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RAMANUJAN'S SUMS AND CYCLOTOMIC POLYNOMIALS

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This paper consists of three parts. First, we shall show that some formulas about Ramanujan's sum and cyclotomic polynomials are equivalent to the definition of cyclotomic polynomials. Next, we shall show also that cyclotomic polynomials are strictly increasing for $x \geq \frac{3}{2}$. The last part is to show that values of some classical series are equivalent to first three coefficients of s in the inverse $\frac{1}{\zeta(s+1)}$ of the zeta function.

Let Γ_n be the cyclic group of n th roots of unity in the complex number field $\mathbb{C}$ and let Δ_n be the set of primitive n th roots of unity. Then we have

$$\Gamma_n = \{\zeta_n^k \mid 0 \leq k \leq n-1\} \quad \text{and} \quad \Delta_n = \{\zeta_n^k \mid (k,n) = 1, 1 \leq k \leq n\}$$

where $\zeta_n = e^{\frac{2\pi i}{n}}$ and (k,n) is the greatest common divisor of k and n.

Classifying elements of Γ_n by the orders, Γ_n is the disjoint union of subsets Δ_d with $d|n$, namely,

$$\Gamma_n = \bigcup_{d|n} \Delta_d.$$

Let n be a natural number and let k be an integer. Then Ramanujan's sum $c_n(k)$ is defined as follows (see [1, p.55-56]):

$$c_n(k) = \sum_{\eta \in \Delta_n} \eta^k.$$

§1. From the definition of Ramanujan's sum, we can see immediately some statements in Lemma 1. For these statements, we use the following notation

$$\delta_{k|n} = \begin{cases} 1 & \text{if } k|n \\ 0 & \text{otherwise.} \end{cases}$$

From (1) in Lemma 1, it is natural to consider a polynomial with coefficients of Ramanujan's sums, namely,

$$R_n(x) = c_n(0) + c_n(1)x + c_n(2)x^2 + \cdots + c_n(n-1)x^{n-1}.$$

Lemma 1. *Let φ denote Euler's function, let μ be Möbius function. Then we have*

(1) $c_n(k) = c_n(\ell)$ *if* $k \equiv \ell \bmod n$.

(2) $R_n(0) = c_n(0) = \varphi(n)$.

(3) $c_n(k\ell) = c_n(\ell)$ *if* $(k,n) = 1$.

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(4) $c_n(k) = c_n(-k) = c_n(n-k)$.
(5) $R_n(1) = \sum_{k=0}^{n-1} c_n(k) = 0$.
(6) $\sum_{d|n} c_d(k) = \delta_{n|k} n$, *namely*, $c_n(k) = \sum_{d|n} \mu(\frac{n}{d}) d\delta_{d|k}$ (*see* [1, p.237]).
(7) $c_n(k) = c_n(\ell)$ *if* $(k,n) = (\ell, n)$.
(8) $c_n(k) = c_n(1) = \mu(n)$ *if* $(k,n) = 1$.
(9) $c_{st}(k) = c_s(k)c_t(k)$ *if* $(s,t) = 1$ (*see* [1, p.56]).
(10) $\frac{R_n(x)}{1-x^n} = \sum_{k\geq 0} c_n(k)x^k = \sum_{d|n} \mu(\frac{n}{d}) \frac{d}{1-x^d}$.

Proof. (1) $\sim$ (4) are easy from the definition.

(5) follows from the next equations.

$$\sum_{k=0}^{n-1} c_n(k) = \sum_{k=0}^{n-1} \sum_{\eta\in\Delta_n} \eta^k = \sum_{\eta\in\Delta_n} \{\sum_{k=0}^{n-1} \eta^k\} = 0$$

where η runs through the primitive nth roots of unity.

(6): Classifying elements by orders in the cyclic group $\Gamma_n = \bigcup_{d|n} \Delta_d$, we have

$$\sum_{d|n} c_d(k) = \sum_{d|n} \sum_{\theta\in\Delta_d} \theta^k = \sum_{\theta\in\Gamma_n} \theta^k = \sum_{\ell=0}^{n-1} (\zeta_n^k)^\ell = \delta_{n|k} n.$$

(7) and (8) are easy from (6).

