Math. J. Okayama Univ. **46** (2004), 85–104

IGUSA LOCAL ZETA FUNCTIONS OF REGULAR 2-SIMPLE PREHOMOGENEOUS VECTOR SPACES OF TYPE I WITH UNIVERSALLY TRANSITIVE OPEN ORBITS

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1. Introduction

In this paper, we explicitly determine the Igusa local zeta functions of several variables for all but one type regular 2-simple prehomogeneous vector spaces of type I with universally transitive open orbits. As for the remaining one type of space, we give the explicit forms of the Igusa local zeta functions of *one variable* for each of the basic relative invariants.

In [4], [5] and [8], the irreducible, simple or 2-simple regular prehomogeneous vector spaces with universally transitive open orbits were classified. As for the irreducible reduced regular prehomogeneous vector spaces with universally transitive open orbits, J. Igusa gave explicitly their Igusa local zeta functions in [4]. And as for the simple regular prehomogeneous vector spaces with universally transitive open orbits, their Igusa local zeta functions were given explicitly in H. Hosokawa [3] and the author [17]. These results indicate that their p-adic Γ-factors are expressed by the Tate local factor and the b-functions. Here we treat the Igusa local zeta functions of regular 2-simple prehomogeneous vector spaces of type I with universally transitive open orbits, which were classified into the following nine spaces:

(1) $(GL(1)^3 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + \Lambda_1^* \otimes 1)$,
(2) $(GL(1)^3 \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))\,(n > m)$,
(3) $(GL(1)^3 \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1^*))\,(n > m)$,
(4) $(GL(1)^3 \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes (\Lambda_1^* + \Lambda_1^*))\,(n > m)$,
(5) $(GL(1)^2 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)\,(n > m)$,
(6) $(GL(1)^4 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1))\,(n > m)$,
(7) $(GL(1)^4 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*))\,(n > m)$,
(8) $(GL(1)^3 \times Spin(10) \times SL(2), (\text{a half spin rep.}) \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))$,
(9) $(GL(1)^4 \times Spin(10) \times SL(2), (\text{a half spin rep.}) \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$,

(cf. [8]). We can easily reduce calculations of the Igusa local zeta functions of the spaces (2), (4) to a result of [2]. We can immediately get those of the spaces (3), (5), (8), (9) from results of [4] and [3]. So we mainly deal

Mathematics Subject Classification. Primary 11S40; Secondary 11S90, 20G25.

Key words and phrases. Igusa local zeta functions, prehomogeneous vector spaces, universally transitive open orbits.

with those of the spaces (1), (6), (7), which are non-trivial at all. For the Igusa local zeta function of a prehomogeneous vector space, the number of its variables is the same as that of the basic relative invariants. As for the spaces (6), (7), we calculate explicitly the Igusa local zeta functions of several variables. As an approach of calculations of that of the space (1), we calculate explicitly those of one variable for each of three basic relative invariants. In order to calculate these Igusa local zeta functions, we use some results of spherical functions of alternating forms of [2], and integrals of [18]. In [18], we calculated integrals on some fibers of $Sp(n)$-invariant maps by the results of [2].

We shall mention that the generalized Iwasawa-Tate theory holds for the spaces (1) $\sim$ (9) (cf. [6] and [9]). In such a space, a global zeta function has an Euler product, and their local factors of finite places are expressed by the Igusa local zeta function. In [7], T. Kimura calculated explicitly the Fourier transform of the complex power over $\mathbb{R}$ for a simple prehomogeneous vector space by using the explicit form of the Igusa local zeta function which was given in [3]. We apply this method to the spaces (2) $\sim$ (9), and get their b-functions.

We found some exceptional properties of the space (1) as against other reduced spaces with universally transitive open orbits. The spaces (2) $\sim$ (9) have the following two specific properties (i) and (ii):

(i) All roots of their b-functions of the basic relative invariants are negative integers.

(ii) Each basic relative invariant $f(x)$ is of the form

$$f(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} c_{i_1 i_2 \cdots i_m} \cdot x_{i_1} x_{i_2} \cdots x_{i_m}$$

with $c_{i_1 i_2 \cdots i_m} \in \mathbb{R}$.

In [4] and [5], J. Igusa mentioned these properties (i), (ii) hold for the reduced irreducible prehomogeneous vector spaces with universally transitive open orbits. However the non-irreducible reduced space (1) satisfies neither (i) nor (ii).

The plan of this paper is as follows. In Section 2, we review some known properties of prehomogeneous vector spaces with universally transitive open orbits. In Section 3, we give our main result on explicit forms of the Igusa local zeta functions of several variables for the spaces (2) $\sim$ (9). In Section 4, we prove our main result. In Section 5, we give explicit forms of the Igusa local zeta functions of one variable for each basic relative invariants of the space (1).

Notation. Let K be a p-adic field i.e. a finite extension of $\mathbb{Q}_p$, and $\mathcal{O}_K$ the ring of integers in K. We fix a prime element π in $\mathcal{O}_K$, and then $\pi\mathcal{O}_K$ is the ideal of nonunits of $\mathcal{O}_K$. The cardinality of the residue field $\mathcal{O}_K/\pi\mathcal{O}_K$ is denoted by q. We denote by $|\ |_K$ the absolute value of K normalized as $|\pi|_K = q^{-1}$. For a commutative ring R, we denote by $M(m,n;R)$ the totality of $m \times n$ matrices over R, and by $Alt(n;R)$ the totality of $n \times n$ alternating matrices over R ($m,n \in \mathbb{Z}_{>0}$). We denote by $\det(x)$ the determinant of $x \in M(n,n;R)$. For any $x \in M(m,n;R)$, ${}^t x$ is the transpose of x. We denote by $\mathrm{Pf}(y)$ the Pfaffian of $y \in Alt(2n;R)$. For an element $y \in Alt(l;R)$ and an integer i ($1 \leq 2i \leq l$), we denote by $\mathrm{Pf}_i(y)$ the Pfaffian of the upper left $2i$ by $2i$ block of y. If $i = n$ and $l = 2n$, then $\mathrm{Pf}_n(y)$ is the Pfaffian of y. For any positive integer n, $\mathfrak{S}_n$ is the symmetric group in n latters. For any positive integer n, we set

$$J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in Alt(2n),$$

and $Sp(n) = \left\{ g \in GL(2n);\ {}^t g J_n g = J_n \right\}$.

2. Preliminaries

In this section, we review some known properties of prehomogeneous vector spaces with universally transitive open orbits. For details, we refer to [7] and [13].

Let G be a connected linear algebraic group defined over $\mathbb{Q}$, V a finite dimensional vector spaces with $\mathbb{Q}$-structure, and $\rho : G \to GL(V)$ a rational representation of G on V defined over $\mathbb{Q}$. Throughout this section, for simplicity, we assume that (G,ρ,V) is one of irreducible, simple or 2-simple of type I regular prehomogeneous vector spaces with a finitely many adelic open orbits, which were classified in [6] and [9]. The spaces (1) $\sim$ (9) are contained in their spaces. Let $f_1,\ldots,f_l$ be the basic relative invariants of (G,ρ,V), and χ_i the rational character of G corresponding to f_i, i.e. $f_i(\rho(g)v) = \chi_i(g) f_i(v)$ for all $g \in G$ and all $v \in V$. Then any relative invariant in $\mathbb{Q}(V)$ can be written uniquely as $c f_1^{\nu_1} \cdots f_l^{\nu_l}$ with $c \in \mathbb{Q}^\times$, $\nu_1,\ldots,\nu_l \in \mathbb{Z}$. The group of rational characters of G corresponding to relative invariants is a free abelian group of rank l generated by $\chi_1,\ldots,\chi_l$. Let K be a p-adic field i.e. a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K$ the ring of integers in K, dv the Haar measure on $V(K)$ normalized by $\int_{V(\mathcal{O}_K)} dv = 1$, and $\mathcal{S}(V(K))$ the Schwartz-Bruhat space of $V(K)$. We denote by $|\ |_K$ the absolute value of K normalized as $|\pi|_K = q^{-1}$. For the basic relative invariants $f_1,\ldots,f_l$, and $\Phi \in \mathcal{S}(V(K))$, we put

$$Z(s;\Phi) = \int_{V(K)} \prod_{i=1}^{l} |f_i(v)|^{s_i} \Phi(v) dv \quad (s = (s_1,\ldots,s_l) \in \mathbb{C}^l,\ \mathrm{Re}(s_i) > 0).$$

It is known that we can express this integral $Z(s;\Phi)$ by

$$Z(s;\Phi)=\frac{P(q^{-s_1},\dots,q^{-s_l})}{\prod_{i=1}^{h}(1-q^{-\sum_{j=1}^{l}a_{ij}s_j-b_i})},$$

where $P(x_1,\dots,x_l)\in\mathbb{Q}[x_1^{\pm1},\dots,x_l^{\pm1}]$ and $a_{ij},b_i\in\mathbb{Z}$ (see, e.g. [1]). Let Φ_0 be the characteristic function of $V(\mathcal{O}_K)$. We put $Z(s)=Z(s;\Phi_0)$. This local zeta function $Z(s)$ is called the Igusa local zeta function of (G,ρ,V).

