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SOME METRIC INVARIANTS OF SPHERES AND ALEXANDROV SPACES I

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Abstract. A metric invariant a_k is defined, and we have that $a_k(X) \leq a_k(S^n)$ holds in an Alexandrov space X with curvature ≥ 1. And the borderline case when $a_3(X) = a_3(S^n)$ and $a_k(S^1)$ are studied.

1. Introduction

The purpose of this paper is to study the behavior of some metric invariants on spheres and Alexandrov spaces. Let X be a compact metric space, where the distance between $x, y \in X$ will be denoted by $dist(x, y)$. Then the metric invariants, e.g., the diameter $diamX = \max_{x,y \in X} dist(x, y)$, the radius $radX = \min_{x \in X} \max_{y \in X} dist(x, y)$ played an important role in Riemannian Alexandrov geometry([G-P1],[B-G-P]). Now, S.Shteingold introduced the notion of k-covering radius $cov_k X = \min_{x_1,\ldots,x_k \in X} \max_{x \in X} min_{i=1,\ldots,k} dist(x_i, x)$ and studied its behavior in Alexandrov spaces with curvature ≥ 1([S]). Here we introduce the following metric invariant $a_k(X)$ related to the k-covering radius.

Definition 1.1. *For a positive integer k, we define the metric invariant $a_k(X)$ of X as follows:*

$$a_k(X) = \min_{x_1,\ldots,x_k \in X} \max_{x \in X} \frac{1}{k} \sum_{i=1}^{k} dist(x_i, x). \tag{1.1}$$

Note that $a_1(X) = \min_{x_1 \in X} \max_{x \in X} dist(x_1, x)$ is nothing but the radius of X, and we have $a_1(X) \geq a_k(X) \geq cov_k(X)$.

We want to study $a_k(X)$ in an Alexandrov space X with curvature ≥ 1, and begin with the case of the n-dimensional unit sphere S^n of constant curvature 1 as the model space. We have as our first result for S^1 the following theorem.

Theorem 1.1. (1) *For $k = 2p - 1$, we have*

$$a_k(S^1) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi. \tag{1.2}$$

$a_k(S^1)$ is realized if and only if a configuration $(x_1, \cdots, x_k)$ of k points is equally spaced in S^1, and $\max_{x \in S^1}(1/k) \sum_{i=1}^{k} dist(x, x_i)$ is attained exactly at the antipodal points of $x_i (1 \leq i \leq k)$.

(2) *For $k = 2p$, we have*

$$a_k(S^1) = \frac{1}{2}\pi. \tag{1.3}$$

$a_k(S^1)$ is realized if and only if a configuration of k points consists of pairs of antipodal points, and in the case we have $(1/k)\sum_{i=1}^{k} dist(x, x_i) \equiv \pi/2$.

In case of S^n of general dimension, we give the following theorems in this paper.

Theorem 1.2.

$$a_3(S^n) = a_3(S^1) = \frac{5}{9}\pi, \tag{1.4}$$

where $a_3(S^n)$ is realized if and only if 3 points are equally spaced on a great circle, and $\max_{x\in S^n}(1/3)\sum_{i=1}^{3} dist(x, x_i)$ is attained exactly at the antipodal points of $x_i (1 \leq i \leq 3)$.

Theorem 1.3. *For $k = 2p$, we have*

$$a_k(S^n) = \frac{1}{2}\pi. \tag{1.5}$$

Moreover, $a_k(S^n)$ is realized if and only if a configuration of k points consists of pairs of the antipodal points, and in the case we have $(1/k)\sum_{i=1}^{k} dist(x, x_i) \equiv \pi/2$. We say that this configuration is symmetric.

For $k = 2p - 1$, we conjecture that

$$a_k(S^n) = a_k(S^1) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi \tag{1.6}$$

holds, where $a_{2p-1}(S^n)$ is realized if and only if a configuration $(x_1, \cdots, x_{2p-1})$ of $2p-1$ points is equally spaced in a great circle S^1 of S^n, and $\max_{x\in S^n}(1/k)\cdot\sum_{i=1}^{k} dist(x, x_i)$ is attained exactly at the antipodal points of $x_i (1 \leq i \leq k)$.

Next we will explain Alexandrov spaces([B-G-P]). Alexandrov spaces are finite-dimensional, locally compact, and complete intrinsic metric spaces with a lower curvature bound in the local triangle sense. Let $(X, dist)$ be an Alexandrov space. A geodesic or a segment is a curve whose length is equal to the distance between its ends. In a locally compact complete space with intrinsic metric any two points can be joined by a geodesic, which is not necessarily a unique segment. A collection of three points $p, q, r \in X$ and three geodesics pq, qr, rp is called a geodesic triangle $\triangle pqr$. We associate a geodesic triangle $\tilde{\triangle} pqr = \triangle \tilde{p}\tilde{q}\tilde{r}$ on the k-plane M_k^2 with vertices $\tilde{p}, \tilde{q}, \tilde{r}$ and sides of lengths $dist(\tilde{p}, \tilde{q}) = dist(p, q), dist(\tilde{q}, \tilde{r}) = dist(q, r)$, and $dist(\tilde{r}, \tilde{p}) = dist(r, p)$, where a k-plane is a 2-dimensional complete simply-connected Riemannian manifold of constant sectional curvature k.

The most basic tool in Alexandrov geometry is the following Toponogov comparison theorem([B-G-P],[G-W]).

Let X be an $n(\geq 2)$-dimensional Alexandrov space with *curvature* $\geq k$. Then we have the following comparison theorems:

(1) For any triple (p_1, p_2, p_3) in X, there is a unique (up to isometry) triple $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ in M_k^2 with $dist(p_i, p_j) = dist(\tilde{p}_i, \tilde{p}_j)(i, j = 1, 2, 3)$. For a segment p_2p_3:$[0, dist(p_2, p_3)] \longrightarrow X$ and a segment $\tilde{p}_2\tilde{p}_3$ in M_k^2, we have

$$(1.7) \qquad dist(p_1, p_2p_3(t)) \geq dist(\tilde{p}_1, \tilde{p}_2\tilde{p}_3(t))(0 < t < dist(p_2, p_3)).$$

(2) If equality holds in (1.7) for some $0 < t_0 < dist(p_2, p_3)$ and c_{t_0} is a segment from p_1 to $p_2p_3(t_0)$, then $c_{t_0}(s), 0 < s \leq dist(p_1, p_2p_3(t_0))$, is joined to p_2 and p_3 by unique segments. Moreover, these segments, together with their limit segments from p_1 to p_2 and p_3, form a surface which has totally geodesic interior and which is isometric to the triangular surface in M_k^2 with vertices $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$.

(3) For any hinge (p_1p_2, p_1p_3) in X with $0 < \sphericalangle(p_1p_2, p_1p_3) < \pi$, we have

$$(1.8) \qquad dist(p_2, p_3) \leq dist(\tilde{p}_2, \tilde{p}_3),$$

where $(\tilde{p}_1\tilde{p}_2, \tilde{p}_1\tilde{p}_3)$ is the corresponding hinge in M_k^2 satisfying $dist(p_1, p_i) = dist(\tilde{p}_1, \tilde{p}_i)(i = 2, 3)$, and $\sphericalangle(p_1p_2, p_1p_3) = \sphericalangle(\tilde{p}_1\tilde{p}_2, \tilde{p}_1\tilde{p}_3)$.

(4) If equality holds in (1.8), then (p_1p_2, p_1p_3) spans a surface which has totally geodesic interior and is isometric to the triangular surface in M_k^2 spanned by $(\tilde{p}_1\tilde{p}_2, \tilde{p}_1\tilde{p}_3)$. In fact, any such surface is determined uniquely by a segment in X between interior points of the segments p_1p_2 and p_1p_3.