(9): It is clear that $(k, st) = (k,s)(k,t)$ and for every divisor d of st, there exist d_1, d_2 such that $d = d_1 d_2,\ d_1 \mid s$ and $d_2 \mid t$, and conversely. Thus (9) follows from (6) and the following equations.

$$c_{st}(k) = \sum_{d|(k,st)} \mu(\frac{st}{d}) d = (\sum_{d_1|(k,s)} \mu(\frac{s}{d_1}) d_1)(\sum_{d_2|(k,t)} \mu(\frac{t}{d_2}) d_2) = c_s(k)c_t(k).$$

(10) follows from the following equations.

$$\begin{aligned} R_n(x) &= \sum_{k=0}^{n-1} (\sum_{\eta\in\Delta_n} \eta^k) x^k = \sum_{\eta\in\Delta_n} \sum_{k=0}^{n-1} (\eta x)^k \\ &= \sum_{\eta\in\Delta_n} \frac{1-(\eta x)^n}{1-\eta x} = (1-x^n) \sum_{\eta\in\Delta_n} \sum_{k\geq 0} (\eta x)^k \\ &= (1-x^n) \sum_{k\geq 0} c_n(k) x^k. \end{aligned}$$

Moreover, we have from (6).

$$\begin{aligned} R_n(x) &= \sum_{k=0}^{n-1} c_n(k)x^k = \sum_{k=0}^{n-1}\sum_{d|n} \mu(\frac{n}{d})d\delta_{d|k}x^k = \sum_{d|n}\mu(\frac{n}{d})d\sum_{k=0}^{n-1}\delta_{d|k}x^k \\ &= \sum_{d|n}\mu(\frac{n}{d})d\sum_{\ell=0}^{\frac{n}{d}-1} x^{d\ell} = \sum_{d|n}\mu(\frac{n}{d})d\frac{x^n-1}{x^d-1}. \end{aligned}$$

□

Remarks. 1). The formula in Lemma 1 (6) is very useful.

$$c_n(k) = \sum_{d|n}\mu(\frac{n}{d})d\delta_{d|k}.$$

This formula shows that $c_n(k)$ is an integer and $c_n(k)$ represents important arithmetic functions, in fact, about Möbius μ function, we can see $\mu(n) = c_n(1)$ from this and we have the well known important property:

$$\sum_{d|n}\mu(d) = \sum_{d|n} c_d(1) = n\delta_{n|1} = \begin{cases} n & \text{for } n = 1 \\ 0 & \text{for } n > 1. \end{cases}$$

About Euler φ function, we have $\varphi(n) = c_n(0)$ from the definition of $c_n(k)$ and we have usual formula

$$\varphi(n) = c_n(0) = \sum_{d|n}\mu(\frac{n}{d})d \text{ and } n = n\delta_{n|0} = \sum_{d|n} c_d(0) = \sum_{d|n}\varphi(d).$$

2). From Lemma 1 (6) and (9), we can compute $c_n(k)$ by the prime factorizations of n and k as follows: In case $n = p^r$ is a power of a prime p and $k = p^s k_0$ with $(k_0, p) = 1$, we have

$$c_{p^r}(k) = p^r\delta_{p^r|k} - p^{r-1}\delta_{p^{r-1}|k} = \begin{cases} p^r - p^{r-1} & \text{for } s \ \geq r \\ -p^{r-1} & \text{for } s = r-1 \\ 0 & \text{for } s < r-1. \end{cases}$$

In general, for prime factorizations of $n = \prod_{i=1}^{s} p_i^{r_i}$, using Lemma 1 (9), we have $c_n(k) = \prod_{i=1}^{s} c_{p_i^{r_i}}(k)$. Thus we have the next formula (see [1, p. 238]) from this consideration,

$$c_n(k) = \frac{\mu(\frac{n}{(k,n)})\varphi(n)}{\varphi(\frac{n}{(k,n)})}.$$

This means that Ramanujan's sums $c_n(m)$ are equivalent to Euler's function $\varphi(n)$ + Möbius function $\mu(n)$.