We shall review local functional equations of $Z(s;\Phi)$. Let V^* be the dual space of V, and ρ^* the contragredient representation of ρ. It is known that (G,ρ^*,V^*) is also a prehomogeneous vector space, and there exist the basic relative invariants $f_1^*,f_2^*,\dots,f_l^*$ of (G,ρ^*,V^*) such that the character χ_i^{-1} corresponds to f_i^*. Let dv^* be the Haar measure on $V^*(K)$ normalized by $\int_{V^*(\mathcal{O}_K)}dv^*=1$, and $S(V^*(K))$ the Schwartz-Bruhat space of $V^*(K)$. We define the p-adic local zeta function $Z^*(s;\Phi^*)$ of (G,ρ^*,V^*) by

$$Z^*(s;\Phi)=\int_{V^*(K)}\prod_{i=1}^{l}|f_i^*(v^*)|_K^{s_i}\Phi^*(v^*)dv^*$$
$$(s=(s_1,\dots,s_l)\in\mathbb{C}^l,\ \mathrm{Re}(s_i)>0)$$

where $\Phi^*\in S(V^*(K))$. Let ψ be an additive character of K such that ψ is non-trivial on $\pi^{-1}\mathcal{O}_K$ and trivial on $\mathcal{O}_K$. We define the Fourier transform $\widehat{\Phi}^*$ of $\Phi^*\in S(V^*(K))$ by $\widehat{\Phi}^*(v)=\int_{V^*(K)}\Phi^*(v^*)\psi(v^*(v))dv^*$. By the regularity of (G,ρ,V), there exists an element $\kappa=(\kappa_1,\kappa_2,\dots,\kappa_l)\in(1/2)\cdot\mathbb{Z}^l$ satisfying $\det(\rho(g))^2=\chi_1(g)^{2\kappa_1}\cdots\chi_l(g)^{2\kappa_l}$ (cf. [14], [12]). By [7, Theorem 3.3], we have the functional equation

$$Z(s-\kappa;\widehat{\Phi}^*)=\gamma(s)Z^*(-s;\Phi^*),$$

where $s-\kappa=(s_1-\kappa_1,\dots,s_l-\kappa_l)$ and $\gamma(s)$ is independent of Φ^*. We call $\gamma(s)$ the *p-adic Γ-factor* of (G,ρ,V). For p-adic local functional equations of prehomogeneous vector spaces which do not have universally transitive open orbits, we refer to [13]. Since the Fourier transform $\widehat{\Phi}_0$ of Φ_0 is equal to Φ_0, we have $\gamma(s)=Z(s-\kappa)/Z(-s)$.

We shall define the b-function of (G,ρ,V). We put $f^m=\prod_{i=1}^{l}f_i^{m_i}$, $f^{*m}=\prod_{i=1}^{l}f_i^{*m_i}$ for $m=(m_1,m_2,\dots,m_l)\in\mathbb{Z}^l$. Fix a $\mathbb{Q}$-basis of V, and identify $V(\mathbb{Q})$ with $\mathbb{Q}^n$ ($\dim V=n$). We also identify $V^*(\mathbb{Q})$ with $\mathbb{Q}^n$ by the basis of V^* dual to the fixed basis of V. We put $v=(v_1,\dots,v_n)$, $\mathrm{grad}_v=(\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_n})$. Then for any l-tuple $m=(m_1,m_2,\dots,m_l)\in(\mathbb{Z}_{\geq0})^l$, there exists a polynomial $b_m(s)$ such that $f^{*m}(\mathrm{grad}_v)f^{s+m}(v)=b_m(s)f^s(v)$, where $s+m=(s_1+m_1,\dots,s_l+m_l)$. The polynomial $b_m(s)$ does not depend on $v\in V$. We call $b_m(s)$ the b-function of (G,ρ,V). The coefficient of the part of the highest degree of the b-function depends on only constant factors of the basic relative invariants. So we treat the b-function $b_m(s)$ except its coefficient of the part of the highest degree.

3. Main result

In this section, we give explicit forms of the Igusa local zeta functions and the p-adic Γ-factors for the spaces (2) $\sim$ (9).

We shall define some notations. We define the irreducible representation Λ_1 (resp. Λ_1^*) of $SL(n)$ by $\Lambda_1(g)x = gx$ (resp. $\Lambda_1(g)x = {}^t g^{-1}x$) for $g \in SL(n)$, $x \in M(n, 1)$, and the irreducible representation Λ_1 of $Sp(n)$ by $\Lambda_1(g)x = gx$ for $g \in Sp(n)$, $x \in M(2n, 1)$. For a half-spin representation of $Spin(10)$, we refer to [14]. For each space, we denote by l the number of the basic relative invariants, and $\chi_1, \ldots, \chi_l$ the characters corresponding to the basic relative invariants. These characters will be given in the form $\chi_i(g) = a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k}$ for $g = (a_1, a_2, \ldots, a_k, A, B) \in G = GL(1)^k \times G_1 \times G_2$, where $m_1, m_2, \ldots, m_k$ are integers depending only on i, and G_1, G_2 are simple algebraic groups.

Theorem 3.1. *Let (G, ρ, V) be one of the prehomogeneous vector spaces (2) $\sim$ (9) of Introduction, and $Z(s)$ the associated Igusa local zeta function. Then, we have*

$$Z(s) = \prod_{j=1}^{N} \frac{1 - q^{-\alpha_j}}{1 - q^{-\eta_j(s)}},$$

where the constants $\alpha_1, \ldots, \alpha_N$ and the forms

$$\eta_j(s) = \sum_{i=1}^{l} \eta_{ij} s_i + \alpha_j, \quad (\eta_{ij} = 0 \text{ or } 1, \;\; \alpha_j \in \mathbb{Z}_{>0})$$

are given in each case as follows:

(2) $(GL(1)^3 \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))$ $(n > m)$. $l = 2$. $\chi_1(g) = a_1^{2m}$, $\chi_2(g) = a_1^{2m-2} a_2 a_3$.

$$\left\{ \begin{matrix} s_1 + 1,\, s_1 + 2n - 2m + 2,\, s_2 + 1,\, s_2 + 2m, \\ s_1 + s_2 + 2j + 1,\, s_1 + s_2 + 2n - 2j + 2 \end{matrix} \;;\, j = 1, 2, \ldots, m - 1 \right\}.$$

(3) $(GL(1)^3 \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1^*))$ $(n > m)$. $l = 2$. $\chi_1(g) = a_1^{2m}$, $\chi_2(g) = a_2 a_3$.

$$\{s_1 + 2j - 1,\, s_1 + 2n - 2j + 2,\, s_2 + 1,\, s_2 + 2m \,;\, j = 1, 2, \ldots, m\}.$$

(4) $(GL(1)^3 \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes (\Lambda_1^* + \Lambda_1^*))$ $(n > m)$. $l = 2$. $\chi_1(g) = a_1^{2m}$, $\chi_2(g) = a_1^2 a_2 a_3$.

$$\left\{ \begin{matrix} s_1 + 2j - 1,\, s_1 + 2n - 2j,\, s_2 + 1,\, s_2 + 2m, \\ s_1 + s_2 + 2m - 1,\, s_1 + s_2 + 2n \end{matrix} \;;\, j = 1, 2, \ldots, m - 1 \right\}.$$

(5) $(GL(1)^2 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$ $(n > m)$, $l = 1$. $\chi_1(g) = a_1^{2m+1} a_2$.

$$\{s_1 + 2j - 1,\ s_1 + 2n - 2j + 2\,;\ j = 1, 2, \ldots, m+1\}.$$

(6) $(GL(1)^4 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1))$ $(n > m)$. $l = 4$. $\chi_1(g) = a_1^{2m+1} a_2$, $\chi_2(g) = a_1^{2m-1} a_2 a_3 a_4$, $\chi_3(g) = a_1^{2m} a_3$, $\chi_4(g) = a_1^{2m} a_4$.

$$\left\{\begin{array}{c} s_i + 1,\ s_1 + 2n - 2m,\ s_2 + 2m, \\ s_1 + s_2 + s_3 + s_4 + 2j + 1, \\ s_1 + s_2 + s_3 + s_4 + 2n - 2j + 2 \end{array} ;\ i = 1, 2, 3, 4,\ j = 1, 2, \ldots, m \right\}.$$

(7) $(GL(1)^4 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*))$ $(n > m)$. $l = 4$. $\chi_1(g) = a_1^{2m+1} a_2$, $\chi_2(g) = a_1^2 a_3 a_4$, $\chi_3(g) = a_1 a_2 a_3$, $\chi_4(g) = a_1 a_2 a_4$.