We also use the generalized Toponogov comparison theorem for quasigeodesics([Pe]). First we explain quasigeodesics. A curve $\tilde{\gamma}$ in M_k^2 is called (locally) convex at the point $\tilde{\gamma}(t)$ with respect to $\tilde{p} \in M_k^2$ if there exists $\varepsilon > 0$ such that the following triangle is convex. The sides of this triangle are the curve $\tilde{\gamma}(t) \mid_{t-\varepsilon}^{t+\varepsilon}$ and the two segments $\tilde{\gamma}(t-\varepsilon)\tilde{p}$ and $\tilde{\gamma}(t+\varepsilon)\tilde{p}$. Let $\gamma : [a, b] \longrightarrow X$ be a curve in X. For $p \in X$, a curve $\tilde{\gamma} : [a, b] \longrightarrow M_k^2$ is called an unfolding of γ with respect to p if the following conditions are satisfied:

$$\begin{cases} 1)\ \tilde{\gamma}(t) \text{ is parameterized by arc length,} \\ 2)\ \text{there exists } \tilde{p} \in M_k^2 \text{ such that } dist(\tilde{\gamma}(t), \tilde{p}) = dist(\gamma(t), p) \text{ for every } t, \\ 3)\ \text{the direction from } \tilde{p} \text{ to } \tilde{\gamma}(t) \text{ turns monotonically with increasing } t. \end{cases}$$

A curve γ in X is called k-convex if for all $p \in X$ there exists a curve $\tilde{\gamma}$ in M_k^2 that satisfies the following conditions:

1) $\tilde{\gamma}$ is an unfolding of γ with respect to p,
2) $\tilde{\gamma}$ is a locally convex curve with respect to $\tilde{p}$ at all $\tilde{\gamma}(t)$ such that $dist(\tilde{p}, \tilde{\gamma}(t)) < \pi(k)$.

In the above we set $\pi(k){=}\pi/\sqrt{k}$ for $k > 0$ and $\pi(k){=}\infty$ for $k \le 0$.

Then a k-convex curve $\gamma : [a,b] \longrightarrow X$ parameterized by arc length is called a k-quasigeodesic, or simply quasigeodesic. We can take a quasigeodesic emanating from p in any direction v. Let $\gamma : [a,b] \longrightarrow X$ be a quasigeodesic. Then for any $p \in X$ and $t_0 \in [a,b]$ the angle $\tilde{\sphericalangle}(\gamma(t_0)\gamma(t), \gamma(t_0)p)$ is nonincreasing in $t(t \ge t_0)$, where $\tilde{\sphericalangle}(\gamma(t_0)\gamma(t), \gamma(t_0)p) = \sphericalangle(\widetilde{\gamma(t_0)\gamma(t)}, \widetilde{\gamma(t_0)\tilde{p}})$ is the corresponding angle of the model triangle $\widetilde{\triangle^*}p\gamma(t_0)\gamma(t)$ in M_k^2 with sides of lengths $dist(p,\gamma(t_0))$,$dist(p,\gamma(t))$, and $t - t_0$.

From this property of quasigeodesics we have the following Generalized Toponogov comparison theorem ([Pe]):

Let X be an $n(\ge 2)$-dimensional Alexandrov space with curvature $\ge k$, and let $\gamma : [0,t] \longrightarrow X$ be a quasigeodesic. For $p \in X$ and $t_0 \in [0,t]$, take a geodesic triangle $\triangle^*\widetilde{\gamma(t_0)}\widetilde{\gamma(t)}\tilde{p}$ in M_k^2 that denotes a triangle with sides $\gamma|_{[t_0,t]}, p\gamma(t_0), p\gamma(t)$, corresponding to the triangle $\triangle^*\gamma(t_0)\gamma(t)p$, satisfying $dist(p,\gamma(t_0)) = dist(\tilde{p},\widetilde{\gamma(t_0)})$, $L(\gamma|_{[t_0,t]}) = dist(\widetilde{\gamma(t_0)},\widetilde{\gamma(t)}) = t - t_0$, and $dist(p,\gamma(t)) = dist(\tilde{p},\widetilde{\gamma(t)})$. In the above we denote by $L(\gamma|_{[t_0,t]})$ the length of a curve $\gamma|_{[t_0,t]}$. Then we have

$$\sphericalangle(\gamma|_{[t_0,t]}, \gamma(t_0)p) \ge \sphericalangle(\widetilde{\gamma(t_0)}\widetilde{\gamma(t)}, \widetilde{\gamma(t_0)}\tilde{p}), \tag{1.9}$$

where the angle $\sphericalangle(\gamma|_{[t_0,t]}, \gamma(t_0)p) = \lim_{t\to 0} \tilde{\sphericalangle}(\gamma(t_0)\gamma(t), \gamma(t_0)p)$. Now for any hinge $(\gamma|_{[t_0,t]}, \gamma(t_0)p)$ in X, take the corresponding hinge $(\widetilde{\gamma(t_0)}\tilde{q}, \widetilde{\gamma(t_0)}\tilde{p})$ in M_k^2 such that $L(\gamma|_{[t_0,t]}) = dist(\widetilde{\gamma(t_0)},\tilde{q}) = t - t_0$, $dist(\gamma(t_0),p) = dist(\widetilde{\gamma(t_0)},\tilde{p})$, and $\sphericalangle(\gamma|_{[t_0,t]}, \gamma(t_0)p) = \sphericalangle(\widetilde{\gamma(t_0)}\tilde{q}, \widetilde{\gamma(t_0)}\tilde{p})$.

Then we have from (1.9)

$$dist(\gamma(t),p) \le dist(\tilde{q},\tilde{p}). \tag{1.10}$$

By using this property of quasigeodesics, i.e., the generalized Toponogov comparison theorem, we get the following theorem.

Theorem 1.4. *Let X be an n-dimensional Alexandrov space with curvature ≥ 1, then we have*

$$a_k(X) \le a_k(S^n). \tag{1.11}$$

Especially we have

$$a_{2p}(X) \le a_{2p}(S^n) = \frac{\pi}{2}. \tag{1.12}$$

Next we explain the notion of the spherical suspension([B-G-P]).

Definition 1.2. *The spherical suspension of a metric space* Y *is the quotient space*

$$\sum_1 Y = Y \times [0,\pi]/\sim, \tag{1.13}$$

where the equivalence relation $\sim$ *is given by*

$$(x_1,a_1)\sim(x_2,a_2)\Leftrightarrow\begin{cases}x_1=x_2,\ 0<a_1=a_2<\pi \text{ or}\\ a_1=a_2=0 \text{ or } a_1=a_2=\pi,\end{cases} \tag{1.14}$$

and is equipped with the canonical metric

$$\cos dist(\hat{x}_1,\hat{x}_2)=\cos a_1\cos a_2+\sin a_1\sin a_2\cos dist(x_1,x_2), \tag{1.15}$$

where we set $\hat{x}_1=(x_1,a_1)$, $\hat{x}_2=(x_2,a_2)$.

Further we define $\sum_k Y=\sum_{k-1}(\sum_1 Y)$ to be a k-times repeated spherical suspension. Then for an Alexandrov space X we have $X=\sum_k Y$ if and only if S^{k-1} is isometrically embedded in X.

Now we ask what happens when equality holds in (1.11). If $k=1$ this means that $radX=\pi$ and X is isometric to S^n. We want to know whether an Alexandrov space X admits a similar structure to S^n if equality holds in (1.11) for general k. By using the generalized Toponogov comparison theorem we get the following theorem for the case of $k=3$. We also give a partial result for $k=2$(see proposition 4.1).

Theorem 1.5. *Let* X *be an* n*-dimensional Alexandrov space with curvature* ≥ 1. *Suppose* $a_3(X)=a_3(S^n)=5\pi/9$. *Then we have* $diamX=\pi$. *If* n=$dim\,X\geq 2$ *then* $X=\sum_2 Z$, *where* Z *is an* $(n-1)$*-dimensional Alexandrov space with curvature*$Z\geq 1$.

2. Proof of Theorem1.1

In this section we are concerned with $a_k(S^1)$. A k-tuple $(x_1,\cdots,x_k)$ of points $x_i(i=1,\cdots,k)$ of S^1 located in counterclockwise order is called a configuration, where each x_i is called a vertex of the configuration. The antipodal point of $x\in S^1$ will be denoted by $\bar{x}$.

