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ON STRONG APPROXIMATION OF FUNCTIONS BY CERTAIN LINEAR OPERATORS

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Abstract. This note is motivated by the results on the strong approximation of 2π-periodic functions by means of trigonometric Fourier series. In this note is investigated certain class of positive linear operators in the polynomial weighted spaces. We introduce the strong differences of functions and their operators and we give the Jackson type theorems for them. We give also some corollaries.

1. Introduction

1.1. The problem of strong approximation of functions connected with Fourier series was examined in many papers presented by G. Alexits, K. Tandori, L. Leindler, R. Taberski, V. Totik and other authors (see [5]).

The monograph [5] is devoted to the strong approximation of 2π-periodic functions belonging to various classes by the means of trigonometric Fourier series.

For example, if $S_n(f;x)$ is the n-th partial sum of trigonometric Fourier series of f, then the n-th $(C,1)$-mean of this series is defined by the formula

$$\sigma_n(f;x) := \frac{1}{n+1}\sum_{k=0}^{n} S_k(f;x), \qquad n \in N_0 = \{0,1,...\}.$$

The n-th strong $(C,1)$-mean of this series is defined as follows

$$H_n^q(f;x) := \left\{\frac{1}{n+1}\sum_{k=0}^{n} |S_k(f;x) - f(x)|^q\right\}^{\frac{1}{q}}, \qquad n \in N_0,$$

where q is a fixed positive number. It is clear that

$$|\sigma_n(f;x) - f(x)| \le H_n^1(f;x)$$

and

$$H_n^q(f;x) \le H_n^p(f;x), \qquad 0 < q < p < \infty,$$

for all $x \in R$ and $n \in N_0$. The last inequalities show that examination of the strong means of Fourier series is useful.

The purpose of this note is to show that investigation of the strong approximation connected with linear operators is also useful.

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1.2. In [2] were examined approximation properties of the Szász-Mirakjan operators ([6])

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1}$$

and the Baskakov operators ([1])

$$V_n(f;x) := \sum_{k=0}^{\infty} \binom{n-1+k}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right), \tag{2}$$

$n \in N = \{1, 2, ...\}$, $x \in R_0 = [0, \infty)$, for the functions f belonging to the polynomial weighted spaces C_p, $p \in N_0$. The space C_p, $p \in N_0$, is associated with the weighted function

$$w_0(x) := 1, \qquad w_p(x) := (1 + x^p)^{-1} \quad \text{if} \quad p \geq 1, \tag{3}$$

and it is the set of all real-valued functions f for which $w_p f$ is uniformly continuous and bounded on R_0 and the norm is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in R_0} w_p(x)\, |f(x)|. \tag{4}$$

The author proved in [2] that for every $p \in N_0$ there exists a positive constant $M(p)$ depending only on p such that for every $f \in C_p$ there holds

$$w_p(x)\, |V_n(f;x) - f(x)| \leq M(p)\, \omega_2\left(f; \sqrt{(x+x^2)/n}\right) \quad n \in N,\ x \in R_0, \tag{5}$$

where $\omega_2(f;\cdot)$ is the second modulus of smoothness of f. From (5) it follows that

$$\lim_{n\to\infty} V_n(f;x) = f(x), \qquad x \in R_0, \quad f \in C_p, \tag{6}$$

and this convergence is uniform on every interval $[x_1, x_2]$, $x_1 \geq 0$.

The analogous results for the Szász-Mirakyan operators are given in [2] also.

In this note we introduce certain class of linear operators in the spaces C_p and we define the strong differences for them. We give two theorems and some corollaries on these strong differences.

We shall denote by $M_k(\alpha, \beta)$, $k \in N$, suitable positive constants depending only on indicated parameters α, β.