$$\left\{\begin{array}{c} s_i + 1,\ s_1 + 2n - 2m,\ s_1 + 2j + 1, \\ s_1 + 2n - 2j,\ s_2 + 2m, \\ s_1 + s_2 + s_3 + s_4 + 2m + 1, \\ s_1 + s_2 + s_3 + s_4 + 2n \end{array} ;\ i = 1, 2, 3, 4,\ j = 1, 2, \ldots, m-1 \right\}.$$

(8) $(GL(1)^3 \times Spin(10) \times SL(2), (\text{a half spin rep.}) \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))$. $l = 2$. $\chi_1(g) = a_1^4$, $\chi_2(g) = a_2 a_3$.

$$\{s_1 + 1,\ s_1 + 4,\ s_1 + 5,\ s_1 + 8,\ s_2 + 1,\ s_2 + 2\}.$$

(9) $(GL(1)^4 \times Spin(10) \times SL(2), (\text{a half spin rep.}) \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$. $l = 4$. $\chi_1(g) = a_1^4$, $\chi_2(g) = a_2 a_3$, $\chi_3(g) = a_3 a_4$, $\chi_4(g) = a_4 a_2$.

$$\{s_1 + 1,\ s_1 + 4,\ s_1 + 5,\ s_1 + 8,\ s_2 + 1,\ s_3 + 1,\ s_4 + 1,\ s_2 + s_3 + s_4 + 2\}.$$

Corollary 3.2. *Let (G, ρ, V) be one of the spaces (2) $\sim$ (9). From the set $\{\eta_j(s);\ j = 1, \ldots, N\}$ of Theorem 3.1, the p-adic Γ-factor $\gamma(s)$ of (G, ρ, V) is given by $\gamma(s) = \prod_{j=1}^{N} \gamma^T(\eta_j(s - \kappa))$, where their κ are given in each case as follows: (2) $\kappa = (2n - 2m + 2, 2m)$, (3) $\kappa = (2n, 2m)$, (4) $\kappa = (2n - 2, 2m)$, (5) $\kappa = 2n$, (6) $\kappa = (2n - 2m, 2m, 1, 1)$, (7) $\kappa = (2n - 2, 2m, 1, 1)$, (8) $\kappa = (8, 2)$, (9) $\kappa = (8, 1, 1, 1)$. Here we put $\gamma^T(s) = (1 - q^{-(1-s)})/(1 - q^{-s})$. This $\gamma^T(s)$ is called the Tate local factor.*

By this corollary and the method of [7], we can get the Γ-factor over $\mathbb{R}$ for the spaces (2) $\sim$ (9). Furthermore by [12, p.459 (5-8)], we can get b-functions from the Γ-factors over $\mathbb{R}$.

Corollary 3.3. *Let (G, ρ, V) be one of the spaces (2) $\sim$ (9). From the set $\{\eta_j(s);\ j = 1, \ldots, N\}$ of Theorem 3.1, the b-function of (G, ρ, V) is given by $b_m(s) = \prod_{j=1}^{N} \Gamma(\eta_j(s + m))/\Gamma(\eta_j(s))$, where $\Gamma(s)$ is the gamma function.*

In [15] and [19], these b-functions were calculated explicitly by the method of [16]. From Corollary 3.2 and 3.3, we see that the p-adic Γ-factors of the spaces (2) $\sim$ (9) are expressed by the Tate local factor and the set $\{\eta_j(s); j = 1, \ldots, N\}$, which are determined by the b-functions.

Example 3.4. We illustrate our result in the case of (2) $m = 2$, $n = 4$: $(GL(1)^2 \times Sp(4) \times SL(4), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1), M(8,4) \oplus M(4,1) \oplus M(4,1))$. For the basic relative invariants of this space, we give their explicit forms in Subsection 4.1. From Theorem 3.1, the Igusa local zeta function of this case is given by

$$\begin{aligned} Z(s_1, s_2) &= \frac{1-q^{-1}}{1-q^{-s_1-1}} \times \frac{1-q^{-6}}{1-q^{-s_1-6}} \times \frac{1-q^{-1}}{1-q^{-s_2-1}} \times \frac{1-q^{-4}}{1-q^{-s_2-4}} \\ &\quad \times \frac{1-q^{-3}}{1-q^{-s_1-s_2-3}} \times \frac{1-q^{-8}}{1-q^{-s_1-s_2-8}}. \end{aligned}$$

We see that this expression corresponds to the form of p-adic local zeta functions which was given in Section 2. From Corollary 3.2, we have the p-adic Γ-factor

$$\gamma(s_1, s_2) = \gamma^T(s_1)\gamma^T(s_1-5)\gamma^T(s_2)\gamma^T(s_2-3)\gamma^T(s_1+s_2-2)\gamma^T(s_1+s_2-7),$$

where $\gamma^T(s) = (1-q^{-(1-s)})/(1-q^{-s})$. From Corollary 3.3 and $\Gamma(t+1) = t\Gamma(t)$, we have the b-function

$$\begin{aligned} b_{(m_1,m_2)}(s_1, s_2) &= \prod_{i=0}^{m_1-1} (s_1+1+i)(s_1+6+i) \\ &\quad \times \prod_{i=0}^{m_2-1} (s_2+1+i)(s_2+4+i) \\ &\quad \times \prod_{i=0}^{m_1+m_2} (s_1+s_2+3+i)(s_1+s_2+8+i). \end{aligned}$$

4. Proof of main result

In this section, we shall calculate explicitly the Igusa local zeta functions of several variables for the spaces (2), (4), (6), (7). As for the spaces (3), (5), (8) and (9), we can immediately obtain their Igusa local zeta functions from results of [4] and [3]. Because their Igusa local zeta functions are given by products of Igusa local zeta functions of reduced irreducible or simple regular prehomogeneous vector spaces with universally transitive open orbits. On the contrary, the cases (2), (4), (6), (7) are not given as such products. As for the cases (2) and (4), we can easily reduce their calculations to that of a local zeta function which was given in [2]. As for the cases (6) and (7), we need to use the integral formulas of [2] and [18] in order to calculate their Igusa local zeta functions.

4.1. **The spaces** (2) **and** (4)**.** In the space (2), the group $G = GL(1)^3 \times Sp(n) \times SL(2m)$ acts on $V = M(2n, 2m) \oplus M(2m, 1) \oplus M(2m, 1)$ by

$$(x, y, z) \mapsto (a_1 g x\, {}^t h, a_2 h y, a_3 h z)$$

for $(x, y, z) \in V$ and $(a_1, a_2, a_3, g, h) \in G$. The basic relative invariants f_1 and f_2 of this space are given by

$$f_1(x) = \mathrm{Pf}({}^t x J_n x), \quad f_2(x, y, z) = \mathrm{Pf}\begin{pmatrix} {}^t x J_n x & y & z \\ -{}^t y & 0 & 0 \\ -{}^t z & 0 & 0 \end{pmatrix}$$

where $(x, y, z) \in V$ (cf. [10]). Let dx be the Haar measure on $M(2n, 2m; K)$ normalized by $\int_{M(2n,2m;\mathcal{O}_K)} dx = 1$, and dy, dz the Haar measure on $M(2m, 1; K)$ normalized by $\int_{M(2n,1;\mathcal{O}_K)} dy = \int_{M(2n,1;\mathcal{O}_K)} dz = 1$. We put $X'_{2m} = \{x \in M(2n, 2m; \mathcal{O}_K);\ \mathrm{Pf}_i({}^t x J_n x) \neq 0\, (1 \le i \le m)\}$, and set

$$\Phi(s) = \Phi(s_1, \ldots, s_m) = \int_{X'_{2m}} \prod_{i=1}^{m} |\mathrm{Pf}_i({}^t x J_n x))|_K^{s_i} dx.$$

This $\Phi(s)$ is absolutely convergent for $\mathrm{Re}(s_1), \ldots, \mathrm{Re}(s_{m-1}) \ge 0$, and have analytic continuation to rational functions in $q^{-s_1}, \ldots, q^{-s_m}$. In [2, Section 3], this integral $\Phi(s)$ was given by

$$\begin{aligned}\Phi(s_1, \ldots, s_m) = &\prod_{i=1}^{m} \frac{(1 - q^{-2i+1})(1 - q^{-2n+2m-2i})}{(1 - q^{-(s_i+\cdots+s_m+2m-2i+1)})(1 - q^{-(s_i+\cdots+s_m+2n-2i+2)})} \\ &\times \prod_{k=1}^{m-1} \frac{1 - q^{-1}}{1 - q^{-2k-1}} \times \prod_{1 \le i < j \le m} \frac{1 - q^{-(s_i+\cdots+s_{j-1})-2(j-i)-1}}{1 - q^{-(s_i+\cdots+s_{j-1})-2(j-i)+1}}.\end{aligned}$$

