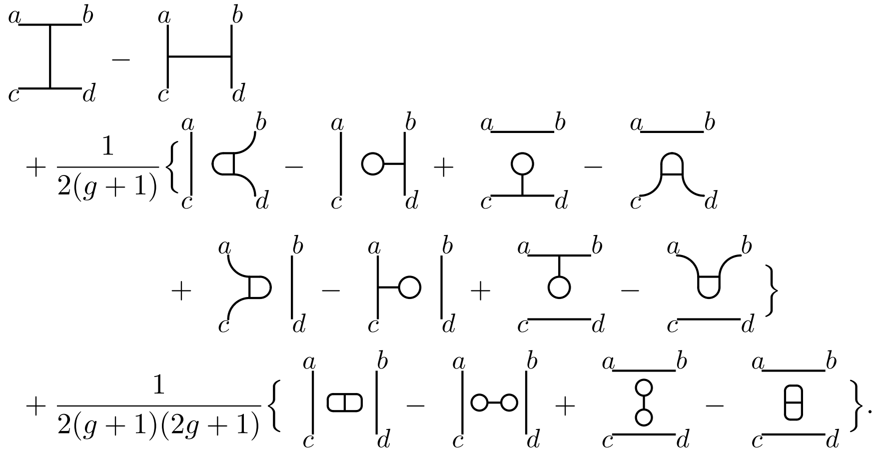
(ii) $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}}\big(\langle \underline{h},b|-\underline{h},d\rangle - \langle \underline{h},-\underline{h}|d,b\rangle\big) = 0.$

(iii) $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}}\big(\langle \underline{h},\underline{f}|-\underline{h},-\underline{f}\rangle - \langle \underline{h},-\underline{h}|-\underline{f},\underline{f}\rangle\big) = 0.$

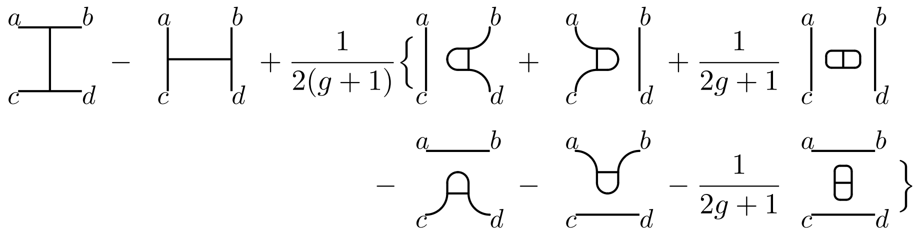
4.4. **Completion of the proof of Theorem 1.2 (and Theorem 1.1).** Returning to the situation §4.2 (4.1), we shall consider the $\mathfrak{sp}$-invariants arising in the sequence of surjections

$$H^{\otimes 6m} \longrightarrow \mathrm{Im}(\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}} \circ f_{IH}) \otimes \bigwedge^{2m-2}(\bigwedge^3 H) \longrightarrow J_{2m} \subset \bigwedge^{2m}(\bigwedge^3 H).$$

Taking into consideration the $\wedge$-admissibility, we may translate Lemma 4.3 (i) into the language of graph invariants as follows:



This means the type of the graph invariants generating the ideal IH^*. Since any graphs having loops vanish in $(\bigwedge U)^{\mathfrak{sp}}$, we get the type of graph invariants generating $IH_0^{bis} = ([2^2]_{\mathfrak{sp}})^{\mathfrak{sp}} \subset \bigwedge U$:



which appeared in the theorem of [3]. This completes the proofs of Theorem 1.1 and Theorem 1.2. □

5. Three type IH-relations

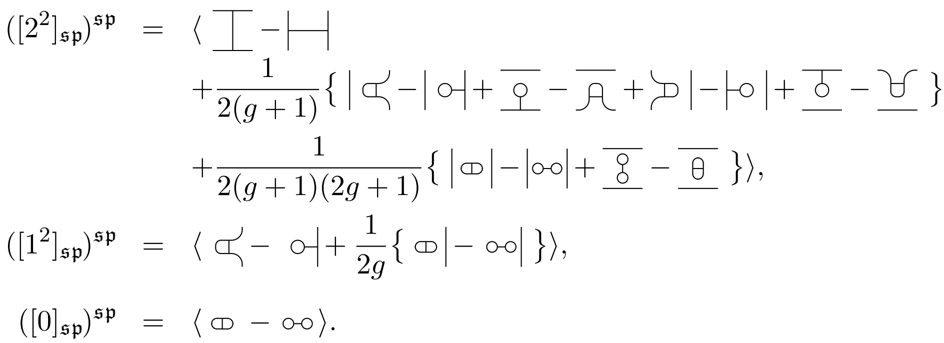
As shown in §4.1, we can compute the projections from $[2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}$ into the $[1^2]_{\mathfrak{sp}}$ or $[0]_{\mathfrak{sp}}$-part as