2. Definitions and preliminary results

2.1. Let Ω be the set of all infinite matrices $A = [a_{nk}]$, $n \in N$, $k \in N_0$, of functions in C_0 having the following properties:

(i) $a_{nk}(x) \geq 0$ for $x \in R_0$, $n \in N$, $k \in N_0$,

(ii) $\sum_{k=0}^{\infty} a_{nk}(x) = 1$ for $x \in R_0$, $n \in N$, (*iii*) for every $n, r \in N$ the series $\sum_{k=0}^{\infty} k^r a_{nk}(x)$ is uniformly convergent on R_0 and its sum is a function belonging to the space C_r,

(*iv*) for every $r \in N$ there exists positive constant $M_1(r, A)$ independent on $x \in R_0$ and $n \in N$ such that for the functions

$$T_{n,2r}(x; A) := \sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n} - x\right)^{2r}, \qquad x \in R_0, \tag{7}$$

(belonging to C_{2r}) there holds

$$\|T_{n,2r}(\,\cdot\,; A)\|_{2r} \leq M_1(r, A)\, n^{-r}, \qquad n \in N.$$

Choosing $A \in \Omega$ and $p \in N_0$ we define for $f \in C_p$ the following positive linear operators

$$L_n(f; A; x) := \sum_{k=0}^{\infty} a_{nk}(x)\, f\left(\frac{k}{n}\right), \qquad n \in N, \quad x \in R_0. \tag{8}$$

The properties (i)-(iv) of the matrix A imply that the operators $L_n(f; A)$ are well-defined and

$$L_n(1; A; x) = 1 \qquad \text{for} \quad x \in R_0, \quad n \in N, \tag{9}$$

and by (8) and (9) we have

$$L_n(f; A; x) - f(x) = \sum_{k=0}^{\infty} a_{nk}(x) \left(f\left(\frac{k}{n}\right) - f(x)\right). \tag{10}$$

For $L_n(f; A)$ and $f \in C_p$ we define the strong difference with the power $q > 0$ as follows:

$$H_n^q(f; A; x) := \left\{\sum_{k=0}^{\infty} a_{nk}(x) \left|f\left(\frac{k}{n}\right) - f(x)\right|^q\right\}^{\frac{1}{q}}, \quad x \in R_0,\ n \in N. \tag{11}$$

Then we see that by the properties (i)-(iv) of A the $H_n^q(f; A)$ are well-defined for every $f \in C_p$, $p \in N_0$, and $q > 0$. Moreover (10) and (11) imply that

$$H_n^q(f; A; x) = \left\{L_n\left(|f(t) - f(x)|^q; A; x\right)\right\}^{\frac{1}{q}}, \tag{12}$$

$$|L_n(f; A; x) - f(x)| \leq H_n^1(f; A; x), \tag{13}$$

and by the Hölder inequality and (12) and (9)

$$H_n^q(f; A; x) \leq H_n^r(f; A; x), \qquad 0 < q < r < \infty, \tag{14}$$

for every $f \in C_p$, $x \in R_0$ and $n \in N$.

2.2. First we shall give some properties of the operators $L_n(f;A)$.

Lemma 2.1. *Let $A \in \Omega$, $p \in N_0$ and $q > 0$ be fixed. Then there exists $M_2 \equiv M_2(p,q,A) > 0$ such that*

$$\| \left(L_n\left((w_p(t))^{-q};A;\cdot\right)\right)^{\frac{1}{q}} \|_p \le M_2, \qquad n \in N, \tag{15}$$

and for every $f \in C_p$

$$\| \left(L_n\left(|f|^q;A;\cdot\right)\right)^{\frac{1}{q}} \|_p \le M_2 \|f\|_p, \qquad n \in N. \tag{16}$$

Proof. a) Let $q = 1$. From (3), (8) and (9) we get

$$\begin{aligned} L_n(1/w_p(t);A;x) &= 1 + L_n\left(t^p;A;x\right) \\ &\le 1 + 2^p\left(L_n\left(|t-x|^p;A;x\right) + x^p\right) \\ &\le 2^p\left((w_p(x))^{-1} + L_n\left(|t-x|^p;A;x\right)\right), \ x \in R_0, \ n \in N. \end{aligned}$$

By the Hölder inequality and (9), we have

$$L_n\left(|t-x|^p;A;x\right) \le \left(L_n\left((t-x)^{2p};A;x\right)\right)^{\frac{1}{2}}$$

and by the inequality $(w_p(x))^2 \le w_{2p}(x)$ for $x \in R_0$, we get

$$w_p(x)\, L_n(1/w_p(t);A;x) \le 2^p\left(1 + \left(w_{2p}(x)\, L_n\left((t-x)^{2p};A;x\right)\right)^{\frac{1}{2}}\right).$$