We put $e_i = {}^t(0, \ldots, 0, 1, 0, \ldots, 0) \in M(l, 1)$ where 1 appears only at the i-th place, and $U(l) = M(l, 1; \mathcal{O}_K) \setminus \pi M(l, 1; \mathcal{O}_K)$. We put $W(i, j) = W'(i) \oplus W''(j)$, $W'(i) = \pi^i U(2m - 1) \oplus \mathcal{O}_K$, and $W''(j) = \pi^j U(2m)$. Then we have $M(2m, 2; \mathcal{O}_K) = \bigcup_{i,j=0}^{\infty} W(i, j)$ (disjoint union). Hence we get

$$\begin{aligned}&Z(s_1, s_2) \\ &= \sum_{i,j=0}^{\infty} \int_{M(2n,2m;\mathcal{O}_K)\oplus W(i,j)} |f_1(x)|_K^{s_1} |f_2(x, z_{2m,1} e_{2m} + \pi^i e_{2m-1}, \pi^j e_{2m})|_K^{s_2} dv \\ &= \frac{(1 - q^{-2m+1})(1 - q^{-2m})}{(1 - q^{-2m+1-s_2})(1 - q^{-2m-s_2})}\end{aligned}$$

$$\times \int_{M(2n,2m;\mathcal{O}_K)\oplus W} |f_1(x)|_K^{s_1} |f_2(x, z_{2m,1}e_{2m} + e_{2m-1}, e_{2m})|_K^{s_2} dv$$
$$= \frac{(1-q^{-2m+1})(1-q^{-2m})}{(1-q^{-2m+1-s_2})(1-q^{-2m-s_2})} \Phi(0, \dots, s_2, s_1).$$

Hence we get an explicit form of the Igusa local zeta function of the space (2).

The basic relative invariants f_1 and f_2 of the space (4) are given by $f_1(x) = \mathrm{Pf}({}^t x J_n x)$, $f_2(x, y, z) = {}^t y x z$ where $(x, y, z) \in V$ (cf. [10]). By an argument similar to that of the space (2), we have

$$Z(s_1, s_2) = \frac{(1-q^{-2m+1})(1-q^{-2m})}{(1-q^{-2m+1-s_2})(1-q^{-2m-s_2})} \Phi(s_2, 0, \dots, 0, s_1).$$

Hence we also get an explicit form of the Igusa local zeta function of the space (4).

4.2. **Some lemmas.** In order to calculate the Igusa local zeta functions of the spaces (6) and (7), we shall give some lemmas.

First we review Hall-Littlewood polynomials. For details, we refer to [11]. For a positive integer m, we put

$$\Lambda_m^+ = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{Z}^m : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0\},$$
$$|\lambda| = \sum_{i=1}^m \lambda_i, \quad n(\lambda) = \sum_{i=1}^m (i-1)\lambda_i.$$

For $\lambda, \mu \in \Lambda_m^+$, we write $\lambda \subset \mu$ if $\lambda_i \le \mu_i$ for all $i \ge 1$. For a non-negative integer i and $\lambda \in \Lambda_m^+$, the number $m_i(\lambda)$ of λ_j's which are equal to i is called the multiplicity of i in λ. For a non-negative integer m, we put $w_m(t) = \prod_{j=1}^m (1-t^j)$, $(w_0(t) = 1)$. For $\lambda \in \Lambda_m^+$, we put $w_\lambda^{(m)}(t) = \prod_{i=0}^{+\infty} w_{m_i(\lambda)}(t)$. The Hall-Littlewood polynomial $P_\lambda(x;t)$ is defined by

$$\begin{aligned} P_\lambda(x;t) &= P_\lambda(x_1, x_2, \dots, x_m; t) \\ &= \frac{(1-t)^m}{w_\lambda^{(m)}(t)} \cdot \sum_{\sigma \in S_m} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(m)}^{\lambda_m} \prod_{1 \le i < j \le m} \frac{x_{\sigma(i)} - t x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} \end{aligned}$$

for each $\lambda \in \Lambda_m^+$. For $\lambda \in \Lambda_m^+$, $P_\lambda(x;t)$ is a polynomial in $x_1, \dots, x_m$ and t, and the set $\{P_\lambda(x;t); \lambda \in \Lambda_m^+\}$ forms a $\mathbb{Z}[t]$-basis of the ring $\mathbb{Z}[t][x_1, \dots, x_m]^{\mathfrak{S}_m}$ of symmetric polynomials in $x_1, \dots, x_m$ with coefficients in $\mathbb{Z}[t]$. We denote by $f_{\mu\nu}^\lambda(t)$ the structure constants of the ring $\mathbb{Z}[t][x_1, \dots, x_m]^{\mathfrak{S}_m}$ with respect to the basis $\{P_\lambda(x;t); \lambda \in \Lambda_m^+\}$:

$$P_\mu(x;t) \cdot P_\nu(x;t) = \sum_\lambda f_{\mu\nu}^\lambda(t) \cdot P_\lambda(x;t) \quad (f_{\mu\nu}^\lambda(t) \in \mathbb{Z}[t]).$$

Unless $|\lambda| = |\mu| + |\nu|$ and $\mu, \nu \subset \lambda$, we have $f^{\lambda}_{\mu\nu}(t) = 0$.

For $\lambda \in \Lambda^+_m$, we put

$$(\pi^\lambda)_{2m} = \begin{pmatrix} 0 & \pi^{\lambda_1} \\ -\pi^{\lambda_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\lambda_m} \\ -\pi^{\lambda_m} & 0 \end{pmatrix} \in Alt(2m; \mathcal{O}_K),$$

$$(\pi^\lambda)_{2m+1} = \begin{pmatrix} (\pi^\lambda)_{2m} & 0 \\ 0 & 0 \end{pmatrix} \in Alt(2m+1; \mathcal{O}_K).$$

The group $GL(l)$ acts on $Alt(l)$ by $h \cdot y = hy\,{}^th$ for $h \in GL(l)$ and $y \in Alt(l)$. Let dy be the Haar measure on $Alt(l; K)$ normalized by $\int_{Alt(l;\mathcal{O}_K)} dy = 1$. By [2, Corollary of Lemma 2.7] or [18, Section 5], we have the following lemma.

Lemma 4.1. *For $\lambda \in \Lambda^+_m$, we have*

$$\int_{GL(2m;\mathcal{O}_K)\cdot(\pi^\lambda)_{2m}} dy = q^{-4n(\lambda)-|\lambda|} \cdot w_{2m}(q^{-1}) \cdot \left(w^{(m)}_\lambda(q^{-2})\right)^{-1},$$

$$\int_{GL(2m+1;\mathcal{O}_K)\cdot(\pi^\lambda)_{2m+1}} dy = q^{-4n(\lambda)-3|\lambda|} \cdot (1-q^{-1})^{-1} \cdot w_{2m+1}(q^{-1}) \times \left(w^{(m)}_\lambda(q^{-2})\right)^{-1}.$$

The group $Sp(n) \times GL(2m)$ acts on $M(n, 2m)$ by $(g, h) \cdot x = gx\,{}^th$ for $(g, h) \in Sp(n) \times GL(2m)$ and $x \in M(n, 2m)$. Let dx be the Haar measure on $M(2n, 2m; K)$ normalized by $\int_{M(2n,2m;\mathcal{O}_K)} dx = 1$.

Lemma 4.2. [18, Theorem 4.3]. *For any $\mathbb{C}$-valued continuous function F on $Alt(2m; \mathcal{O}_K)$, we have*

$$\int_{M(2n,2m;\mathcal{O}_K)} F({}^tx J_n x)dx = \sum_{\lambda\in\Lambda^+_m} c(\lambda) \cdot \int_{GL(2m;\mathcal{O}_K)\cdot(\pi^\lambda)_l} F(y)dy,$$

where

$$c(\lambda) = \frac{w_n(q^{-2})}{w_{n-m}(q^{-2})} \sum_{\mu,\nu\in\Lambda^+_m} f^\lambda_{\mu\nu}(q^{-2}) q^{2(n(\lambda)-n(\mu)-n(\nu))-(2n-2m+1)|\mu|}.$$

For $y \in Alt(l : \mathcal{O}_K)$, we put $H_{l,y} = \{h \in GL(l; \mathcal{O}_K) : \mathrm{Pf}_i(h \cdot y) \neq 0\, (1 \leq 2i \leq l)\}$. For $l = 2m$ or $l = 2m + 1$, $s \in \mathbb{C}^m$, we set

$$\zeta_l(y; s) = \zeta_l(y; s_1, \dots, s_m) = \int_{H_{l,y}} \prod_{i=1}^m |\mathrm{Pf}_i(h \cdot y)|_K^{s_i} dh$$

where dh is the Haar measure on $GL(l; K)$ normalized by $\int_{GL(l;\mathcal{O}_K)} dh = 1$. When $\mathrm{Re}(s_1), \dots, \mathrm{Re}(s_{m-1}) \geq 0$, the integrals $\zeta_l(y; s)$ is absolutely convergent and has an analytic continuation to a rational function in $q^{-s_1}, \dots, q^{-s_m}$ by the theory of complex powers of polynomial functions.