3). Moreover, from Lemma 1 (6), we have (see [1, p. 250-251]).

$$\zeta(s)\sum_{n\geq 1}\frac{c_n(k)}{n^s} = \sum_{n\geq 1}\frac{\sum_{d|n}c_d(k)}{n^s} = \sum_{n\geq 1}\frac{\delta_{n|k}n}{n^s}$$
$$= \begin{cases} \frac{\sigma_{s-1}(k)}{k^{s-1}} & \text{for } k>0 \text{ and } s>1 \\ \zeta(s-1) & \text{for } k=0 \text{ and } s>2 \end{cases}$$

where $\zeta(s)$ is the Riemann zeta function and $\sigma_\ell(m)=\sum_{d|m}d^\ell$.

In particular,

$$\sum_{n\geq 1}\frac{c_n(k)}{n^2}=\frac{6}{\pi^2}\frac{\sigma_1(k)}{k} \text{ for } k>0 \quad\text{and}\quad \sum_{n\geq 1}\frac{\varphi(n)}{n^s}=\frac{\zeta(s-1)}{\zeta(s)} \text{ for } s>2.$$

4). In Lemma 1, (6) and (10) are equivalent. We can see already that (10) follows from (6). On the other hand, we have (6) from (10) and next equations.

$$\sum_{k\geq 0}c_n(k)x^k=\sum_{d|n}\mu(\frac{n}{d})d\frac{1}{1-x^d}=\sum_{d|n}\mu(\frac{n}{d})d\sum_{\ell\geq 0}x^{d\ell}=\sum_{\ell\geq 0}(\sum_{d|n}\mu(\frac{n}{d})d)x^{d\ell}.$$

In the next theorem we shall show some formulas are equivalent to the definition of cyclotomic polynomials.

Theorem 1. *The next formulas are all equivalent for monic polynomials $\Phi_n(x)$ over $\mathbb{C}$.*

(1) $x^n-1=\prod_{d|n}\Phi_d(x)$, *namely,* $\Phi_n(x)=\prod_{d|n}(x^d-1)^{\mu(\frac{n}{d})}$.
(2) $\Phi_n(x)=\prod_{\eta\in\Delta_n}(x-\eta)$.
(3) $\sum_{m\geq 0}\frac{c_n(m)}{x^m}=x\frac{\Phi'_n(x)}{\Phi_n(x)}$, *where* $\Phi'_n(x)=\frac{d\Phi_n(x)}{dx}$.
(4) $\frac{R_n(x)}{1-x^n}=\frac{1}{x}\frac{\Phi'_n(\frac{1}{x})}{\Phi_n(\frac{1}{x})}$.
(5) $x\frac{\Phi'_n(x)}{\Phi_n(x)}=\sum_{d|n}\mu(\frac{n}{d})\frac{dx^d}{x^d-1}$, *namely,* $\frac{nx^n}{x^n-1}=\sum_{d|n}x\frac{\Phi'_d(x)}{\Phi_d(x)}$.

Proof. (1) ⇒ (2): Let α be of order n. Then there exists a divisor d of n with $\Phi_d(\alpha)=0$ and so $\alpha^d-1=0$, namely, $d=n$. Conversely, let d be the order of β and assume $\Phi_n(\beta)=0$. Then d is a divisor of n. If $d<n$, then we obtain

$$(x^d)^{\frac{n}{d}-1}+\cdots+x^d+1=\frac{x^n-1}{x^d-1}=\frac{\prod_{h|n}\Phi_h(x)}{\prod_{h|d}\Phi_h(x)}=\Phi_n(x)g(x)$$

where $g(x)$ is 1 or a product of cyclotomic polynomials. Thus we have (2) from the next contradiction.

$$\frac{n}{d}=(\beta^d)^{\frac{n}{d}-1}+\cdots+\beta^d+1=\Phi_n(\beta)g(\beta)=0.$$

(2) ⇒ (3): The next equations show (3) from (2)

$$\begin{aligned} x\frac{\Phi_n'(x)}{\Phi_n(x)} &= \sum_{\eta\in\Delta_n}\frac{x}{x-\eta} = \sum_{\eta\in\Delta_n}\frac{1}{1-\frac{\eta}{x}} = \sum_{\eta\in\Delta_n}\sum_{m\geq 0}\frac{\eta^m}{x^m} \\ &= \sum_{m\geq 0}\sum_{\eta\in\Delta_n}\frac{\eta^m}{x^m} = \sum_{m\geq 0}\frac{c_n(m)}{x^m}. \end{aligned}$$

