A STUDY OF F_q-FUNCTIONS CONNECTED WITH RAMANUJAN'S TENTH ORDER MOCK THETA FUNCTIONS

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ABSTRACT. We have defined generalized functions which reduce to Ramanujan's mock theta functions of order ten. We have shown that they are F_q-functions. We have given their integral representation and multibasic expansions.

1. INTRODUCTION

In this paper, we shall consider a generalization of Ramanujan's tenth order mock theta functions. The mock theta functions were briefly described by Ramanujan in his last letter to Hardy [6, p.354-355]. In this letter, Ramanujan gave a list of seventeen functions and called them mock theta functions of order three, five and seven. Later in the Lost Notebook [7] seven more mock theta functions were found and Andrews and Hickerson [2] considered them and called them of order six. Recently Choi [3] considered four more mock theta functions found in the Lost Notebook and called them of order ten. However, these mock theta functions are mysterious functions and no one including Ramanujan has ever proved that the mock theta functions exist.

By placing these mock theta functions in the family of F_q-functions and representing them as q-integrals and giving their series expansion, I feel, will be helpful in knowing more about these functions. In a later paper, the author has given a modular transformation for these mock theta functions.

Truesdell [8] in his book has tried to unify the theory of special functions-n^{th} derivative formulae, transformations, contour integrals, miscellaneous relations etc. He calls the functions which satisfy the functional equation

$$(1.1) \qquad \frac{\partial}{\partial z}F(z,\alpha) = F(z,\alpha+1),$$

F-functions.

The q-analogue of this is the q-differential difference equation

$$(1.2) \qquad D_{q,z}F(z,\alpha) = F(z,\alpha+1),$$

where

$$(1.3) \qquad zD_{q,z}F(z,\alpha) = F(z,\alpha) - F(zq,\alpha),$$

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and the functions which satisfy this equation are called F_q-functions.

In this paper (§5 and §6), we have defined four generalized functions which reduce to Ramanujan's mock theta functions of order ten and have shown they are F_q-functions and deduced certain simple properties.

In §7, we give an integral representation for these functions.

In §8, we represent these functions as multibasic series.

2. NOTATIONS

We shall use the following usual basic hypergeometric notations:
For $|q^k| < 1$,

$$(a, q^k)_n = (1 - a)(1 - aq^k) \cdots (1 - aq^{k(n-1)}), \quad n \geq 1$$

$$(a, q^k)_0 = 1,$$

$$(a, q^k)_\infty = \prod_{j=0}^{\infty}(1 - aq^{kj}),$$

$$(a_1, a_2, \ldots, a_m; q^k)_n = (a_1; q^k)_n (a_2; q^k)_n \cdots (a_m; q^k)_n,$$

$$(a, q)_n = (a)_n.$$

$$\phi \begin{bmatrix} a_1, \ldots, a_r : c_{1,1}, \ldots, c_{1,r_1} : \cdots : c_{m,1}, \ldots, c_{m,r_m} \\ b_1, \ldots, b_s : e_{1,1}, \ldots, e_{1,s_1} : \cdots : e_{m,1}, \ldots, e_{m,s_m} \end{bmatrix} ; q, q_1, \ldots, q_m ; z \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} z^n \left[(-1)^n q^{\frac{n^2-n}{2}} \right]^{1+s-r} \right.$$

$$\left. \times \prod_{j=1}^{m} \frac{(c_{j,1}, \ldots, c_{j,r_j}; q_j)_n}{(e_{j,1}, \ldots, e_{j,s_j}; q_j)_n} \left[(-1)^n q^{\frac{n^2-n}{2}} \right]^{s_j-r_j} \right\}.$$

$$_A\phi_{A-1}\left[a_1, a_2, \ldots, a_A; b_1, b_2, \ldots, b_{A-1}; q_1, z \right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1; q_1)_n \cdots (a_A; q_1)_n z^n}{(b_1; q_1)_n \cdots (b_{A-1}; q_1)_n (q_1; q_1)_n}, \quad |z| < 1.$$