Lemma 4.3. [2, Theorem 6]. *For $\lambda \in \Lambda_m^+$, we have*

$$\begin{aligned}\zeta_{2m}((\pi^\lambda)_{2m}; s_1, \dots, s_m) \;=\; & \prod_{k=1}^{m-1} \frac{1-q^{-1}}{1-q^{-2k-1}} \cdot \prod_{1 \le i < j \le m} \frac{1-q^{z_i - z_j - 1}}{1-q^{z_i - z_j + 1}} \\ & \times q^{2n(\lambda)-(m-1)|\lambda|} \cdot \frac{w_\lambda^{(m)}(q^{-2})}{w_m(q^{-2})} \\ & \times P_\lambda(q^{z_1}, \dots, q^{z_m}; q^{-2})\end{aligned}$$

where z is a variables in $\mathbb{C}^m$ which is related with the variable s by

$$\begin{cases} s_i &= \quad z_{i+1} - z_i - 2 \qquad (1 \le i \le m-1) \\ s_m &= \quad (m+1) - z_m - 2 \end{cases}.$$

Lemma 4.4. [18, Lemma 6.8]. *For $\lambda \in \Lambda_m^+$, we have*

$$\zeta_{2m+1}((\pi^\lambda)_{2m+1}; s) = \frac{1-q^{-1}}{1-q^{-2m-1}} \cdot \prod_{i=1}^{m} \frac{1-q^{-(s_i + \cdots + s_m + 2m - 2i + 3)}}{1-q^{-(s_i + \cdots + s_m + 2m - 2i + 1)}} \times \zeta_{2m}((\pi^\lambda)_{2m}; s).$$

4.3. **The space** (7)**.** In the space (7), the group $G = GL(1)^4 \times Sp(n) \times SL(2m+1)$ acts on $V = M(2n, 2m+1) \oplus M(2n, 1) \oplus M(2m+1, 1) \oplus M(2m+1, 1)$ by $(x_1, x_2, z, w) \mapsto (a_1 g x_1 {}^t h, a_2 g x_2, a_3 {}^t h^{-1} z, a_4 {}^t h^{-1} w)$ for $(x_1, x_2, z, w) \in V$ and $(a_1, a_2, a_3, a_4, g, h) \in G$. For $(x_1, x_2, z, w) \in V$ and $x = (x_1 | x_2) \in M(2n, 2m+2)$, the basic relative invariants f_1, f_2, f_3 and f_4 are given by $f_1 = \mathrm{Pf}\left({}^t x J_n x\right)$, $f_2 = {}^t z\, {}^t x_1 J_n x_1 w$, $f_3 = {}^t z x_1 J_n x_2$, $f_4 = {}^t w x_1 J_n x_2$ (cf. [10]). Let dz, dw be the Haar measure on $M(2m+1, 2; K)$ normalized by $\int_{M(2m+1,1;\mathcal{O}_K)} dz = \int_{M(2m+1,1;\mathcal{O}_K)} dw = 1$, and dy the Haar measure on $Alt(2m+2; K)$ normalized by $\int_{Alt(2m+2;\mathcal{O}_K)} dy = 1$. By Lemma 4.2, we have

(4.1)

$$Z(s) = \frac{w_n(q^{-2})}{w_{n-m-1}(q^{-2})} \sum_{\lambda, \mu, \nu \in \Lambda_{m+1}^+} f_{\mu,\nu}^\lambda(q^{-2}) q^{2(n(\lambda)-n(\mu)-n(\nu))-(2n-2m-1)|\mu|} I(\lambda)$$

where we put

$$I(\lambda) = \int_{A_\lambda \oplus M(2m+1,2;\mathcal{O}_K)} \prod_{i=1}^{4} |f_i'(y, z, w)|_K^{s_i} dy dz dw,$$

$A_\lambda = GL(2m+2;\mathcal{O}_K)\cdot(\pi^\lambda)_{2m+2}$, $f_1' = \mathrm{Pf}\,(y)$, $f_2 = {}^t z y_1 w$, $f_3 = {}^t z y_2$, $f_4 = {}^t w y_2$ for $y_1 \in Alt(2m+1)$, $y_2 \in M(2m+1,1)$, $y = \begin{pmatrix} y_1 & y_2 \\ -{}^t y_2 & 0 \end{pmatrix}$. Hence we have only to calculate this $I(\lambda)$. By an argument similar to that of Subsection 4.1, we easily get

$$I(\lambda) = q^{-s_1|\lambda|}\frac{(1-q^{-2m-1})(1-q^{-2m})}{(1-q^{-2m-1-s_2-2_3})}\sum_{l=0}^{\infty} q^{-2ml-s_2 l} J(l,\lambda)$$

where we put

$$J(l,\lambda) = \int_{A_\lambda\oplus\mathcal{O}_K} |y_{1,2}|_K^{s_2}|y_{1,2m+2}|_K^{s_3}|w_1 y_{1,2m+2} + \pi^l y_{2,2m+2}|_K^{s_4} dy dw_1,$$

$y = (y_{i,j}) \in Alt(2m+2)$ $(y_{j,i} = -y_{i,j})$, $w = {}^t(w_1, w_2, \ldots, w_{2m+1})$, and dw_1 is the Haar measure on K normalized by $\int_{\mathcal{O}_K} dw_1 = 1$. We need some lemmas to calculate this $J(l)$. A element $T \in M(k,k';\mathcal{O}_K)$ $(k > k')$ is said to be primitive if it can be extended to a unimodular matrix by complementing $2(k-k')$ column vectors. We set $L = \{T \in M(2m+2,3;\mathcal{O}_K); T \text{ is primitive}\}$, $L(i,\lambda) = \{T \in L;\ {}^t T(\pi^\lambda)_{2m+2}T \in GL(3;\mathcal{O}_K)\cdot(\pi^i)_3\}$ $(i \in \mathbb{Z}_{\geq 0})$. We put $v = {}^t(v_1, v_2, v_3) \in M(3,1;\mathcal{O}_K)$. Let dT be the Haar measure on $M(2m+2,3;K)$ normalized by $\int_{M(2m+2,3;\mathcal{O}_K)} dT = 1$, and dv the Haar measure on $M(3,1;K)$ normalized by $\int_{M(3,1;\mathcal{O}_K)} dv = 1$.

Lemma 4.5. *For every l and λ, we have*

$$J(l,\lambda) = B(\lambda)\times K(l),$$

where we put

$$B(\lambda) = \left(\int_{A_\lambda} dy\right)\left(\int_L dT\right)^{-1}\left(\int_{U(3)} dv\right)^{-1}\sum_{i=0}^{\infty} q^{-(s_2+s_3+s_4)i}\int_{L(i,\lambda)} dT,$$

$$K(l) = \int_{U(3)\oplus\mathcal{O}_K} |v_1|_K^{s_2}|v_2|_K^{s_3}|wv_2 + \pi^l v_3|_K^{s_4} dv dw_1.$$

Proof. Let dg be the Haar measure on $GL(2m+2;K)$ normalized by

$$\int_{GL(2m+2:\mathcal{O}_K)} dg = 1,$$

$$g = {}^t(g_1|g_2|\cdots|g_{2m+2}) \in GL(2m+2;K)\ (g_i \in M(2m+2,1;K)),$$

and

$${}^t T(\pi^\lambda)_{2m+2}T = \begin{pmatrix} 0 & \varphi_3(T) & -\varphi_2(T) \\ -\varphi_3(T) & 0 & \varphi_1(T) \\ \varphi_2(T) & -\varphi_1(T) & 0 \end{pmatrix} \in Alt(3;\mathcal{O}_K).$$

Then we have

$$\begin{aligned}
J(l,\lambda) &= \left(\int_{A_\lambda} dy\right) \\
&\quad \times \int_{GL(2m+2;\mathcal{O}_K)\oplus\mathcal{O}_K} |{}^t g_1(\pi^\lambda)_{2m+2} g_2|_K^{s_2} |{}^t g_1(\pi^\lambda)_{2m+2} g_{2m+2}|_K^{s_3} \\
&\quad \times |z_{1,2}\,{}^t g_1(\pi^\lambda)_{2m+2} g_{2m+2} + \pi^l\,{}^t g_2(\pi^\lambda)_{2m+2} g_{2m+2}|_K^{s_4} dg dw_1 \\
&= \left(\int_{A_\lambda} dy\right)\left(\int_L dT\right)^{-1} \\
&\quad \times \int_{L\oplus\mathcal{O}_K} |\varphi_1(T)|_K^{s_2} |\varphi_2(T)|_K^{s_3} |w_1\varphi_2(T) + \pi^l \varphi_3(T)|_K^{s_4} dT dw_1.
\end{aligned}$$