Now for a configuration $(x_1,\cdots,x_k)$ we set

$$f_{x_1,\cdots,x_k}(x):=\sum_{i=1}^{k} dist(x,x_i). \tag{2.1}$$

Considering S^1 as the unit circle in $\boldsymbol{R}^2$, we take the angle measure $t=t(x)$ of radius Ox as the coordinate of $x\in S^1$. Then we may write

$$dist(x,x_i)=\begin{cases}\mid t-t_i\mid \ \text{ if } 0\leq\mid t-t_i\mid\leq\pi,\\ 2\pi-\mid t-t_i\mid \ \text{ if } \pi\leq\mid t-t_i\mid\leq 2\pi,\end{cases} \tag{2.2}$$

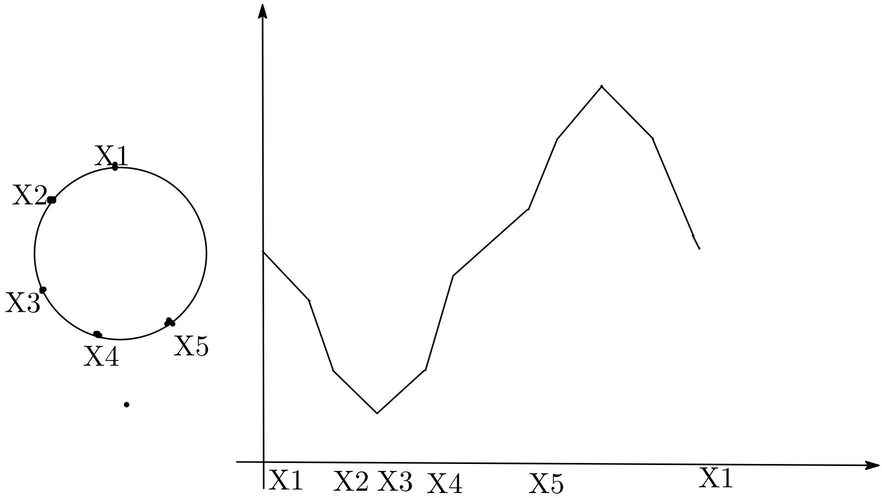

FIGURE 1

where we set $t = t(x), t_i = t(x_i)$. Hence $f_i(x) := dist(x, x_i)$ is smooth except for x_i and $\bar{x}_i$, and the gradient vector $\nabla f_i(x)(x \neq x_i, \bar{x}_i)$ is a unit tangent vector to the minimal circle arc of S^1 from x to x_i. f_i assumes the maximum π(resp., minimum 0) at $\bar{x}_i$(resp., x_i) and its graph is a polygonal line with gradient ± 1. Now for a configuration $(x_1, \cdots, x_k)$, $f(x) = \sum_{i=1}^k dist(x, x_i)$ is smooth except for $x_i, \bar{x}_i (i = 1, \cdots, k)$ and its graph is a polygonal line([figure1]). As x passes through a vertex x_i(resp.,$\bar{x}_i$), the gradient of the graph of $f(x)$ increases (resp., decreases) by 2. Then we easily see that $f(x)$ is constant if and only if the configuration consists of pairs of antipodal points.

Lemma 2.1. *We have for any* $x \in S^1$

$$f(x) + f(\bar{x}) = k\pi. \tag{2.3}$$

Proof. For any fixed vertex x_i we have $dist(x, x_i)+$ $dist(\bar{x}, x_i)=\pi$ for any $x \in S^1$. Then (2.3) follows by taking sum with respect to i. □

First we will prove Theorem 1.1 for odd $k = 2p-1$ by induction. If $p = 1$, we have $max_{x \in S^1} f(x) = \pi$ for any (x_1). We assume that (1.2) holds for $k = 2p - 3$. Suppose a configuration $(x_1, \cdots, x_k)$, $k = 2p - 1$ is given.

Lemma 2.2. $f(x)$ *assumes a maximal value at the antipodal* $\bar{x}_i$ *of a vertex* x_i. *Then* $f(x)$ *assumes a minimal value at the vertex* x_i.

Proof. First we show that $f(x)$ cannot assume an extremal value at $x(\neq x_i, \bar{x}_i)(i = 1, \cdots, k)$. Indeed, otherwise we have $\nabla f(x) = 0$, since f is smooth at x. On the other hand we have $\nabla f(x) = \sum_{i=1}^k \nabla f_i(x)$, $f_i(x) =$

$dist(x, x_i)$, where $\nabla f_i(x)$ are unit vectors of $R \cong T_x(S^1)$. Then $\sum_{i=1}^{k} \nabla f_i(x) \neq 0$, because k is odd. This also implies that the gradient of the graph of $f(x)$ at $x(\neq x_i, \bar{x}_i)$ is an odd integer, and that $f(x)$ is not locally constant. Now the graph of the gradient of $f(x)$ is a polygonal line and the gradient of the graph changes the sign from plus to minus at a maximal point. Hence f may assume a maximal value only at the antipodal $\bar{x}_i$ of some vertex x_i. From Lemma 2.1 f assumes a minimal value at the vertex x_i. □

In case of $k = 2p-1$, we say that the polygonal line ,which is the graph of f, forms a peak(resp., trough) at $\bar{x}_i$(resp., x_i) when f assumes a maximal value(resp., minimal value) at $\bar{x}_i$(resp., x_i).

Lemma 2.3. *For a given configuration $(x_1, \cdots, x_k)$, $k = 2p-1$, suppose that vertices differ from one another and that the antipodal of any vertex is not a vertex of the configuration. If f assumes the minimum value at a vertex x_i and therefore the maximum value at $\bar{x}_i$, then around the peak at $\bar{x}_i$ and the trough at x_i the graph of f consists of two segments whose gradients are* 1 *and* -1.

Proof. The gradient of the polygonal lines, which is the graph of f, is an odd integer, and changes the sign at x_i(resp.,$\bar{x}_i$) and decrease (resp., increase) by 2 because of the assumptions. □

Lemma 2.4. *When there is a vertex x_i whose antipodal point $\bar{x}_i$ is a vertex of the configuration, the maximum value of $f(x)$ is larger than $\frac{(2p^2-2p+1)\pi}{2p-1}$ which is the maximum of $f(x)$ determined by the configuration whose vertices are equally spaced.*

Proof. Suppose $x_i = \bar{x}_j (1 \leq i, j \leq 2p-1, i \neq j)$. Then for any x, we have $dist(x, x_i) + dist(x, x_j) = \pi$ and $f(x) = \pi + \sum_{k \neq i,j} dist(x, x_k)$. By the induction assumption we have

$$max_{x \in S^1} \sum_{k \neq i,j} dist(x, x_k) \geq \frac{2(p-1)^2 - 2(p-1) + 1}{2p-3} \pi.$$

It follows that

$$\begin{aligned} \max_{x \in S^1} \{\pi + \sum_{k \neq i,j} dist(x, x_k)\} &\geq \pi + \frac{2(p-1)^2 - 2(p-1) + 1}{2p-3} \pi \\ &= \frac{2p^2 - 4p + 2}{2p-3} \pi > \frac{2p^2 - 2p + 1}{2p-1} \pi. \end{aligned} \tag{2.4}$$

□

Recall that for a given configuration $(x_1, \cdots, x_k)$ vertices x_i are counter-clockwise arranged. If $x_{i+l} \neq x_i (l > 0)$ we write $x_i < x_{i+l}$, and $x_i < x < x_{i+l}$

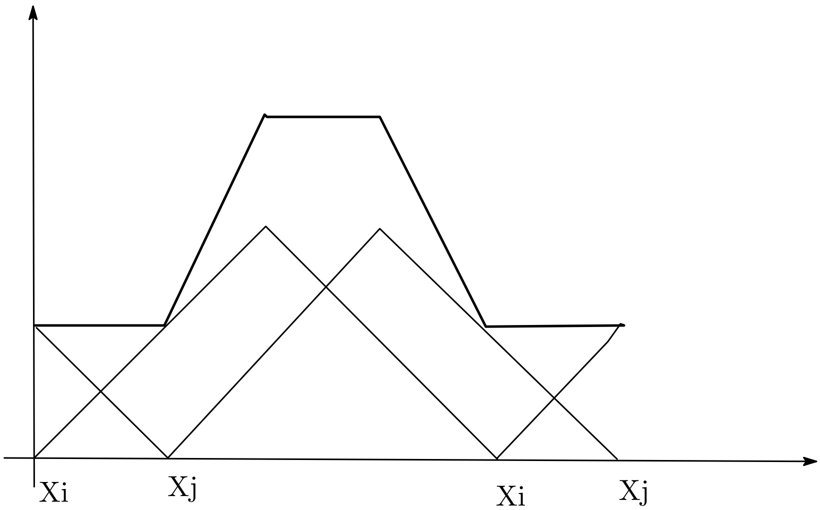

FIGURE 2

means that x is contained in the arc from x_i to x_{i+l} in S^1. Here we show that the maximum value of f can be made smaller by moving the overlapped vertices.