3. TENTH ORDER MOCK THETA FUNCTIONS

The four tenth order mock theta functions as defined by Ramanujan are

$$(3.1) \qquad \Phi(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_{n+1}},$$

$$(3.2) \qquad \Psi(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q; q^2)_{n+1}},$$

$$(3.3) \qquad X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}},$$

and

$$(3.4) \qquad \chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}.$$

4. GENERALIZED FUNCTIONS

We define the following four functions

$$(4.1) \qquad \Phi(z,\alpha) := \frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n q^{\frac{1}{2}n(n-1)+n\alpha}}{(q;q^2)_{n+1}},$$

$$(4.2) \qquad \Psi(z,\alpha) := \frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n q^{\frac{1}{2}n(n+1)+n\alpha}}{(q;q^2)_{n+1}},$$

$$(4.3) \qquad X(z,\alpha) := \frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n (z)_n q^{n^2+n\alpha-n}}{(-q)_{2n}},$$

$$(4.4) \qquad \chi(z,\alpha) := \frac{q}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n (z)_n q^{n^2+n\alpha+n}}{(-q)_{2n+1}}.$$

For $z = 0$, $\alpha = 1$, these functions reduce to the four tenth order mock theta functions $\Phi(q)$, $\Psi(q)$, $X(q)$, $\chi(q)$.

5. GENERALIZED FUNCTIONS AS F_q-FUNCTIONS

Theorem 1. $\Phi(z,\alpha)$, $\Psi(z,\alpha)$, $X(z,\alpha)$, and $\chi(z,\alpha)$ are F_q-Functions.

Proof.

$$\Phi(z,\alpha) = \frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n q^{\frac{1}{2}n(n-1)+n\alpha}}{(q;q^2)_{n+1}}.$$

By definition

$$zD_{q,z}\Phi(z,\alpha) = \frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n q^{\frac{1}{2}n(n-1)+n\alpha}}{(q;q^2)_{n+1}} - \frac{1}{(zq)_{\infty}} \sum_{n=0}^{\infty} \frac{(zq)_n q^{\frac{1}{2}n(n-1)+n\alpha}}{(q;q^2)_{n+1}}$$

$$= \frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n q^{\frac{1}{2}n(n-1)+n\alpha}}{(q;q^2)_{n+1}} \left(1 - (1 - zq^n)\right)$$

$$= \frac{z}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n q^{\frac{1}{2}n(n-1)+n(\alpha+1)}}{(q;q^2)_{n+1}}$$

$$= z\Phi(z, \alpha + 1).$$

Hence $\Phi(z, \alpha)$ is a F_q-Function.

Similarly it can be proved that $\Psi(z, \alpha)$, $X(z, \alpha)$ and $\chi(z, \alpha)$ are F_q-Functions. □

6. SIMPLE PROPERTIES

(i) $D_{q,z}\Phi(z, \alpha) = \Psi(z, \alpha)$,

(ii) $D_{q,z}\Phi(z, \alpha)|_{z=0} = \Psi(q)$,

(iii) $qD_{q,z}^2[X(z, \alpha) - \chi(z, \alpha)] = \chi(z, \alpha)$,

(iv) $qD_{q,z}^2[X(z, \alpha) - \chi(z, \alpha)]_{z=0} = \chi(q)$.

proof of (i).

$$D_{q,z}\Phi(z, \alpha) = \Phi(z, \alpha + 1)$$

$$= \frac{1}{(z)_\infty} \sum_{n=0}^\infty \frac{(z)_n q^{\frac{1}{2}n(n-1)+n(\alpha+1)}}{(q; q^2)_{n+1}}$$

$$= \Psi(z, \alpha),$$

which proves (i). □

proof of (ii). Put $z = 0$, $\alpha = 1$, in (i). □

proof of (iii).