Let dh be the Haar measure on $GL(3;K)$ normalized by $\int_{GL(3;\mathcal{O}_K)} dh = 1$, $h = (h_{ij}) \in GL(3)$. Here we identify $Alt(3)$ as $M(3,1)$. Then we have

$$\begin{aligned}
&\int_{L\oplus\mathcal{O}_K} |\varphi_1(T)|_K^{s_2} |\varphi_2(T)|_K^{s_3} |w_1\varphi_2(T) + \pi^l \varphi_3(T)|_K^{s_4} dT dw_1 \\
&= \sum_{i=0}^{\infty} \int_{L(i,\lambda)\oplus\mathcal{O}_K} |\varphi_1(T)|_K^{s_2} |\varphi_2(T)|_K^{s_3} |w_1\varphi_2(T) + \pi^l \varphi_3(T)|_K^{s_4} dT dw_1 \\
&= \sum_{i=0}^{\infty} \int_{L(i,\lambda)\oplus\mathcal{O}_K} \int_{GL(3;\mathcal{O}_K)} |\varphi_1(T\,{}^t h)|_K^{s_2} |\varphi_2(T\,{}^t h)|_K^{s_3} |w_1\varphi_2(T\,{}^t h) \\
&\qquad\qquad + \pi^l \varphi_3(T\,{}^t h)|_K^{s_4} dh dT dw_1 \\
&= \sum_{i=0}^{\infty} \int_{L(i,\lambda)} dT \times \int_{GL(3;\mathcal{O}_K)\oplus\mathcal{O}_K} |\pi^i h_{11}|_K^{s_2} |\pi^i h_{21}|_K^{s_3} |\pi^i h_{21} w_1 \\
&\qquad\qquad + \pi^{l+i} h_{31}|_K^{s_4} dh dw_1 \\
&= \left(\int_{U(3)} dv\right)^{-1} \sum_{i=0}^{\infty} q^{-(s_2+s_3+s_4)i} \int_{L(i,\lambda)} dT \times K(l).
\end{aligned}$$

Hence we get the above equality. □

Lemma 4.6. *For every $\lambda \in \Lambda_{2m+2}^+$, we have*

$$B(\lambda) = \frac{(1-q^{-2m-1-s_2-s_3-s_4})}{(1-q^{-2m-1})(1-q^{-s_2-s_3-s_4-3})} q^{-2n(\lambda)-(m+1)|\lambda|} \times \frac{w_{2m+2}(q^{-1})}{w_{m+1}(q^{-2})} P_\lambda(q^z;q^{-2}),$$

where

$$(z_1+m+1, z_2+m+1, \ldots, z_{m+1}+m+1) = (-s_2-s_3-s_4+1, 3, \ldots, 2m+1).$$

Proof. By Lemma 4.4, we have

$$\begin{aligned}
&\zeta_{2m+2}((\pi^\lambda)_{2m+2}; s, 0, \dots, 0)\\
&\qquad = \left(\int_L dT\right)^{-1} \int_L |\mathrm{Pf}_1({}^tT(\pi^\lambda)_{2m+2}T)|_K^s dT\\
&\qquad = \left(\int_L dT\right)^{-1} \sum_{j=0}^{\infty} \int_{L(j,\lambda)} dT \cdot \zeta_3((\pi^j)_3; s)\\
&\qquad = \left(\int_L dT\right)^{-1} \sum_{j=0}^{\infty} \int_{L(j,\lambda)} dT \cdot q^{-js} \frac{(1-q^{-1})(1-q^{-s-3})}{(1-q^{-3})(1-q^{-s-1})}.
\end{aligned}$$

By Lemma 4.3, this integral $\zeta_{2m+2}((\pi^\lambda)_{2m+2}; s, 0, \dots, 0)$ is given explicitly. Hence if we set $s = s_2 + s_3 + s_4$, then we can get an explicit form of $\left(\int_L dT\right)^{-1} \sum_{j=0}^{\infty} \int_{L(j,\lambda)} dT \cdot q^{-j(s_2+s_3+s_4)}$. Therefore if we put together this result, Lemma 4.1 and $\int_{U(3)} dv = 1 - q^{-3}$, we get the above lemma. $\square$

Lemma 4.7.

$$\sum_{l=0}^{\infty} q^{-2ml-s_2 l} K(l) = \prod_{i=2}^{4} \frac{1-q^{-1}}{1-q^{-s_i-1}} \times \frac{(1-q^{-s_2-s_3-s_4-3})(1-q^{-s_2-s_3-2m-1})}{(1-q^{-s_2-2m})(1-q^{-s_2-s_3-s_4-2m-1})}.$$

Proof. We consider $l \geq 1$. We divide the domain $U(3)$ as $\mathcal{O}_K \oplus \mathcal{O}_K \oplus U(1)$, $\mathcal{O}_K \oplus U(1) \oplus \pi\mathcal{O}_K$ and $U(1) \oplus \pi\mathcal{O}_K \oplus \pi\mathcal{O}_K$. Then we have

$$\begin{aligned}
K(l) = \frac{(1-q^{-1})^2}{(1-q^{-1-s_2})} K'(l) + \frac{q^{-1}(1-q^{-1})^3}{(1-q^{-1-s_2})(1-q^{-1-s_4})}\\
+ q^{-2-s_3-s_4}(1-q^{-1})K''(l),
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
K'(l) &= \int_{\mathcal{O}_K{}^2} |v_2|_K^{s_3} |w_1 v_2 + \pi^l|_K^{s_4} dv dw_1,\\
K''(l) &= \int_{\mathcal{O}_K{}^3} |v_2|_K^{s_3} |w_1 v_2 + \pi^l v_3|_K^{s_4} dv dw_1.
\end{aligned}$$

And we have

$$\begin{aligned}
K'(l) &= \frac{(1-q^{-1})^2\{1-(q^{-s_3-s_4-1})^{l+1}\}}{(1-q^{-s_4-1})(1-q^{-s_3-s_4-1})} + q^{-1-s_3}(q^{-s_3-s_4-1})^l \frac{1-q^{-1}}{1-q^{-1-s_3}},\\
K''(l) &= \frac{(1-q^{-1})^2\{1-(q^{-s_3-s_4-1})^{l}\}}{(1-q^{-s_4-1})(1-q^{-s_3-s_4-1})} + \frac{(1-q^{-1})^2(q^{-s_3-s_4-1})^l}{(1-q^{-1-s_3})(1-q^{-1-s_4})}.
\end{aligned}$$

Since

$$K(0) = \frac{(1-q^{-1})^2(1-q^{-3-s_2-s_3-s_4})}{(1-q^{-1-s_2})(1-q^{-1-s_3})(1-q^{-1-s_4})},$$

we can also apply Equation 4.2 to the case $l = 0$. Therefore by putting together the above results, we get this lemma. $\square$

From Lemma 4.5, 4.6 and 4.7, we get the following lemma.

Lemma 4.8.

$$\begin{aligned} I(\lambda) &= \prod_{i=2}^{4} \frac{1-q^{-1}}{1-q^{-s_i-1}} \times \frac{1-q^{-2m}}{1-q^{-s_2-2m}} \times \frac{w_{2m+2}(q^{-1})}{w_{m+1}(q^{-2})} \\ &\quad \times q^{-2n(\lambda)} P_\lambda(q^{-s_1-s_2-s_3-s_4-2m-1}, q^{-s_1-2m+1}, \dots, q^{-s_1-1}; q^{-2}). \end{aligned}$$

By this lemma, Equation 4.1 and

$$\sum_{\lambda \in \Lambda_n^+} t^{n(\lambda)} P_\lambda(x_1, x_2, \dots, x_n; t) = \prod_{i=1}^{n} (1-x_i)^{-1} \tag{4.3}$$

(cf. [11, Chapter 3, Section 4, Example 1]), we get an explicit form of the Igusa local zeta functions of the space (7).