(3) ⇒ (4): From (3) and Lemma 1 (1) (or Lemma 1 (10)), we obtain

$$\frac{R_n(x)}{1-x^n} = \sum_{k\geq 0} R_n(x)x^{nk} = \sum_{k\geq 0} c_n(k)x^k = \frac{1}{x}\frac{\Phi_n'(\frac{1}{x})}{\Phi_n(\frac{1}{x})}.$$

(4) ⇒ (5): It follows from (4) and Lemma 1 (6) that

$$\begin{aligned} x\frac{\Phi_n'(x)}{\Phi_n(x)} &= \frac{R_n(\frac{1}{x})}{1-\frac{1}{x^n}} = \sum_{k\geq 0}\frac{c_n(k)}{x^k} = \sum_{k\geq 0}\sum_{d|n}\mu(\frac{n}{d})d\frac{\delta_{d|k}}{x^k} \\ &= \sum_{d|n}\mu(\frac{n}{d})d\sum_{k\geq 0}\frac{\delta_{d|k}}{x^k} = \sum_{d|n}\mu(\frac{n}{d})d\sum_{\ell\geq 0}\frac{1}{x^{d\ell}} \\ &= \sum_{d|n}\mu(\frac{n}{d})d\frac{1}{1-\frac{1}{x^d}} = \sum_{d|n}\mu(\frac{n}{d})\frac{dx^d}{x^d-1}. \end{aligned}$$

(5) ⇒ (1): We set $S(x) = \prod_{d|n}\Phi_d(x)$. It follows from (5) that

$$S(x)\frac{nx^{n-1}}{x^n-1} = \sum_{d|n} S(x)\frac{\Phi_d(x)'}{\Phi_d(x)} = S(x)' \text{ and so } S(x)'(x^n-1) = S(x)nx^{n-1}.$$

Thus we have

$$\left(\frac{S(x)}{x^n-1}\right)' = 0 \text{ and hence } S(x) = c(x^n-1).$$

Since $S(x)$ is monic, we have our assertion. □

Remark. We shall show again the formulas Lemma 1 (6) from Theorem 1. We can see the next equations from Theorem 1 (5).

$$x\frac{\Phi_n'(x)}{\Phi_n(x)} = \sum_{d|n}\mu(\frac{n}{d})d\frac{1}{1-\frac{1}{x^d}} = \sum_{d|n}\mu(\frac{n}{d})d\sum_{k\geq 0}\frac{1}{x^{kd}} = \sum_{k\geq 0}\sum_{d|n}\frac{\mu(\frac{n}{d})d}{x^{kd}}.$$

Thus we obtain from Theorem 1 (3)

$$\sum_{m\geq 0}\frac{c_n(m)}{x^m} = \sum_{k\geq 0}\sum_{d|n}\frac{\mu(\frac{n}{d})d}{x^{kd}}.$$

The formula in Lemma 1 (6) also follows from the above equation.

$$c_n(m) = \sum_{d|m,\, d|n} \mu(\frac{n}{d})d = \sum_{d|n} \mu(\frac{n}{d})d\delta_{d|m}.$$

§2. We stated in [5, Corollary] that cyclotomic polynomials are strictly increasing for $x > 1$. However, it was incorrect and we restated in [6, Theorem 3] that this hold for $x \geq 2$. We firmly believe that it is valid for $x > 1$. However we cannot prove now it. It is equivalent to $R_n(x) > 0$ for $1 > x > 0$ by Theorem 1 (4), and it is also equivalent to $R_n(x)' \leq 0$ for $1 > x > 0$ because $R_n(0) = \varphi(n)$ and $R_n(1) = 0$. In this section, we shall prove that it is true for $x \geq \frac{3}{2}$.