$$D_{q,z}^2[X(z, \alpha) - \chi(z, \alpha)] = X(z, \alpha + 2) - \chi(z, \alpha + 2),$$

since

$$D_{q,z}^2 F(z, \alpha) = D_{q,z}(D_{q,z}F(z, \alpha)) = D_{q,z}F(z, \alpha + 1) = F(z, \alpha + 2).$$

So

$$D_{q,z}^2[X(z, \alpha) - \chi(z, \alpha)] = \frac{1}{(z)_\infty} \sum_{n=0}^\infty \frac{(-1)^n (z)_n q^{n^2+n(\alpha+2)-n}}{(-q)_{2n}}$$

$$- \frac{q}{(z)_\infty} \sum_{n=0}^\infty \frac{(-1)^n (z)_n q^{n^2+n(\alpha+2)+n}}{(-q)_{2n+1}}$$

$$= \frac{1}{(z)_\infty} \sum_{n=0}^\infty \frac{(-1)^n (z)_n q^{n^2+n\alpha+n}}{(-q)_{2n+1}}(1 + q^{2n+1})$$

$$- \frac{1}{(z)_\infty} \sum_{n=0}^\infty \frac{(-1)^n (z)_n q^{n^2+n\alpha+3n+1}}{(-q)_{2n+1}}$$

$$= \frac{q^{-1}}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(z)_n q^{n^2+n\alpha+n+1}}{(-q)_{2n+1}}$$

$$= \frac{1}{q}\chi(z,\alpha),$$

which proves (iii).   □

proof of (iv). Put $z = 0$, $\alpha = 1$, in (iii).   □

7. Integral Representation

Thomae [4, p.19] and Jackson [4, p.19] defined the q-integral as

$$\int_0^1 f(t)\, d_q t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n.$$

Theorem 2.

(i) $\Phi(q^z, \alpha) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \displaystyle\int_0^1 t^{z-1}(tq;q)_\infty \Phi(0, at)\, d_q t,$

(ii) $\Psi(q^z, \alpha) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \displaystyle\int_0^1 t^{z-1}(tq;q)_\infty \Psi(0, at)\, d_q t,$

(iii) $X(q^z, \alpha) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \displaystyle\int_0^1 t^{z-1}(tq;q)_\infty X(0, at)\, d_q t,$

(iv) $\chi(q^z, \alpha) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \displaystyle\int_0^1 t^{z-1}(tq;q)_\infty \chi(0, at)\, d_q t,$

where

$$\Phi(0, at) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q;q^2)_{n+1}}(at)^n, \qquad \Psi(0, at) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{(q;q^2)_{n+1}}(at)^n,$$

$$X(0, at) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)}}{(-q;q)_{2n}}(at)^n, \quad \chi(0, at) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(-q;q)_{2n+1}}(at)^n.$$

Proof. Limiting case of the q-beta integral [4, p.19(1.11.7)] is

(7.1) $\dfrac{1}{(q^x; q)_\infty} = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \displaystyle\int_0^1 t^{x-1}(tq;q)_\infty\, d_q t.$

Now

$$\Phi(z, \alpha) = \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{\frac{1}{2}n(n-1)+n\alpha}}{(q;q^2)_{n+1}}.$$

Writing q^z for z and $q^\alpha = a$, we have

$$\Phi(q^z, \alpha) = \frac{1}{(q^z)_\infty} \sum_{n=0}^{\infty} \frac{(q^z)_n q^{\frac{1}{2}n(n-1)+n\alpha}}{(q; q^2)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)+n\alpha}}{(q; q^2)_{n+1}(q^{n+z})_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)+n\alpha}}{(q; q^2)_{n+1}} \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^{n+z-1}(tq; q)_\infty \, d_q t, \quad \text{by (7.1)}$$

(7.2) $$= \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^{z-1}(tq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q; q^2)_{n+1}}(at)^n \, d_q t.$$