$$\mathrm{Pr}_{[1^2]_{\mathfrak{sp}}} = -\frac{g+1}{g}C^2 + \frac{2g+1}{g}C,$$

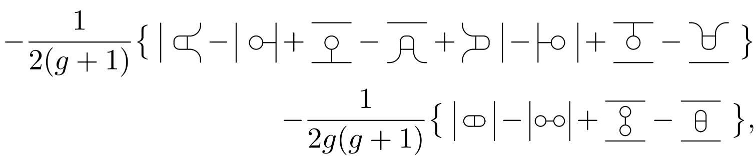
$$\mathrm{Pr}_{[0]_{\mathfrak{sp}}} = \frac{(g+1)^2}{g(2g+1)}C^2 - \frac{(g+1)(3g+1)}{g(2g+1)}C + 1,$$

respectively. Now, we summarize the images of the three types IH-relation via the projections to each part of $\mathrm{Im} f_{IH}$ as the following theorem.

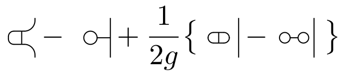
Theorem 5.1. *The ideals* $([2^2]_{\mathfrak{sp}})^{\mathfrak{sp}}$, $([1^2]_{\mathfrak{sp}})^{\mathfrak{sp}}$, $([0]_{\mathfrak{sp}})^{\mathfrak{sp}} \subset (\mathrm{Im} f_{IH})^{\mathfrak{sp}} \subset (\bigwedge(\bigwedge^3 H))^{\mathfrak{sp}}$ *are respectively generated by* $\mathfrak{sp}$*-invariants corresponding to the graphs of following types:*



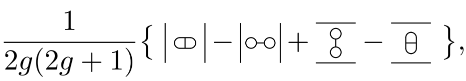
Proof. The first line comes from the proof of Theorem 1.2. The other two lines follow from similar computations. In fact, the $\mathfrak{sp}$-invariants of the images of the projection $\mathrm{Pr}_{[1^2]_{\mathfrak{sp}}}$ are respectively given by the graph



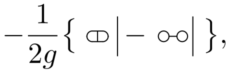
for the type IH_0, and

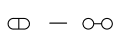


for the type IH_1, and 0 for the type IH_2. See §4.2 for the definitions of IH_0, IH_1, IH_2. Since the graph for IH_0 consists of the graphs for IH_1, the second line follows. The $\mathfrak{sp}$-invariants of the images of the projection $\mathrm{Pr}_{[0]_{\mathfrak{sp}}}$ are respectively translated into the graph

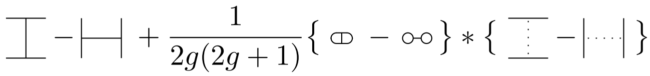


for the type IH_0, and

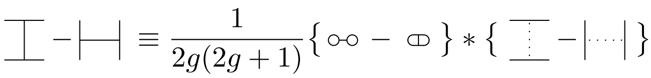


for the type IH_1, and  for the type IH_2. The third line follows similarly to the second. Thus we complete the proof. □

Remark. The third line of Theorem 5.1 is given in [5, Proposition 10.2]. If we argue in the similar way using $\mathrm{Pr}_{[2^2]_{\mathfrak{sp}}\oplus[1^2]_{\mathfrak{sp}}}$, then we see that the graph invariants



generate $([2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}})^{\mathfrak{sp}}$. Namely, we have



in $(\bigwedge(\bigwedge^3 H)/([2^2]_{\mathfrak{sp}}\oplus[1^2]_{\mathfrak{sp}}))^{\mathfrak{sp}}$. This gives an alternative proof of the proposition presented by N. Kawazumi and S. Morita in [5, Proposition 11.1].

By Theorem 5.1, we obtain the following:

Corollary 5.2. *Let IH_0, IH_1, IH_2 denote the ideal generated by the graph invariants for the type IH_0, IH_1, IH_2 of §4.2 respectively. Then we stably have*

$$\begin{aligned}(\mathrm{Im} f_{IH})^{\mathfrak{sp}} = ([2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}})^{\mathfrak{sp}} &\cong IH_2 + IH_1 + IH_0 = IH,\\ ([1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}})^{\mathfrak{sp}} &\cong IH_2 + IH_1,\\ ([0]_{\mathfrak{sp}})^{\mathfrak{sp}} &\cong IH_2.\end{aligned}$$

Acknowledgement

The present paper has been submitted to the Graduate School of Natural Science and Technology at Okayama University as part of the author's doctral thesis. The author would like to thank his advisor Professor Hiroaki Nakamura very much for valuable comments and suggestions.

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Hiroki Akazawa
Meisei Gakuin High School,
3-5-1 Tsushima-Nishizaka,
Okayama, 700-0086 Japan

(*Received March 14, 2005*)