Proposition 1. *Let $n > 1$ be a natural number and let r be the number of distinct odd prime factors in n. Then we have $\Phi_n(x)$ is strictly increasing for $x \geq 1 + \frac{1}{2 \cdot r!}$. In particular, $\Phi_n(x)$ is strictly increasing for $x \geq \frac{3}{2}$.*

Proof. Let t be the product of distinct primes dividing n. Then we have $\Phi_n(x) = \Phi_t(x^{\frac{n}{t}})$. Thus we may assume n is square free. The condition $x \geq 1 + \frac{1}{2 \cdot r!}$ is equivalent to

$$2 \cdot r! \geq \frac{1}{x-1}.$$

We set $f(x) = x^d - 1 - dx + d$. Then $f(1) = 0$ and $f(x)$ is not decreasing for $x \geq 1$. Thus we have the next inequality

$$\frac{1}{x-1} \geq \frac{d}{x^d - 1} > \frac{d}{x^d + 1} \quad \text{for a natural number } d.$$

We set $n = p_0 m$ and $m = p_1 p_2 \cdots p_r$ where $p_0 = 1$ or 2, and $p_1, p_2, \ldots, p_r$ are distinct odd primes. Then we have

$$\varphi(n) = \varphi(m) = (p_1 - 1)(p_2 - 1) \cdots (p_r - 1) \geq 2 \cdot 4 \cdot 6 \cdots 2r = 2^r \cdot r!.$$

In case $p_0 = 1$, we have the next inequality from the above and $\Phi_n(x) = \prod_{d|n}(x^d - 1)^{\mu(\frac{n}{d})}$.

$$\begin{aligned} x\frac{\Phi_n'(x)}{\Phi_n(x)} &= \sum_{d|n} \mu(\frac{n}{d})\frac{dx^d}{x^d - 1} = \sum_{d|n} \mu(\frac{n}{d})d(1 + \frac{1}{x^d - 1}) \\ &= \varphi(n) + \sum_{d|n} \mu(\frac{n}{d})\frac{d}{x^d - 1} \\ &> \varphi(n) - \sum_{\mu(\frac{n}{d}) = -1} \frac{1}{x-1} = \varphi(n) - \frac{2^{r-1}}{x-1} \\ &\geq \varphi(n) - 2^r \cdot r! \geq 0. \end{aligned}$$

In case $p_0 = 2$, we have the next inequality from the above and $\Phi_n(x) = \Phi_{2m}(x) = \prod_{d|m}(x^d+1)^{\mu(\frac{m}{d})}$.

$$\begin{aligned} x\frac{\Phi_n'(x)}{\Phi_n(x)} &= \sum_{d|m}\mu(\frac{m}{d})\frac{dx^d}{x^d+1} = \sum_{d|m}\mu(\frac{m}{d})d(1-\frac{1}{x^d+1}) \\ &= \varphi(m) - \sum_{d|m}\mu(\frac{m}{d})\frac{d}{x^d+1} \\ &> \varphi(m) - \sum_{\mu(\frac{m}{d})=1}\frac{1}{x-1} = \varphi(n) - \frac{2^{r-1}}{x-1} \\ &\geq \varphi(n) - 2^r \cdot r! \geq 0. \end{aligned}$$

□

§3. In this section, we shall show that values of next series $u(k)$ for $k = 0, 1, 2$ are equal to first three coefficients of s in the inverse $\frac{1}{\zeta(s+1)}$ of the zeta function. These results are closely related to the prime number theorem (see [3]).

We define the next series and a sum. Since $u(k)$ is convergent (see [3, pp.594-595, §158]), $r_\ell(k)$ is convergent (see Lemma 2 (2)).

$$r_\ell(k) := \sum_{n\geq 1}\frac{c_n(\ell)(\log n)^k}{n}.$$

$$u(k) := r_1(k) = \sum_{n\geq 1}\frac{\mu(n)(\log n)^k}{n}.$$

$$v_\ell(k) := \sum_{d|\ell}(\log d)^k.$$

Concerning three values $u(k)$ for $k = 0, 1, 2$, $u(0) = 0$ was proved by Mangoldt 1898 [4], $u(1) = -1$ by Landau 1899 [2, 3] and these general, but equivalent, results were proved by Ramanujan 1918 [7], in addition to $u(2) = -2\gamma$ where γ is Euler's constant. However, we can see from Theorem 3 that these values were already found by Mangoldt 1898 ([4, p.440]). These are combined by the next lemma.