But

$$\Phi(0, \alpha) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)+n\alpha}}{(q; q^2)_{n+1}},$$

and since $q^\alpha = a$,

$$\Phi(0, a) = \sum_{n=0}^{\infty} \frac{(a)^n q^{\frac{1}{2}n(n-1)}}{(q; q^2)_{n+1}}.$$

Hence

(7.3) $$\Phi(0, at) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q; q^2)_{n+1}}(at)^n.$$

By (7.3), (7.2) can be written as

$$\Phi(q^z, \alpha) = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^{x-1}(tq; q)_\infty \Phi(0, at) \, d_q t,$$

which proves (i).

Similarly we can prove (ii), (iii) and (iv). $\qquad\square$

Putting $a = q$, $\Phi(0, at)$ reduces to the mock theta function $\Phi(q)$. Similarly for other functions.

8. Multibasic Expansions

By using the summation formula [4, p.71, (3.6.7)] and [5, Lemma 10, p.57], we have the multibasic expansion

(8.1) $$\sum_{k=0}^{\infty} \frac{(1 - ap^k q^k)(1 - bp^k q^{-k})(a, b; p)_k (c, a/bc; q)_k q^k}{(1-a)(1-b)(q, aq/b; q)_k (ap/c, bcp; p)_k} \sum_{m=0}^{\infty} \alpha_{m+k}$$

$$= \sum_{m=0}^{\infty} \frac{(ap, bp; p)_m (cq, aq/bc; q)_m}{(ap/c, bcp; p)_m (q, aq/b; q)_m} \alpha_m.$$

Corollary 1. *Letting $q \to q^2$ and $c \to \infty$ in (8.1), we have*

$$(8.2) \quad \sum_{k=0}^{\infty} \frac{(1 - ap^k q^{2k})(1 - bp^k q^{-2k})(a, b; p)_k q^{k^2+k}}{(1-a)(1-b)(q^2, aq^2/b; q^2)_k b^k p^{\frac{k^2+k}{2}}} \sum_{m=0}^{\infty} \alpha_{m+k}$$

$$= \sum_{m=0}^{\infty} \frac{(ap, bp; p)_m q^{m^2+m}}{(q^2, aq^2/b; q^2)_m b^m p^{\frac{m^2+m}{2}}} \alpha_m.$$

Corollary 2. *Letting $q \to q^3$ and $c \to \infty$ in (8.1), we have*

$$(8.3) \quad \sum_{k=0}^{\infty} \frac{(1 - a^k p^k q^{3k})(1 - bp^k q^{-3k})(a, b; p)_k q^{\frac{3k^2+3k}{2}}}{(1-a)(1-b)(q^3, aq^3/b; q^3)_k b^k p^{\frac{k^2+k}{2}}} \sum_{m=0}^{\infty} \alpha_{m+k}$$

$$= \sum_{m=0}^{\infty} \frac{(ap, bp; p)_m q^{\frac{3m^2+3m}{2}}}{(q^3, aq^3/b; q^3)_m b^m p^{\frac{m^2+m}{2}}} \alpha_m.$$

Theorem 3.

(i) $\displaystyle \Phi(x, \alpha) = \frac{(1-q)^{-1}}{(x)_\infty} \sum_{k=0}^{\infty} \frac{(1 - xq^{3k-1})(1 - q^{-k+1})(x)_{k-1} q^{\frac{k^2-k}{2}+k\alpha}}{(1 - q^{k+1})(q^3; q^2)_k}$

$$\times \phi \left[\begin{matrix} q, 0 & : xq^{2k}, q^{2k+2} \\ q^{k+2} & : q^{2k+3}, 0 \end{matrix} \; ; \, q, q^2 \; ; \, q^\alpha \right],$$