Lemma 2.5. *If $dist(x_i, x_j)(i < j)$ increases, the maximum value of the sum $dist(x, x_i) + dist(x, x_j)$ decreases.*

Proof. The sum of the gradients of the graphs of $dist(x, x_i)$ and $dist(x, x_j)$ is 0 for $x_i \leq x \leq x_j$ or $\bar{x}_i \leq x \leq \bar{x}_j$. The sum $dist(x, x_i) + dist(x, x_j)$ assumes the maximum value which is equal to $2\pi - dist(x_i, x_j)$ for $\bar{x}_i \leq x \leq \bar{x}_j$ and assumes the minimum value which is equal to $dist(x_i, x_j)$ for $x_i \leq x \leq x_j$. Therefore if $dist(x_i, x_j)(i < j)$ increases, the maximum value of the sum $dist(x, x_i) + dist(x, x_j)$ decreases([figure 2]). □

Lemma 2.6. *When vertices x_i, x_j in S^1 are moved equally in the opposite directions, the sum $dist(x, x_i) + dist(x, x_j)$ assumes the same value independent of the position of x_i, x_j for $\bar{x}_j \leq x \leq x_i$ or $\bar{x}_i \geq x \geq x_j$.*

Proof. When vertices x_i, x_j are moved equally in the opposite directions for $\bar{x}_j \leq x \leq x_i$ or $\bar{x}_i \geq x \geq x_j$,the increase(resp.,decrease) of $dist(x, x_i)$ is equal to the decrease(resp.,increase) of $dist(x, x_j)$ for $\bar{x}_j \leq x \leq x_i$ or $\bar{x}_i \geq x \geq x_j$.Therefore the sum $dist(x, x_i) + dist(x, x_j)$ assumes the same value independent of the position of vertices x_i, x_j for $\bar{x}_j \leq x \leq x_i$ or $\bar{x}_i \geq x \geq x_j$. □

Lemma 2.7. *When vertices overlap, the maximum value cannot be made greater by moving the overlapped vertices.*

Proof. First suppose that the maximum value of $f(x)$ is realized at the antipodal point of overlapped vertices. From Lemma 2.5, 2.6 the maximum value is made smaller by moving the overlapped points equally in the different directions slightly. If the maximum value of $f(x)$ is realized at a point different from the antipodal point of the overlapped vertices, the maximum value is kept constant by moving the overlapped points in the same manner as Lemma 2.5, 2.6. □

In the following we consider the case where $k(= 2p - 1)$ vertices are different from one another, and there is no vertex whose antipodal point is a vertex.

Lemma 2.8. *Suppose a minimum of $f(x)$ is assumed at x_i, and consequently a maximum of $f(x)$ is assumed at $\bar{x}_i$. Then we have*

$$x_{p+i-1} < \bar{x}_i < x_{p+i}, \tag{2.5}$$

where $p + i$, $p + i - 1$ are counted modulo k.

Proof. Suppose $x_{p+i} < \bar{x}_i$ or $x_{p+i} = \bar{x}_i$. Then the gradient of the polygonal line $f(x)$ at the left side of $\bar{x}_i$ is greater than or equal to $(p+1)-(p-2) = 3$. From Lemma 2.3 it contradicts that the gradient of polygonal line $f(x)$ at the left side of $\bar{x}_i$ is 1. Next suppose $x_{p+i-1} > \bar{x}_i$ or $x_{p+i-1} = \bar{x}_i$. Then the gradient of polygonal line $f(x)$ at the right side of $\bar{x}_i$ is greater than or equal to $(p + 1) - (p - 2) = 3$. From Lemma 2.3 it contradicts that the gradient of a polygonal line $f(x)$ at the right side of $\bar{x}_i$ is 1. □

In case of $k = 2p - 1$, the configuration $(x_1, \cdots, x_k)$ of k points on S^1 is called balanced, if we have $x_i < \bar{x}_{i+p} < x_{i+1}$ for any i, where $i+p$ is counted modulo k. For a balanced configuration $(x_1, \cdots, x_k)$, the gradient of the graph of $f(x) = f_{x_1,\cdots,x_k}(x)$ is equal to ± 1 and there are k peaks where $f(x)$ assumes maximal values at the antipodal point $\bar{x}_i$. The maximum value is one of the peak values([figure 3]). Now we will show that the configuration such that the maximum value is minimum is the configuration such that k points are equally spaced. Indeed, the following lemma 2.9 shows that it suffices to consider balanced configurations. Finally in Lemma 2.9 we show the above assertion for the family of balanced configurations.

Lemma 2.9. *In case of $k = 2p-1$, a_k is realized for a balanced configuration.*

Proof. We may assume $k(= 2p - 1)$ vertices do not overlap and there are no vertices whose antipodal points are vertices. When there is no antipodal point between x_i and x_{i+1} for some i in an imbalanced configuration, this configuration is changed into the configuration a vertex of which is antipodal of a vertex by moving points $\bar{x}_i$ and $\bar{x}_{i+1}$ equally in opposite directions till either reaches the most nearby vertex. Then the maximum of $f(x)$ is

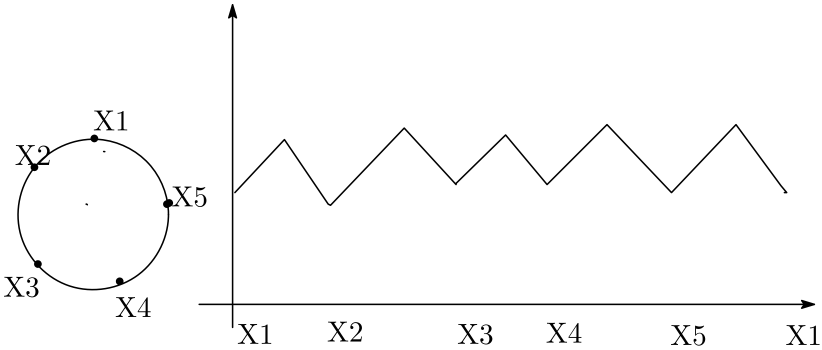

FIGURE 3

kept constant or made smaller from Lemma 2.5, 2.6. From Lemma 2.4 the maximum of the sum of distance from x is made greater than $\frac{(2p^2-2p+1)\pi}{2p-1}$. Therefore a_k is realized for a balanced configuration. □

Lemma 2.10. *In the family of balanced configurations $a_k(S^1)$ is realized if and only if $k(=2p-1)$ points are equally spaced.*

Proof. Let $(x_1, \cdots, x_k)$, $k = 2p-1$, be a balanced configuration and set $M_{x_1,\cdots,x_k} := \max_{x\in S^1} f_{x_1,\cdots,x_k}(x)$. Then $M_{x_1,\cdots,x_k} = \max_{1\le i\le k} f_{x_1,\cdots,x_k}(\bar{x}_i)$ by Lemma 2.2. Since $(x_1, \cdots, x_k)$, $k = 2p-1$, is balanced, we have

$$x_i < \bar{x}_{i+p} < x_{i+1} < \bar{x}_{i+p+1} < \cdots x_{i+p-1} < \bar{x}_i, \tag{2.6}$$

$$\bar{x}_i < x_{i+p} < \bar{x}_{i+1} < x_{i+p+1} < \cdots \bar{x}_{i+p-1} < x_i. \tag{2.7}$$