Applying (7) and the inequality given in (iv), we obtain

$$w_p(x)\, L_n(1/w_p(t);A;x) \le M_2(p,A) \qquad \text{for} \quad x \in R_0, \quad n \in N,$$

which by (4) implies (15).

b) Let $q \ge 2$ be integer. From (3) we get the following inequalities

$$(w_p(x))^q \le w_{pq}(x), \qquad (w_p(x))^{-q} \le 2^q (w_{pq}(x))^{-1}, \tag{17}$$

for $x \in R_0$. Applying (17) we can write

$$\begin{aligned} w_p(x)\left(L_n\left((w_p(t))^{-q};A;x\right)\right)^{\frac{1}{q}} &\le 2\ \left(w_{pq}(x) L_n\left(1/w_{pq}(t);A;x\right)\right)^{\frac{1}{q}} \\ &\le 2\ \left(\|L_n\left(1/w_{pq}(t);A;\cdot\right)\|_{pq}\right)^{\frac{1}{q}}, \end{aligned}$$

and we can apply (15) for the last norm. This implies (15).

c) Let $0 < q \notin N$. Then by the Hölder inequality and (9) we get

$$\left(L_n\left((w_p(t))^{-q};A;x\right)\right)^{\frac{1}{q}} \le \left(L_n\left((w_p(t))^{-r};A;x\right)\right)^{\frac{1}{r}}, \qquad x \in R_0,$$

for every $0 < q < r < \infty$. In particular setting $r = [q]+1$ ($[q]$ denotes the integral part of q), we have

$$\left\|\left(L_n\left((w_p(t))^{-q};A;\cdot\right)\right)^{\frac{1}{q}}\right\|_p \le \left\|\left(L_n\left((w_p(t))^{-r};A;\cdot\right)\right)^{\frac{1}{r}}\right\|_p,$$

and by the case b) we obtain (15) for $0 < q \notin N$. Thus the proof of (15) is completed.

If $f \in C_p$ and $q > 0$, then by (8) and (4) we get

$$\left\| \left(L_n\left(|f|^q; A; \cdot\right)\right)^{\frac{1}{q}} \right\|_p \le \|f\|_p \left\| \left(L_n\left((w_p(t))^{-q}; A; \cdot\right)\right)^{\frac{1}{q}} \right\|_p,$$

and by (15) we obtain (16). $\square$

Lemma 2.2. *Let A, p and q be as in Lemma 2.1. Then there exists $M_3 \equiv M_3(p,q,A) > 0$ such that*

$$w_p(x) \left\{ L_n\left(\left(\frac{|t-x|}{w_p(t)} \right)^q; A; x \right) \right\}^{\frac{1}{q}} \le M_3 \left(L_n\left((t-x)^{2s}; A; x\right)\right)^{\frac{1}{2s}} \tag{18}$$

for all $x \in R_0$ and $n \in N$, where

$$s = \begin{cases} q & \text{if} \quad q \in N, \\ [q]+1 & \text{if} \quad 0 < q \notin N. \end{cases} \tag{19}$$

Proof. By (8) and by the Hölder inequality we get

$$w_p(x) \left(L_n\left(\left(\frac{|t-x|}{w_p(t)} \right)^q; A; x \right) \right)^{\frac{1}{q}} \le w_p(x) \left(L_n\left((w_p(t))^{-2q}; A; x\right)\right)^{\frac{1}{2q}} \times$$
$$\times \left(L_n\left((t-x)^{2q}; A; x\right)\right)^{\frac{1}{2q}}$$

for all $x \in R_0$, $n \in N$. Applying (15) and the inequality

$$\left(L_n\left(|t-x|^q; A; x\right)\right)^{\frac{1}{q}} \le \left(L_n\left(|t-x|^r; A; x\right)\right)^{\frac{1}{r}}, \qquad x \in R_0, \quad n \in N, \tag{20}$$

for $0 < q < r < \infty$, we easily obtain the desired estimation (18). $\square$

Lemma 2.2 and the property (iv) of A imply the following

Corollary 1. For every matrix $A \in \Omega$, $p \in N_0$ and $q > 0$ there exists $M_4 \equiv M_4(p,q,A) > 0$ such that

$$w_p(x) \left\{ L_n\left(\left(\frac{|t-x|}{w_p(t)} \right)^q; A; x \right) \right\}^{\frac{1}{q}} \le M_4 \frac{1+x}{\sqrt{n}}$$

for all $x \in R_0$ and $n \in N$.