4.4. **The space** (6)**.** In the space (6), the group $G = GL(1)^4 \times Sp(n) \times SL(2m+1)$ acts on $V = M(2n, 2m+1) \oplus M(2n, 1) \oplus M(2m+1, 1) \oplus M(2m+1, 1)$ by $(x_1, x_2, z, w) \mapsto (a_1 g x_1 {}^t h, a_2 g x_2, a_3 h z, a_4 h w)$ for $(x_1, x_2, z, w) \in V$ and $(a_1, a_2, a_3, a_4, g, h) \in G$. For $y \in Alt(2m+2)$, we denote by $\Delta(y)$ the copfaffian of y, i.e. $y\Delta(y) = \Delta(y)y = -\mathrm{Pf}(y) I_{2m+2}$, and we set $\Delta(y) = \begin{pmatrix} \Delta_1(y) & \Delta_2(y) \\ -{}^t\Delta_2(y) & 0 \end{pmatrix}$, $\Delta_1(y) \in Alt(2m+1)$, $\Delta_2(y) \in M(2m+1, 1)$. For $(x_1, x_2, z, w) \in V$ and $x = (x_1|x_2) \in M(2n, 2m+2)$, the basic relative invariants f_1, f_2, f_3 and f_4 are given by $f_1 = \mathrm{Pf}\left({}^t x J_n x\right)$, $f_2 = {}^t z \Delta_1({}^t x J_n x) w$, $f_3 = {}^t z \Delta_2({}^t x J_n x)$, $f_4 = {}^t w \Delta_2({}^t x J_n x)$ (cf. [10]). Throughout this subsection, we assume that the notations dy, dz, dw, dg, A_λ and $I(\lambda)$ are the same as those of Subsection 4.3. By Lemma 4.2, we have

$$\begin{aligned} Z(s) &= \frac{w_n(q^{-2})}{w_{n-m-1}(q^{-2})} \\ &\quad \times \sum_{\lambda,\mu,\nu \in \Lambda_{m+1}^+} f_{\mu,\nu}^{\lambda}(q^{-2}) q^{2(n(\lambda)-n(\mu)-n(\nu))-(2n-2m-1)|\mu|} I'(\lambda), \end{aligned} \tag{4.4}$$

where we put

$$I'(\lambda) = \int_{A_\lambda \oplus M(2m+1,2;\mathcal{O}_K)} \prod_{i=1}^{4} |f_i'(y,z,w)|_K^{s_i} dy dz dw,$$

$f_1' = \mathrm{Pf}(y)$, $f_2' = {}^t z \Delta_1(y) w$, $f_3' = {}^t z \Delta_2(y)$, $f_4' = {}^t w \Delta_2(y)$. Hence we have only to calculate this integral $I'(\lambda)$. We put

$$\tau = (|\lambda| - \lambda_{m+1}, |\lambda| - \lambda_m, \dots, |\lambda| - \lambda_1) \in \Lambda_{m+1}^+.$$

Since $\Delta(y) \in A_\tau$ for $y \in A_\lambda$, we get $I'(\lambda) = \left(\int_{A_\lambda} dy\right) \left(\int_{A_\tau} dy\right)^{-1} I(\tau)$. Then from $4n(\tau) + |\tau| = m^2|\lambda| + 4n(\lambda)$, Lemma 4.1 and Lemma 4.8, we get

$$I'(\lambda) = \prod_{i=2}^{4} \frac{1-q^{-1}}{1-q^{-s_i-1}} \times \frac{1-q^{-2m}}{1-q^{-s_2-2m}} \times \frac{w_{2m+2}(q^{-1})}{w_{m+1}(q^{-2})} \times q^{-2n(\lambda)}$$
$$\times P_\lambda(q^{-s_1-1}, q^{-(s_1+s_2+s_3+s_4+3)}, \ldots, q^{-(s_1+s_2+s_3+s_4+2m+1)}; q^{-2}).$$

Here we use the equality

$$P_\tau(x_1, x_2, \ldots, x_{m+1}; t) = (x_1 x_2 \cdots x_{m+1})^{|\lambda|} P_\lambda(x_1^{-1}, x_2^{-1}, \ldots, x_{m+1}^{-1}; t).$$

Therefore by Equation 4.3 and 4.4, we get an explicit form of the Igusa local zeta function of the space (6).

5. The space (1)

In this section, we calculate explicitly the Igusa local zeta function of one variable for each basic relative invariant of the space (1). In this space, the group $G = GL(1)^3 \times SL(5) \times SL(2)$ acts on $V = Alt(5) \oplus Alt(5) \oplus M(5,1) \oplus M(5,1)$ by $(x, y, z, w) \mapsto (a_1(gx\,{}^tg, gy\,{}^tg)\,{}^th, a_2\,{}^tg^{-1}z, a_3\,{}^tg^{-1}w)$ for $(x, y, z, w) \in V$ and $(a_1, a_2, a_3, g, h) \in G$. Let $x^{(i)} \in Alt(4)$ be the alternating matrix obtained from $x \in Alt(5)$ by subtracting the i-th row and column $(1 \le i \le 5)$. Then the basic relative invariants f_1, f_2 and f_3 are given by

$$f_1 = \det \begin{pmatrix} \beta(x)yz & \beta(x)yw \\ -\beta(y)xz & -\beta(y)xw \end{pmatrix},$$
$$f_2 = \det \begin{pmatrix} \beta(x)yz & {}^tzxw \\ -\beta(y)xz & {}^tzyw \end{pmatrix}, \quad f_3 = \det \begin{pmatrix} \beta(x)yw & {}^tzxw \\ -\beta(y)xw & {}^tzyw \end{pmatrix},$$

where we put $\beta(x) = {}^t(\beta_1(x), \ldots, \beta_5(x))$, $\beta_i(x) = (-1)^{i-1}\mathrm{Pf}(x^{(i)})$ (cf. [10]). We set $Z_{f_1}(s_1) = Z(s_1, 0, 0)$, $Z_{f_2}(s_2) = Z(0, s_2, 0)$, $Z_{f_3}(s_3) = Z(0, 0, s_3)$.

Proposition 5.1.

$$Z_{f_1}(s) = \frac{(1-q^{-1})(1-q^{-2})^2(1-q^{-3})(1-q^{-4})(1-q^{-5})}{(1-q^{-1-s})(1-q^{-2-s})^2(1-q^{-3-s})(1-q^{-4-2s})(1-q^{-5-2s})},$$

$$Z_{f_2}(s) = Z_{f_3}(s)$$
$$= \frac{(1-q^{-1})(1-q^{-2})(1-q^{-3})(1-q^{-4})(1-q^{-5})}{(1-q^{-1-s})(1-q^{-2-s})(1-q^{-3-s})(1-q^{-4-2s})(1-q^{-5-2s})}.$$

Proof. Let dx, dy be the Haar measure on $Alt(5; K)$ normalized by $\int_{Alt(5;\mathcal{O}_K)} dx = \int_{Alt(5;\mathcal{O}_K)} dy = 1$, dz, dw the Haar measure on $M(5,1; K)$ normalized by $\int_{M(5,1;\mathcal{O}_K)} dz = \int_{M(5,1;\mathcal{O}_K)} dw = 1$, dg' the Haar measure on $GL(5; K)$ normalized by $\int_{GL(5;\mathcal{O}_K)} dg' = 1$, and dg the Haar measure on $GL(4; K)$ normalized by $\int_{GL(4;\mathcal{O}_K)} dg = 1$. First we shall calculate the integral $Z_{f_1}(s)$ For $\lambda \in \Lambda_2^+$, we put $A'_\lambda = GL(5;\mathcal{O}_K)\cdot(\pi^\lambda)_5$ and $V(\lambda) = A'_\lambda \oplus Alt(5;\mathcal{O}_K) \oplus M(5,2;\mathcal{O}_K)$. Then we have

$$\begin{aligned}
Z_{f_1}(s) &= \sum_{\lambda\in\Lambda_2^+} \int_{V(\lambda)} \int_{GL(5;\mathcal{O}_K)} |f_1(g'(\pi^\lambda)_5\,{}^tg', y, z, w)|_K^s dg'dxdydzdw \\
&= \sum_{\lambda\in\Lambda_2^+} \int_{V(\lambda)} \int_{GL(4;\mathcal{O}_K)} |f_1(x_0, y, z, w)|_K^s dgdxdydzdw \\
&= \frac{1-q^{-4}}{1-q^{-4-2s}} \\
&\quad \times \sum_{\lambda\in\Lambda_2^+} \int_{V(\lambda)} \int_{GL(4;\mathcal{O}_K)} \left|f_1\left(x_0, \begin{pmatrix} y_{ij} & e_4 \\ -e_4 & 0 \end{pmatrix}, z, w\right)\right|_K^s dgdxdydzdw \\
&= \frac{1-q^{-4}}{1-q^{-4-2s}}\frac{1-q^{-3}}{1-q^{-3-s}} \\
&\quad \times \sum_{\lambda\in\Lambda_2^+} \int_{V(\lambda)} \int_{GL(4;\mathcal{O}_K)} |f_1(x_0, y_0, z, w)|_K^s dgdxdydzdw,
\end{aligned}$$

where we put

$$x_0 = \begin{pmatrix} g(\pi^\lambda)_4\,{}^tg & \\ & 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 1 & 0 & y_{14} & 0 \\ -1 & 0 & 0 & y_{24} & 0 \\ 0 & 0 & 0 & y_{34} & 0 \\ -y_{14} & -y_{24} & -y_{34} & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