Lemma 2. *We obtain assertions.*

(1) $$\frac{1}{\zeta(s+1)} = \sum_{k=0}^{m}(-1)^k u(k)\frac{s^k}{k!} + O(s^{m+1}) \text{ as } s \to 0.$$

(2) $$r_\ell(k) = \sum_{t=0}^{k}\binom{k}{t}u(t)v_\ell(k-t).$$

Proof. (1) We set $f(s)=\frac{1}{\zeta(s+1)}$. It is well known that

$$f^{(k)}(t)=(-1)^k\sum_{n\geq 1}\frac{\mu(n)(\log n)^k}{n^{t+1}} \text{ for } t>0.$$

Since $u(k)$ is convergent from [3, pp.594-595, §158] and $f^{(k)}(t)$ is right continuous at a point 0 from [3, p.107, §30], we have

$$\lim_{t\to+0}f^{(k)}(t)=(-1)^k u(k).$$

On the other hand, for $s>0$ in the neighbourhood of $t>0$,

$$f(s)=\sum_{k=0}^{m}f^{(k)}(t)\frac{(s-t)^k}{k!}+O((s-t)^{m+1}).$$

Hence, making $t\to+0$, we have

$$f(s)=\sum_{k=0}^{m}(-1)^k u(k)\frac{s^k}{k!}+O(s^{m+1}) \text{ as } s\to 0.$$

(2) Using Lemma 1 (6), we obtain (2) from the next equations.

$$\begin{aligned}
r_\ell(k) &= \sum_{n\geq 1}\frac{c_n(\ell)(\log n)^k}{n}=\sum_{n\geq 1}\{\sum_{h|n,h|\ell}\mu(\frac{n}{h})h\}\frac{(\log n)^k}{n}\\
&= \sum_{h'\geq 1}\sum_{h|\ell}\mu(h')\frac{(\log h'+\log h)^k}{h'}\\
&= \sum_{h'\geq 1}\sum_{h|\ell}\frac{\mu(h')}{h'}\sum_{t=0}^{k}\binom{k}{t}(\log h')^t(\log h)^{k-t}\\
&= \sum_{t=0}^{k}\binom{k}{t}\{\sum_{h'\geq 1}\frac{\mu(h')}{h'}(\log h')^t\}\{\sum_{h|\ell}(\log h)^{k-t}\}\\
&= \sum_{t=0}^{k}\binom{k}{t}u(t)v_\ell(k-t).
\end{aligned}$$

□

The next was stated in [4, p. 440].

Theorem 2 (Mangoldt). $\dfrac{1}{\zeta(s+1)}=s-\gamma s^2+O(s^3)$ *as* $s\to 0$.

Proof. It is well known from [4, p. 440](see also [3, p.164]) that

$$\zeta(s+1) = \frac{1}{s} + \gamma + O(s) \text{ as } s \to 0.$$

Thus we have

$$\frac{1}{\zeta(s+1)} = \frac{s}{1+\gamma s + O(s^2)} = s - \gamma s^2 + O(s^3) \text{ as } s \to 0.$$

□

The following theorem is our purpose in this section.

Theorem 3. *Let $d(\ell)$ be the number of positive divisors of ℓ and let γ be Euler's constant. Then we obtain equivalent results.*

(a) $\frac{1}{\zeta(s+1)} = s - \gamma s^2 + O(s^3)$ *as* $s \to 0$.
(b) $u(0) = 0,\ u(1) = -1$ *and* $u(2) = -2\gamma$.
(c) $r_\ell(0) = 0, r_\ell(1) = -d(\ell)$ *and* $r_\ell(2) = -d(\ell)(2\gamma + \log \ell)$.

Proof. (a) ⇔ (b): It is clear from Lemma 2(1).

(b) ⇔ (c): (b) is a special case of (c) because $u(k) = r_1(k)$. The first two equations in (c) are easy from (b) and Lemma 2(2). Using $(\prod_{t|\ell} t)^2 = \prod_{t|\ell} t\frac{\ell}{t} = \ell^{d(\ell)}$, the last equation in (c) follows from (b), Lemma 2(2) and the next equations.

$$\begin{aligned} r_\ell(2) &= u(2)v_\ell(0) + 2u(1)v_\ell(1) + u(0)v_\ell(2) = -2\gamma d(\ell) - 2\sum_{t|\ell} \log t \\ &= -2\gamma d(\ell) - 2\log(\prod_{t|\ell} t) = -2\gamma d(\ell) - d(\ell)\log \ell. \end{aligned}$$

□

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