It follows that

$$f(\bar{x}_i) = \sum_{j=1}^{p-1} dist(x_{i+j}, x_{i+j+p-1}) + \pi. \tag{2.8}$$

Then we have

$$\begin{aligned}(2p-1)M_{x_1,\cdots,x_k} &\ge \sum_{i=1}^{k} f(\bar{x}_i) = 2(p-1)^2\pi + (2p-1)\pi \\ &= (2p^2-2p+1)\pi,\end{aligned} \tag{2.9}$$

namely,

$$M_{x_1,\cdots,x_k} \ge \frac{2p^2-2p+1}{2p-1}\pi. \tag{2.10}$$

If equality holds in (2.10) we have $f(\bar{x}_1) = f(\bar{x}_2) = \cdots = f(\bar{x}_{2p-1}) = \frac{(2p^2-2p+1)\pi}{2p-1}$, which is equivalent to $dist(x_1, x_2) = dist(x_2, x_3) = \cdots = dist(x_i, x_{i+1}) = \cdots = dist(x_{2p-1}, x_1)$. □

Next we show Theorem 1.1 for the case of $k = 2p$. Indeed, Theorem 1.1 for $k = 2p$ may be generalized to n-dimensional case(Theorem 1.3). However here we give a detailed proof for S^1 to show the idea. Let $(x_1, \cdots, x_{2p})$ be a configuration of $k(= 2p)$ points in S^1, and set $f(x) = f_{x_1,\cdots,x_k}(x) = \sum_{i=1}^{k} dist(x, x_i)$ as before. First note that the gradients of a polygonal line which is the graph of f are even integers([figure 4]). Indeed, the gradient vector $\nabla f(x)$ at $x(\neq x_i, \bar{x}_i)$ is the sum of unit tangent vectors in $T_x(S^1)$ of even numbers. From this fact we also see that $x(\neq x_i, \bar{x}_i)$ is a critical point of f if and only if there are the same number of vertices on arcs $x < \bar{x}$ and $\bar{x} < x$. Now for any configuration $(x_1, \cdots, x_{2p})$ we have

$$\int_0^{2\pi} dist(x, x_i)dx = \pi^2, \tag{2.11}$$

and therefore

$$\int_0^{2\pi} f(x)dx = 2p\pi^2. \tag{2.12}$$

And for a configuration $(x_1, \cdots, x_{2p})$ such that the antipodal point of any vertex is a vertex of the configuration(i.e., $x_{p+i} = \bar{x}_i$), $f(x) = f_{x_1,\cdots,x_{2p}}(x)$ is equal to a constant $p\pi$, we call such a configuration symmetric. For any configuration $(x_1, \cdots, x_{2p})$, we have the following lemma.

Lemma 2.11.

$$M_{x_1,\cdots,x_{2p}} := \max_{x\in S^1} f_{x_1,\cdots,x_{2p}}(x) \geq p\pi. \tag{2.13}$$

$$m_{x_1,\cdots,x_{2p}} := \min_{x\in S^1} f_{x_1,\cdots,x_{2p}}(x) \leq p\pi. \tag{2.14}$$

Proof. Suppose $M_{x_1,\cdots,x_{2p}} < p\pi$, then we have $\int_0^{2\pi} f_{x_1,\cdots,x_{2p}}(x) < 2p\pi^2$. It contradicts the assumption. Similarly suppose $m_{x_1,\cdots,x_{2p}} > p\pi$, then we have $\int_0^{2\pi} f_{x_1,\cdots,x_{2p}}(x) > 2p\pi^2$. It contradicts the assumption. □

Lemma 2.12. *Suppose $k = 2p$. Then $a_k(S^1) = \pi/2$ and $a_k(S^1)$ is realized if and only if a configuration consists of pairs of antipodal points $(x_i, \bar{x}_i)$.*

Proof. From Lemma 2.11 we have

$$a_{2p}(S^1) = \frac{1}{2p} \min M_{x_1,\cdots,x_{2p}} \geq \frac{\pi}{2}, \tag{2.15}$$

and for a symmetric configuration we have $M_{x_1,\cdots,x_{2p}} = m_{x_1,\cdots,x_{2p}} = p\pi$. Therefore $a_{2p}(S^1) = \pi/2$. To complete the proof of Theorem 1.1 it suffices to show that a configuration $(x_1, \cdots, x_{2p})$ with $M_{x_1,\cdots,x_{2p}} = p\pi$ must be symmetric. Indeed, in this case we have $M_{x_1,\cdots,x_{2p}} = p\pi$ and $\int_0^{2\pi} f_{x_1,\cdots,x_{2p}}(x) = 2p\pi^2$. Therefore $f_{x_1,\cdots,x_{2p}}(x) \equiv p\pi$ is a constant function and every $x \in S^1$

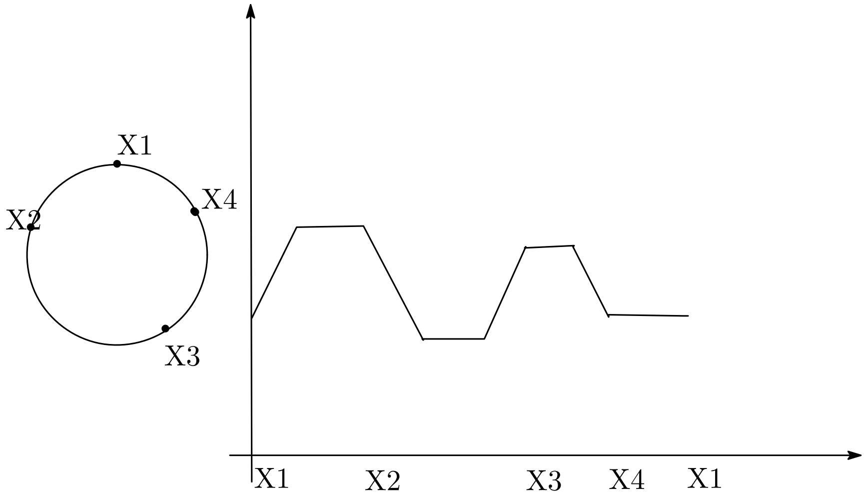

FIGURE 4

is a critical point of $f_{x_1,\cdots,x_{2p}}(x)$. It follows that for any $x(\neq x_i, \bar{x}_i)$, there are p vertecies on arcs $x < \bar{x}$ and $\bar{x} < x$. This happens only for a symmetric configuration. □

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Here we will give a proof of $a_3(S^n) = 5\pi/9$. We begin with the case of $n = 2$ considering S^2 as the unit sphere in $\mathbf{R}^3$. Let x_1, x_2, x_3 be points of S^2 which are contained in a small or great circle. For given $x_1, x_2, x_3 \in S^2$ take a plane Π in R^3 containing these there points. Suppose that $\Pi \cap S^2$ is a small circle C, and let N be the pole of S^n such that $dist(x_i, N)$ is equal to $t (0 \leq t \leq \pi/2)$. Set $f(x) = f_{x_1,x_2,x_3}(x) = \sum_{i=1}^{3} dist(x_i, x)$. We show that

$$\max_{x \in S^2} f(x) > \frac{5}{3}\pi \tag{3.1}$$

holds. Indeed, assuming that $x_1, x_2, x_3 \in C$ are located in counterclockwise order and

$$dist(x_2, x_3) \geq \max\{dist(x_1, x_2), dist(x_1, x_3)\}, \tag{3.2}$$

take $x'_i (i = 1, 2, 3)$ on a great circle S parallel to C which are projections of $x_i (i = 1, 2, 3)$ by great half circles through N. Then we have

$$\begin{cases} dist(x_1, x_2) + dist(x_1, x_3) < dist(x'_1, x'_2) + dist(x'_1, x'_3) \\ \leq 2\pi - dist(x'_2, x'_3) \leq \frac{4}{3}\pi. \end{cases} \tag{3.3}$$

Now for $x = \bar{x}_1$, the antipodal point of x_1, note that $dist(x_1, x) = \pi$, $dist(x_2, x) = \pi - dist(x_1, x_2)$, and $dist(x_3, x) = \pi - dist(x_1, x_3)$. It follows that

$$(3.4) \qquad \begin{cases} dist(x_1, x) + dist(x_2, x) + dist(x_3, x) \\ = 3\pi - dist(x_1, x_2) - dist(x_1, x_3) \\ > 3\pi - \frac{4}{3}\pi = \frac{5}{3}\pi. \end{cases}$$

Therefore the maximum value of the sum of distances from arbitrary three points x_1, x_2, x_3 on any small circle exceeds $5\pi/3$, that is equal to $a_3(S^1)$.