If we put $g = {}^t(g_1|g_2|g_3|g_4) \in GL(4;\mathcal{O}_K)$, then we have

$$\begin{aligned}
&f(x_0, y_0, z, w) \\
&\quad = \det\left(\begin{array}{c|c} z_4 & z_2\cdot{}^tg_1(\pi^\lambda)_4\,g_2 + z_3\cdot{}^tg_1(\pi^\lambda)_4\,g_3 + z_4\cdot{}^tg_1(\pi^\lambda)_4\,g_4 \\ \hline w_4 & w_2\cdot{}^tg_1(\pi^\lambda)_4\,g_2 + w_3\cdot{}^tg_1(\pi^\lambda)_4\,g_3 + w_4\cdot{}^tg_1(\pi^\lambda)_4\,g_4 \end{array}\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_{V(\lambda)} \int_{GL(4;\mathcal{O}_K)} |f(x_0, y_0, z, w)|_K^s dgdxdydzdw \\
&\quad = q^{-s|\lambda|} \int_{V(\lambda)} \int_{GL(4;\mathcal{O}_K)} |\det A|_K^s dgdxdydzdw, \\
&\qquad\qquad \text{where } A = \left(\begin{array}{c|c} z_4 & z_2\cdot{}^tg_1(\pi^\lambda)_4\,g_2 + z_3\cdot{}^tg_1(\pi^\lambda)_4\,g_3 \\ \hline w_4 & w_2\cdot{}^tg_1(\pi^\lambda)_4\,g_2 + w_3\cdot{}^tg_1(\pi^\lambda)_4\,g_3 \end{array}\right) \\
&\quad = q^{-s|\lambda|}\frac{1-q^{-2}}{1-q^{-2-s}} \int_{V(\lambda)} \int_{GL(4;\mathcal{O}_K)} |z_2\cdot{}^tg_1(\pi^\lambda)_4\,g_2 \\
&\qquad\qquad + z_3\cdot{}^tg_1(\pi^\lambda)_4\,g_3|^s dgdxdydzdw
\end{aligned}$$

$$= q^{-s|\lambda|} \frac{(1-q^{-2})^2}{(1-q^{-2-s})^2} \left(\int_{A'_\lambda} dx \right) \int_{GL(4;\mathcal{O}_K)} |{}^t g_1 (\pi^\lambda)_4 \, g_2|^s dg$$
$$= q^{-s|\lambda|} \frac{(1-q^{-2})^2}{(1-q^{-2-s})^2} \left(\int_{A'_\lambda} dx \right) \zeta_4((\pi^\lambda)_4; s, 0).$$

By Lemma 4.1 and 4.3, we have

$$Z_{f_1}(s) = \frac{(1-q^{-1})(1-q^{-2})^2(1-q^{-3})(1-q^{-4})(1-q^{-5})}{(1-q^{-1-s})(1-q^{-2-s})^2(1-q^{-4-2s})} \times \sum_{\lambda \in \Lambda_2^+} q^{-2n(\lambda)} P_\lambda(q^{-2s-5}, q^{-s-3}; q^{-2}).$$

By Equation 4.3, we get an explicit form of $Z_{f_1}(s)$.

Next we shall calculate the integral $Z_{f_2}(s)$. For $\lambda \in \Lambda_2^+$, we put

$$A''_\lambda = GL(2;\mathcal{O}_K) \begin{pmatrix} \pi^{\lambda_1} & 0 & 0 & 0 \\ 0 & \pi^{\lambda_2} & 0 & 0 \end{pmatrix} GL(4;\mathcal{O}_K) \subset M(2,4;\mathcal{O}_K),$$

$$z_0 = {}^t(1,0,0,0,0),$$

$$x(\lambda_1) = \begin{pmatrix} 0 & \pi^{\lambda_1} & 0 & 0 & 0 \\ -\pi^{\lambda_1} & 0 & x_{23} & x_{24} & x_{25} \\ 0 & -x_{23} & 0 & x_{34} & x_{35} \\ 0 & -x_{24} & -x_{34} & 0 & x_{45} \\ 0 & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix},$$

$$y(\lambda_2) = \begin{pmatrix} 0 & 0 & \pi^{\lambda_2} & 0 & 0 \\ 0 & 0 & y_{23} & y_{24} & y_{25} \\ -\pi^{\lambda_2} & -y_{23} & 0 & y_{34} & y_{35} \\ 0 & -y_{24} & -y_{34} & 0 & y_{45} \\ 0 & -y_{25} & -y_{35} & -y_{45} & 0 \end{pmatrix}.$$

We identify $Alt(5)^{\oplus 2}$ as $Alt(4)^{\oplus 2} \oplus M(2,4)$. Then we have

$$Z_{f_2}(s) = \frac{1-q^{-5}}{1-q^{-5-2s}} \times \sum_{\lambda \in \Lambda_2^+} \int_{Alt(4;\mathcal{O}_K)^{\oplus 2} \oplus A''_\lambda \oplus M(5,1;\mathcal{O}_K)} |f(x(\lambda_1), y(\lambda_2), z_0, w)|_K^s dx dy dw$$
$$= \frac{1-q^{-5}}{1-q^{-5-2s}} \sum_{\lambda \in \Lambda_2^+} q^{-|\lambda| s} \left(\int_{A''_\lambda} dv \right) \int_{\mathcal{O}_K^4} |\pi^{\lambda_2} x_{45} w_3 + \pi^{\lambda_1} y_{45} w_2|_K^s dx_{45} dy_{45} dw_2 dw_3,$$

where dv is the Haar measure on $M(2,4;K)$ normalized by $\int_{M(2,4;\mathcal{O}_K)} dv = 1$. Put $\tau = \lambda_1 - \lambda_2 \geq 0$. From [18, Section 3], we have

$$\int_{A''_\lambda} dv = \frac{w_4(q^{-1})}{w_\lambda^{(2)}(q^{-1})} \cdot q^{-3|\lambda|-2n(\lambda)}$$
$$= \begin{cases} (1+q^{-1})(1-q^{-3})(1-q^{-4})q^{-8\lambda_2-3\tau} & (\tau \geq 1) \\ (1-q^{-3})(1-q^{-4})q^{-8\lambda_2} & (\tau = 0) \end{cases}.$$

By simple calculation, for $\tau \geq 1$ we have

$$\int_{\mathcal{O}_K^4} |x_{45}w_3 + \pi^\tau y_{45}w_2|_K^s dx_{45}dy_{45}dw_2dw_3$$
$$= \frac{(1-q^{-1})^2}{(1-q^{-1-s})^2} + \frac{q^{(-1-s)\tau-1}(1-q^{-1})^2(1-q^{-s})}{(1-q^{-1-s})^2(1-q^{-2-s})}.$$

Therefore if we put together the above results, we get an explicit form of $Z_{f_2}(s)$. We also have $Z_{f_3}(s) = Z_{f_2}(s)$ since $f_2(x,y,z,w) = -f_3(x,y,w,z)$. $\square$

Finally we shall discuss the Igusa local zeta function of several variables for the space (1). K. Sugiyama communicated to the author an explicit form of the b-function. The author heard that he calculated the b-function by using the method of [16] and contractions of this space. The author also calculated explicitly this b-function by using the method of [16] and the computer soft *Mathematica*. The b-function of the space (1) is given by $b_m(s) = \prod_{j=1}^8 \Gamma(\eta_j(s+m))/\Gamma(\eta_j(s))$, where

$$\{\eta_j(s);\, j=1,\ldots,8\} = \left\{ \begin{array}{c} s_1+1,\, s_1+2,\, s_2+1,\, s_3+1, \\ s_1+s_2+s_3+2,\, s_1+s_2+s_3+3, \\ 2s_1+2s_2+2s_3+4,\, 2s_1+2s_2+2s_3+5 \end{array} \right\}.$$

By [7] and [12], we see that this explicit form of the b-function must correspond to an explicit form of the b-function, which is given by an explicit form of the Igusa local zeta function $Z(s)$. From this relation, we expect that the factors of the denominator of $Z(s)$ are expressed by $1-q^{-(s_i+a)}$, $1-q^{-(s_1+s_2+s_3+a)}$ or $1-q^{-(2(s_1+s_2+s_3)+a)}$ $(a \in \mathbb{Z}_{>0})$. Furthermore for each known space with a universally transitive open orbit, we have a certain set $\{\eta_j(s)\}$ such that $Z(s) = \prod(1-q^{-\eta_j(0)})/(1-q^{-\eta_j(s)})$ and $b_m(s) = \prod \Gamma(\eta_j(s+m))/\Gamma(\eta_j(s))$. Therefore we expect that the Igusa local zeta function of the space (1) is also given by $Z(s) = \prod_{j=1}^8 (1-q^{-\eta_j(0)})/(1-q^{-\eta_j(s)})$ for the above set $\{\eta_j(s)\}$, and the p-adic Γ-factor of the space (1) is given by $\gamma(s) = \prod_{i=1}^8 \gamma^T(\eta_j(s-\kappa))$, where $\kappa = (2,1,1)$.

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(*Received May 14, 2004*)