3. Theorems and corollaries

3.1. First we shall give two theorems on the strong differences $H_n^q(f;A)$ defined by (11). We shall use the modulus of continuity of $f \in C_p$ ([3])

$$\omega(f;t) = \sup_{0 \le h \le t} \|\Delta_h f(\cdot)\|_p, \qquad t \ge 0, \tag{21}$$

where $\Delta_h f(x) = f(x+h) - f(x)$.

It is known ([3]) that $\lim_{t\to 0^+}\omega(f;t)=0$ for every $f\in C_p$, $p\in N_0$.

Let C_p^1 be the class of all $f\in C_p$ having the first derivative on R_0 and $f'\in C_p$.

Theorem 3.1. *Suppose that $A\in\Omega$, $p\in N_0$ and $q>0$. Then there exist $M_5\equiv M_5(p,q,A)>0$ such that for every $f\in C_p^1$ there holds*

$$w_p(x)\,H_n^q(f;A;x)\le M_5\,\|f'\|_p\,(T_{n,2s}(x;A))^{\frac{1}{2s}}\,, \tag{22}$$

for all $x\in R_0$ and $n\in N$, where $T_{n,2s}(\,\cdot\,;A)$ is defined by (7) *and s is given by* (19).

Proof. For $f\in C_p^1$ and $t,x\in R_0$ we have

$$|f(t)-f(x)|=\left|\int_x^t f'(u)du\right|\le\|f'\|_p\left(\frac{1}{w_p(t)}+\frac{1}{w_p(x)}\right)|t-x|.$$

From this we get

$$H_n^q(f;A;x)\le\|f'\|_p\left(L_n\left(\left(\frac{1}{w_p(t)}+\frac{1}{w_p(x)}\right)^q|t-x|^q;A;x\right)\right)^{\frac1q}$$

and further

$$w_p(x)\,H_n^q(f;A;x)\le 2\,\|f'\|_p\left\{w_p(x)\left(L_n\left(\left(\frac{|t-x|}{w_p(t)}\right)^q;A;x\right)\right)^{\frac1q}+\right.$$
$$\left.+\left(L_n\left(|t-x|^q;A;x\right)\right)^{\frac1q}\right\}$$

for $x\in R_0$ and $n\in N$. Appling Lemma 2.2 and (7) and the inequality (20) with $r=2q$, we obtain

$$w_p(x)\,H_n^q(f;A;x)\le 2\,\left\|f'\right\|_p(T_{n,2s}(x;A))^{\frac{1}{2s}}\,(M_3(p,q,A)+1)$$

for $x\in R_0$, $n\in N$ and s defined by (19). Thus the proof of (22) is completed. □

Theorem 3.2. *Let $A\in\Omega$, $p\in N_0$ and $q>0$. Then there exists $M_6\equiv M_6(p,q,A)=\text{const.}>0$ such that for every $f\in C_p$ we have*

$$w_p(x)\,H_n^q(f;A;x)\le M_6\,\omega\left(f;\frac{1+x}{\sqrt{n}}\right), \tag{23}$$

for all $x\in R_0$ and $n\in N$, where $\omega(f;\,\cdot\,)$ is the modulus of continuity of f, defined by (21).

Proof. Let $q\ge 1$. We shall apply the Stieklov function f_h for $f\in C_p$:

$$f_h(x):=\frac1h\int_0^h f(x+u)du,\qquad x\in R_0,\quad h>0.$$

From this formula and (21) we get for $h > 0$:

$$\|f - f_h\|_p \leq \omega(f; h), \tag{24}$$

$$\|f_h'\|_p \leq h^{-1}\,\omega(f; h), \tag{25}$$

i.e. $f_h \in C_p^1$ if $f \in C_p$. It is obvious that

$$|f(t) - f(x)| \leq |f(t) - f_h(t)| + |f_h(t) - f_h(x)| + |f_h(x) - f(x)|$$

for $x, t \in R_0$ and $h > 0$. This fact and (12), (8) and (9) and the Minkowski inequality imply that