Next suppose that x_1, x_2, x_3 are on a great circle S arranged in counter-clockwise order and y is on the same great circle. If x_1, x_2, x_3 are not equally spaced then from Theorem 1.1 we have

$$\max_{x \in S^2} f_{x_1,x_2,x_3}(x) \geq \max_{x \in S} f_{x_1,x_2,x_3}(x) > 5\pi/3.$$

So we assume that x_1, x_2, x_3 are equally spaced on S. First we show that the maximum value of f is assumed at a point of S. Note that any point $x \in S^2$ lies on a half great circle γ joining a point $y \in S$ and the antipodal point $\bar{y}$ of y and perpendicular to S. We may assume that $x_1 \leq y \leq x_2$ on S. Set $l_i = dist(x, x_i)(i = 1, 2, 3)$, $t = dist(x, y)(0 \leq t \leq \pi)$, and $s = dist(x_1, y)$, where we may assume that $0 \leq s \leq \pi/3$. Then by the cosine formula we have

$$(3.5) \qquad \cos l_1 = \cos t \cos s,$$

$$(3.6) \qquad \cos l_2 = \cos t \cos(\frac{2}{3}\pi - s) = -\cos t \cos(\frac{1}{3}\pi + s),$$

$$(3.7) \qquad \cos l_3 = \cos t \cos(\frac{2}{3}\pi + s) = -\cos t \cos(\frac{1}{3}\pi - s).$$

For a fixed s, $l_i(i = 1, 2, 3)$ are functions of t and we get

$$\begin{aligned} & l_1'(t) + l_2'(t) + l_3'(t) \\ & \quad = \sin t \left\{ \frac{\cos s}{\sqrt{1 - \cos^2 t \cos^2 s}} - \frac{\cos(\frac{\pi}{3} + s)}{\sqrt{1 - \cos^2 t \cos^2(\frac{\pi}{3} + s)}} \right. \\ & \qquad \left. - \frac{\cos(\frac{\pi}{3} - s)}{\sqrt{1 - \cos^2 t \cos^2(\frac{\pi}{3} - s)}} \right\}. \end{aligned}$$

Now set $a = |\cos t|$ and

$$(3.8) \qquad g(u) = \frac{u}{\sqrt{1 - a^2 u^2}}.$$

Set $u_1 = \cos(\pi/3 + s)$, $u_2 = \cos(\pi/3 - s)$, and note that $u_1 + u_2 = \cos s$ holds. Then for $0 \le s \le \pi/6$ noting that

$$0 \le u_1 \le \frac{1}{2},\ \frac{1}{2} \le u_2 \le \frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2} \le u_1 + u_2 \le 1, \tag{3.9}$$

we easily have $g(u_1 + u_2) \ge g(u_1) + g(u_2)$. It follows that

$$l_1'(t) + l_2'(t) + l_3'(t) \ge 0$$

for $0 \le s \le \pi/6$. Next for $\pi/6 \le s \le \pi/3$, noting that

$$-\frac{1}{2} \le u_1 \le 0,\ \frac{\sqrt{3}}{2} \le u_2 \le 1,\ \frac{1}{2} \le u_1 + u_2 \le \frac{\sqrt{3}}{2}, \tag{3.10}$$

we have $g(u_1 + u_2) \le g(u_1) + g(u_2)$. Therefore we have

$$l_1'(t) + l_2'(t) + l_3'(t) \le 0$$

for $\pi/6 \le s \le \pi/3$. Hence $l_1(t) + l_2(t) + l_3(t)$ assumes the maximum value at $t = \pi$ for $0 \le s \le \pi/6$ and at $t = 0$ for $\pi/6 \le s \le \pi/3$. Especially for $s = \pi/6$, we have $l_1(t)+l_2(t)+l_3(t) \equiv 3\pi/2 < 5\pi/3$ and this value is less than $\max_{x\in S^2} f_{x_1,x_2,x_3}(x)$. It follows that $f : S^2 \to \boldsymbol{R}^2$ assumes the maximum at a point of the great circle S. Then we have our assertion by Theorem 1.1. Finally we consider the case of general $n \ge 2$. Let $x_1, x_2, x_3 \in S^n$ be given. If $x_1 = x_2 = x$ then for the antipodal $\bar{x}$ of x we have

$$\max_{x\in S^n} f_{x_1,x_2,x_3}(x) \ge f_{x_1,x_2,x_3}(\bar{x}) \ge 2\pi > \frac{5}{3}\pi. \tag{3.11}$$

Therefore we may assume that x_1, x_2, x_3 are different. Then x_1, x_2, x_3 are on either a small or a great circle of some 2-dimensional sphere S^2. If they are on a small sphere then the above argument implies that $\max_{x\in S^n} f > 5\pi/3$. If they are on a great circle, then for any $x \in S^n$ we may assume that x, x_1, x_2, x_3 are contained in some great 2-dimensional sphere S^2 and the above argument works. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3 and Theorem 1.4

First we show that $a_k(S^n)$ is equal to $\pi/2$ for $k = 2p$. Suppose

$$f_{x_1,x_2,\cdots,x_k}(x) = \sum_{i=1}^{k} dist(x, x_i), \tag{4.1}$$

as before. Then we have

$$a_k(S^n) = \min_{x_1,x_2,\cdots,x_k} \max_{x\in S^n} f_{x_1,x_2,\cdots,x_k}(x) = \min_{x_1,x_2,\cdots,x_k} \| f_{x_1,x_2,\cdots,x_k} \|_\infty . \tag{4.2}$$

Set

$$M_{x_1,\cdots,x_{2p}} := \max_{x\in S^n} f_{x_1,\cdots,x_{2p}}(x), \tag{4.3}$$

$$m_{x_1,\cdots,x_{2p}} := \min_{x\in S^n} f_{x_1,\cdots,x_{2p}}(x), \tag{4.4}$$

as before. Here we have

$$\| f_{x_1,x_2,\cdots,x_{2p}} \|_1 = \int_{S^n} f_{x_1,\cdots,x_{2p}}(x)dx = p\pi vol(S^n). \tag{4.5}$$

Suppose $M_{x_1,\cdots,x_{2p}} < p\pi$, then we have $\int_{S^n} f_{x_1,\cdots,x_{2p}}(x) < p\pi vol(S^n)$. It contradicts the assumption. Therefore we obtain $M_{x_1,\cdots,x_{2p}} \geq p\pi$. Next suppose $m_{x_1,\cdots,x_{2p}} > p\pi$, then we have $\int_{S^n} f_{x_1,\cdots,x_{2p}}(x) > p\pi vol(S^n)$. It contradicts the assumption. Therefore we obtain $m_{x_1,\cdots,x_{2p}} \leq p\pi$. Hence

$$a_{2p}(S^n) = \frac{1}{2p} \min M_{x_1,\cdots,x_{2p}} \geq \frac{\pi}{2}, \tag{4.6}$$

and for a symmetric configuration(see the statement of Theorem 1.3) we have $M_{x_1,\cdots,x_{2p}} = m_{x_1,\cdots,x_{2p}} = p\pi$. Therefore $a_{2p}(S^n) = \pi/2$. To complete the proof of this theorem it suffices to show that a configuration $(x_1,\cdots,x_{2p})$ with $M_{x_1,\cdots,x_{2p}} = p\pi$ must be symmetric. In this case we have $M_{x_1,\cdots,x_{2p}} = p\pi$ and $\int_{S^n} f_{x_1,\cdots,x_{2p}}(x) = p\pi vol(S^n)$. Therefore $f_{x_1,\cdots,x_{2p}}(x) \equiv p\pi$ is a constant function and every $x \in S^n$ is a critical point of $f_{x_1,\cdots,x_{2p}}(x)$. Since $f_{x_1,x_2,\cdots,x_{2p}}(x) = p\pi$, $f_{x_1,x_2,\cdots,x_{2p}}(x)$ is smooth. $dist(x, x_i)$ is differentiable at any point except for x_i and $\bar{x}_i$. If none of points $x_1, x_2, \cdots, x_{2p}$ coincides with $\bar{x}_i$, $f_{x_1,x_2,\cdots,x_{2p}}(x)$ is not differentiable at $\bar{x}_i$. It contradicts that $f_{x_1,x_2,\cdots,x_{2p}}(x)$ is smooth. It happens only for a symmetric configuration. This completes the proof of Theorem 1.3.