$$\begin{aligned} H_n^q(f; A; x) &\leq \left(L_n\left(|f(t) - f_h(t)|^q; A; x\right)\right)^{\frac{1}{q}} + \\ &\quad + \left(L_n\left(|f_h(t) - f_h(x)|^q; A; x\right)\right)^{\frac{1}{q}} + |f_h(x) - f(x)| \\ &:= \sum_{i=1}^{3} Z_{n,i}(x), \end{aligned}$$

for $x \in R_0$, $n \in N$ and $h > 0$. By (24) we have

$$\|Z_{n,3}(\,\cdot\,)\|_p \leq \omega(f; h), \qquad h > 0.$$

Applying (16) and (24), we get

$$\|Z_{n,1}(\,\cdot\,)\|_p \leq M_2(p, q, A)\, \|f - f_h\|_p \leq M_2(p, q, A)\, \omega(f; h), \quad h > 0.$$

By Theorem 3.1 and (25) we have

$$\begin{aligned} w_p(x)\, Z_{n,2}(x) &\leq M_5\, \|f_h'\|_p\, (T_{n,2s}(x; A))^{\frac{1}{2s}} \\ &\leq M_5\, h^{-1}\, \omega(f; h)\, (T_{n,2s}(x; A))^{\frac{1}{2s}}\,. \end{aligned}$$

From the above and by the property (iv) of A we obtain

$$w_p(x)\, H_n^q(f; A; x) \leq M_6(p, q, A)\, \omega(f; h) \left(1 + h^{-1}\, \frac{1 + x}{\sqrt{n}}\right).$$

Setting $h = \frac{1+x}{\sqrt{n}}$, we obtain (23) for $q \geq 1$.

If $0 < q < 1$ then by (14) we have

$$H_n^q(f; A; x) \leq H_n^1(f; A; x), \qquad x \in R_0, \quad n \in N,$$

and by (23) for $q = 1$ we get (23) for $0 < q < 1$. □

Theorem 3.2 implies the following

Corollary 2. If the assumptions of Theorem 3.2 are satisfied, then for every $f \in C_p$, $p \in N_0$, we have

$$\lim_{n\to\infty} H_n^q(f; A; x) = 0 \quad \textit{at every } x \in R_0.$$

This convergence is uniform on every interval $[x_1, x_2]$, $x_1 \geq 0$.

Remark. The inequality (13) shows that results given for $H_n^q(f;A)$ in Theorem 3.1, Theorem 3.2 and Corollary 2 concern also the difference $|L_n(f;A;x)$ $-f(x)|$. Thus the strong approximation for considered operators is more general.

3.2. Now we shall give three examples of operators of the $L_n(f;A)$ type defined by (8).

1. The Szász-Mirakyan operators S_n, $n \in N$, defined by (1) are generated by the matrix $A_1 = [a_{nk}(x)]$ with

$$a_{nk}(x) = e^{-nx} \frac{(nx)^k}{k!}, \qquad n \in N, \quad k \in N_0, \quad x \in R_0.$$

It is easily verified that $A_1 \in \Omega$, i.e. the A_1 satisfies the conditions (i) - (iv).

2. The Baskakov operators with V_n, $n \in N$, defined by (2), are connected with the matrix A_2 on the elements

$$a_{nk}(x) = \binom{n-1+k}{k} x^k (1+x)^{-n-k}, \quad n \in N,\ k \in N_0,\ x \in R_0.$$

We can prove that $A_2 \in \Omega$ also.

3. The Bernstein operators

$$B_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad n \in N,$$

defined for continuous functions f on the interval $[0,1]$ are operators of the type $L_n(f;A)$ with the matrix $A_3 = [a_{nk}(x)]$ where

$$a_{nk}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k} & \text{if} \quad 0 \le k \le n, \\ 0 & \text{if} \quad k > n, \end{cases}$$

for $n \in N$. Here for considered functions $f(\,\cdot\,)$ and $a_n(\,\cdot\,)$ we set: $f(x) = f(1)$ and $a_{nk}(x) = a_{nk}(1)$ for all $x > 1$. We can verify that $A_3 \in \Omega$.

Hence the above lemmas, theorems and corollaries concern also the strong approximation of functions by the Szász-Mirakyan, Baskakov and Bernstein operators.

We remark also that the order of the strong differences given in Theorem 3.2 and Corollary 2 are similar to (5) and (6) for the Baskakov operators.

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