Now we turn to the proof of Theorem 1.4. Let X be an n-dimensional Alexandrov space with *curvature* ≥ 1. First we recall the notion of the exponential map ([G-W],[Pe]). For $p \in X$ we denote by S_p the space of directions at p, that is an $(n-1)$-dimensional Alexandrov space with *curvature* ≥ 1. Note that each $v \in S_p$ determines a quasigeodesic $c_v : [0, \pi] \longrightarrow X$ emanating from p with the initial direction v. Then for the spherical suspension $\sum_1 S_p = S_p \times [0, \pi]/\sim$, the exponential map $\exp_p : \sum_1 S_p - (S_p \times \{\pi\})/\sim \longrightarrow X$ is defined as follows. For $v \in S_p$ we denote by $\bar{c}_v(t) = (t, v)(0 \leq t \leq \pi)$, the corresponding segment in $\sum_1 S_p$ emanating from the vertex $\bar{p}$. Then we set $\exp_p \bar{c}_v(t) = c_v(t)$.

Now we show that $a_k(X)$ does not exceed $a_k(S^n)$ by the generalized Toponogov comparison theorem. Let $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_k$ be points in S^n that realizes $a_k(S^n)$. And take a point $\tilde{p} \in S^n$ different from the antipodal points of $\tilde{x}_i (i = 1, \cdots, k)$. Take a regular point $p \in X$. Then $\sum_1 S_p$ is isometric to S^n, and we identify $\sum_1 S_p$ (resp.,S_p) with $S^n = \sum_1 S_{\tilde{p}}$ (resp.,$S_{\tilde{p}} = S^{n-1}$).

Let x_i be a point such that $\exp_p \bar{c}_{v_i}(t) = c_{v_i}(t)$, where $\bar{c}_{v_i}$ is a geodesic emanating from $\tilde{p}$ with initial direction v_i to $\tilde{x_i} \in \sum_1 S_{\tilde{p}} = S^n (i = 1, \cdots, k)$ and c_{v_i} is a quasigeodesic emanating from p with initial direction v_i to x_i. Take a point $x_0 \in X$ such that

$$a_k(x_1, \cdots, x_k) := \max_{x \in X} \frac{1}{k} \sum_{i=1}^{k} dist(x, x_i) = \frac{1}{k} \sum_{i=1}^{k} dist(x_0, x_i). \tag{4.7}$$

Let $\gamma_0 : [0, dist(p, x_0)] \longrightarrow X$ be a minimal geodesic from p to x_0, and set $\tilde{x_0} = \exp_{\tilde{p}}^{S^n}(dist(p, x_0)\dot{\gamma_0}(0))$. By the generalized Toponogov comparison theorem for $\triangle p x_i x_0$ and $\triangle \tilde{p} \tilde{x}_i \tilde{x}_0$(see (1.10)) we have

$$dist(x_0, x_i) \leq dist(\tilde{x}_0, \tilde{x}_i). \tag{4.8}$$

It follows that

$$\begin{aligned} a_k(X) &\leq a_k(x_1, x_2, \cdots, x_k) = \frac{1}{k} \sum_{i=1}^{k} dist(x_0, x_i) \\ &\leq \frac{1}{k} \sum_{i=1}^{k} dist(\tilde{x}_0, \tilde{x}_i) \leq a_k(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_k) = a_k(S^n). \end{aligned} \tag{4.9}$$

Therefore we have

$$a_k(X) \leq a_k(S^n), \tag{4.10}$$

and the proof of Theorem 1.4 is complete. By Theorem 1.3 and Theorem 1.4 we obtain $a_{2p}(X) \leq a_{2p}(S^n) = \pi/2$.

We show that when M is an n-dimensional Riemannian manifold with some conditions on $a_2(M)$ and the injective radius, M is isometric to the unit n-dimensional sphere.

Proposition 4.1. *Suppose that M is an n-dimensional Riemannian manifold with curvature ≥ 1. Suppose that $a_2(M) = \pi/2$ holds and in addition that the injective radius $i(M)$ of M is greater than $\pi/2$. Then M is isometric to the n-dimensional unit sphere S^n.*

Proof. Let x_1, x_2 be a pair of points in M such that $diam M = dist(x_1, x_2)$. First we show that

$$dist(x_1, x) + dist(x_2, x) \leq \pi \tag{4.11}$$

holds for any point x. Let γ_i be a minimal geodesic from x_i to $x (i = 1, 2)$. Since x_1 is critical for the distance function $y \longrightarrow dist(x_2, y)$ and x_2 is also critical for the distance function $y \longrightarrow dist(x_1, y)$, we may take a minimal geodesics from x_1 to x_2 and from x_2 to x_1, so that we have $\sphericalangle(x_2 x_1, x_2 x) \leq$

$\pi/2$ and $\sphericalangle(x_1x_2, x_1x) \leq \pi/2$ for the angle of hinges. Then by the Toponogov comparison theorem and the cosine formula we obtain

$$\begin{aligned}(4.12)\qquad &\cos dist(x_1,x) \geq \cos dist(x_1,x_2)\cos dist(x_2,x)\\ &+ \sin dist(x_1,x_2)\sin dist(x_2,x)\cos\sphericalangle(x_2x_1,x_2x)\\ &\geq \cos dist(x_1,x_2)\cos dist(x_2,x),\end{aligned}$$

and

$$\begin{aligned}(4.13)\qquad &\cos dist(x_2,x) \geq \cos dist(x_1,x_2)\cos dist(x_1,x)\\ &+ \sin dist(x_1,x_2)\sin dist(x_1,x)\cos\sphericalangle(x_1x_2,x_1x)\\ &\geq \cos dist(x_1,x_2)\cos dist(x_1,x).\end{aligned}$$

Adding these inequalities it follows that

$$(4.14)\qquad \cos\frac{dist(x_1,x)+dist(x_2,x)}{2}\cos\frac{dist(x_1,x)-dist(x_2,x)}{2} \geq 0.$$

Then we get $dist(x_1,x) + dist(x_2,x) \leq \pi$, and therefore

$$(4.15)\qquad \frac{\pi}{2} = a_2(M) \leq \frac{1}{2}\max_{x\in M}\{dist(x_1,x)+dist(x_2,x)\} \leq \frac{\pi}{2}.$$

Hence we can take a point $x_0 \in M$ such that $dist(x_1,x_0) + dist(x_2,x_0) = \pi$. Further for this $x = x_0$ equality holds in (4.12), (4.13). Now suppose $d(x_1,x_2) < \pi$. Then we have $dist(x_1,x_0) = dist(x_2,x_0) = \pi/2$, and $\sphericalangle(x_2x_1,x_2x_0) = \sphericalangle(x_1x_2,x_1x_0) = \pi/2$. Since the injective radius $i(M) > \pi/2$ minimal geodesics γ_i from x_i to x_0 are unique, and we show $\sphericalangle(\dot\gamma_1(\pi/2), \dot\gamma_2(\pi/2)) = \pi$. Otherwise we take a point $x' = \gamma_1(\pi/2+\epsilon)(0 < \epsilon < i(M) - \pi/2)$. Then we have by the triangle inequality

$$\begin{aligned}(4.16)\qquad &dist(x_1,x') + dist(x_2,x')\\ &= \frac{\pi}{2} + \epsilon + dist(x_2,x')\\ &= dist(x_1,x_0) + dist(x_0,x') + dist(x',x_2)\\ &> dist(x_1,x_0) + dist(x_0,x_2) = \pi.\end{aligned}$$

$\triangle x_0x_1x_2$ spans a totally geodesic surface of constant curvature 1. Since equality holds in the Toponogov comparison theorem, it follows that $dist(x_1,x_2) = \pi$. Therefore we have $diam M = \pi$, and $M = S^n$ by the maximal diameter theorem. □

Remark 4.1. (1) *By adding the condition $i(M) > \pi/2$ M is not isometric to the real projective space RP^n or the hemisphere S^+. Since $a_2(RP^n) = \pi/2$ holds for the real projective space RP^n of constant curvature* 1 *and $n \geq 2$ we need an assumption such that $i(M) > \pi/2$ in Proposition* 4.1.
(2) *On the other hand, when X is an Alexandrov space such that curvature $\geq$ 1 and $a_2(X) = \pi/2$, we do not know yet such a structure theorem for X.*

5. Proof of Theorem 1.5

Let X be an n-dimensional Alexandrov space with curvature ≥ 1 and $n \geq 2$. Recall that we have an inequality

$$a_3(X) \leq a_3(S^n) = \frac{5}{9}\pi \tag{5.1}$$

by Theorem 1.2 and Theorem 1.4. Now in this section we show that X is isometric to a spherical double suspension $\sum_2 Z$ when equality holds in (5.1). First we show that X is isometric to a spherical suspension $\sum_1 Y$.

Lemma 5.1. *Let X be an n-dimensional Alexandrov space with curvature ≥ 1. Suppose $a_3(X) = a_3(S^n) = 5\pi/9$. Then X is isometric to $\sum_1 Y$, where Y is an $(n-1)$-dimensional Alexandrov space with curvature ≥ 1.*

Proof. We show that $diam X$ is equal to π.@ Let $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ be points in S^n that realize $a_3(S^n)$. Then the configuration $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is equally spaced on a great circle S^1, and take a point $\tilde{p} \in S^n$ different from the antipodal of $\tilde{x}_i (i = 1, 2, 3)$. Take a regular point $p \in X$. Then $\sum_1 S_p$ is isometric to S^n, and we identify $\sum_1 S_p$ (resp.,S_p) with $S^n = \sum_1 S_{\tilde{p}}$ (resp.,$S_{\tilde{p}} = S^{n-1}$). Let x_i be a point $\exp_p \bar{c}_{v_i}(dist(\tilde{p}, \tilde{x}_i)) = c_{v_i}(dist(\tilde{p}, \tilde{x}_i))$, where $\bar{c}_{v_i}$ is a geodesic emanating from $\tilde{p}$ with initial direction v_i to $\tilde{x}_i \in \sum_1 S_{\tilde{p}} = S^n (i = 1, 2, 3)$ and c_{v_i} is a quasigeodesic emanating from p with initial direction v_i to x_i as in the proof of Theorem 1.4. Take a point $x_0 \in X$ such that

$$a_3(x_1, x_2, x_3) := \max_{x \in X} \frac{1}{3} \sum_{i=1}^{3} dist(x, x_i) = \frac{1}{3} \sum_{i=1}^{3} dist(x_0, x_i). \tag{5.2}$$

Let $\gamma_0 : [0, dist(p, x_0)] \longrightarrow X$ be a minimal geodesic from p to x_0. And set $\tilde{x}_0 = \exp_{\tilde{p}}^{S^n}(dist(p, x_0)\dot{\gamma_0}(0))$. By the generalized Toponogov comparison theorem for $\triangle p x_i x_0$ and $\triangle \tilde{p} \tilde{x}_i \tilde{x}_0$, we have for $i = 1, 2, 3$

$$dist(x_0, x_i) \leq dist(\tilde{x}_0, \tilde{x}_i), \tag{5.3}$$

and hence

$$\begin{cases} a_3(X) \leq a_3(x_1, x_2, x_3) = \frac{1}{3} \sum_{i=1}^{3} dist(x_0, x_i) \\ \qquad \leq \frac{1}{3}\{dist(\tilde{x}_0, \tilde{x}_1) + dist(\tilde{x}_0, \tilde{x}_2) + dist(\tilde{x}_0, \tilde{x}_3)\} \\ \qquad \leq a_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = a_3(S^n) = a_3(X). \end{cases} \tag{5.4}$$

Therefore for any i we obtain

$$dist(x_0, x_i) = dist(\tilde{x}_0, \tilde{x}_i), \tag{5.5}$$

and $a_3(S^n) = a_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = 1/3 \sum_{i=1}^{3} dist(\tilde{x}_0, \tilde{x}_i)$. It follows that $\tilde{x}_0$ must be an antipodal point of some $\tilde{x}_i$, namely,

$$dist(\tilde{x}_0, \tilde{x}_i) = \pi, \tag{5.6}$$

and hence

$$dist(x_0, x_i) = \pi. \tag{5.7}$$

Then $diamX = \pi$ and X is isometric to $\sum_1 Y$ by the Toponogov maximal diameter theorem([G-P2]). □

Next we show that X is isometoric to $\sum_2 Z$ if $\dim X \geq 2$.

Lemma 5.2. *Suppose $X = \sum_1 Y$, where Y is an $(n-1)$-dimensional Alexandrov space with curvature ≥ 1 and $diamY < \pi$ and $n \geq 2$. Let x_1,$x_2 \in X$ be the pole points of the suspension $X = \sum_1 Y$. Then there is no pair of points whose distance is π except for x_1,x_2.*

Proof. Let y_1, y_2 be points in Y. Set $z_1 = (y_1, t_1)(0 \leq t_1 \leq \pi), z_2 = (y_2, t_2)(0 \leq t_2 \leq \pi)$, where t_1, t_2 is the distance from x_1 in $\sum_1 Y$. Suppose $dist(z_1, z_2) = \pi$. By the definition of the spherical suspension we have

$$\begin{aligned} -1 &= \cos dist(z_1, z_2) \\ &= \cos t_1 \cos t_2 + \sin t_1 \sin t_2 \cos dist(y_1, y_2) \\ &\geq \cos(t_1 + t_2) + \sin t_1 \sin t_2 \{\cos dist(y_1, y_2) + 1\} \\ &\geq -1. \end{aligned} \tag{5.8}$$

It follows that we have either $t_1 = \pi, t_2 = 0$ or $t_1 = 0, t_2 = \pi$. Hence there is no pair of points whose distance is π except for x_1, x_2. □

Lemma 5.3. *Let X be an n-dimensional Alexandrov space with curvature ≥ 1 and $n \geq 2$. Suppose $a_3(X) = a_3(S^n) = 5\pi/9$. Then $X = \sum_2 Z$.*

Proof. By Lemma 5.1 we may write $X = \sum_1 Y$. Suppose $diamY < \pi$. In the proof of Lemma 5.1 a point p is an arbitrary regular point. Recall that regular points are dense in X. If the base point $p \in X$ is shifted, the points x_1, x_2, x_3 that realize $a_3(X)$ can be moved. Then $a_3(X)$ is realized by another pair of points $x_0, x_i (i = 1, 2, 3)$ whose distance is equal to π. This contradicts Lemma 5.2. Therefore we have $diamY = \pi$ and $X = \sum_2 Z$. □

By Lemma 5.1, 5.3 the proof of Theorem 1.5 is complete.

Remark 5.1. *By applying the same argument for $k = 2p - 1$ we may show that X is isometric to a spherical suspension if $a_k(X) = \frac{2p^2-2p+1}{(2p-1)^2}\pi$ holds. We also conjecture that X is isometric to $\sum_1 Y$ if $a_2(X) = \pi/2$ and $radX > \pi/2$ hold.*

After the completion of the present paper we settled the conjecture about $a_k(S^n)$ in the introduction. We give a proof and also discuss some results about Remark 5.1 in a forthcoming paper.

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