(ii) $\displaystyle \Psi(x, \alpha) = \frac{(1-q)^{-1}}{(x)_\infty} \sum_{k=0}^{\infty} \frac{(1 - xq^{3k-1})(1 - q^{-k+1})(x)_{k-1} q^{\frac{k^2+k}{2}+k\alpha}}{(1 - q^{k+1})(q^3; q^2)_k}$

$$\times \phi \left[\begin{matrix} q, 0 & : xq^{2k}, q^{2k+2} \\ q^{k+2} & : q^{2k+3}, 0 \end{matrix} \; ; \, q, q^2 \; ; \, q^{\alpha+1} \right],$$

(iii) $\displaystyle X(x, \alpha) = \frac{1}{(x)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (1 - q^{2-2k})(x)_k q^{k^2 - k + k\alpha}}{(1 - q^{k+2})(i\sqrt{q}, -i\sqrt{q}, iq, -iq; q)_k}$

$$\times \phi \left[\begin{matrix} q, xq^k & : 0, 0 & : q^{3k+3} \\ q^{k+3} & : -q^{2k+1}, -q^{2k+2} & : 0 \end{matrix} \; ; \, q, q^2, q^3 \; ; \, -q^\alpha \right],$$

(iv) $\displaystyle \chi(x, \alpha) = \frac{q(1-q)^{-1}}{(x)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (1 - q^{1-2k})(x)_k q^{k^2 + k + k\alpha}}{(1 - q^{k+1})(iq, -iq, iq^{3/2}, -iq^{3/2}; q)_k}$

$$\times \phi \begin{bmatrix} q, xq^k : 0, 0 & : q^{3k+3} \\ q^{k+2} & : -q^{2k+2}, -q^{2k+3} : 0 \end{bmatrix} \; q, q^2, q^3 \; ; \; -q^{\alpha+1} \end{bmatrix}.$$

Proof of (i). Taking $a = \frac{x}{q}$, $b = q$, $p = q$ and $\alpha_n = \frac{(x,q^2;q^2)_n q^{n\alpha}}{(q^2)_n(q^3;q^2)_n}$ in (8.2), we have

$$\sum_{k=0}^{\infty} \frac{(1 - xq^{3k-1})(1 - q^{-k+1})(x/q, q; q)_k q^{\frac{k^2-k}{2}}}{(1 - x/q)(1 - q)(x, q^2, q^2)_k}$$

$$\times \sum_{m=0}^{\infty} \frac{(x; q^2)_{m+k}(q^2; q^2)_{m+k} q^{(m+k)\alpha}}{(q^2; q)_{m+k}(q^3; q^2)_{m+k}} = \sum_{m=0}^{\infty} \frac{(x; q)_m q^{\frac{m^2-m}{2}+m\alpha}}{(q^3; q^2)_m}.$$

$$\text{L.H.S} = \sum_{k=0}^{\infty} \frac{(1 - xq^{3k-1})(1 - q^{-k+1})(x/q, q; q)_k q^{\frac{k^2-k}{2}+k\alpha}}{(1 - x/q)(1 - q)(q^2; q)_k(q^3; q^2)_k}$$

$$\times \sum_{m=0}^{\infty} \frac{(xq^{2k}; q^2)_m(q^{2k+2}; q^2)_m q^{m\alpha}}{(q^{k+2}; q)_m(q^{2k+3}; q^2)_m}$$

$$= \sum_{k=0}^{\infty} \frac{(1 - xq^{3k-1})(1 - q^{-k+1})(x; q)_{k-1} q^{\frac{k^2-k}{2}+k\alpha}}{(1 - q^{k+1})(q^3; q^2)_k}$$

$$\times \phi \begin{bmatrix} q, 0 & : xq^{2k}, q^{2k+2} \\ q^{k+2} & : q^{2k+3}, 0 \end{bmatrix} \; q, q^2 \; ; \; q^{\alpha} \end{bmatrix}.$$

$$\text{R.H.S} = (1 - q) \sum_{m=0}^{\infty} \frac{(x; q)_m q^{\frac{m^2-m}{2}+m\alpha}}{(q; q^2)_{m+1}}$$

$$= (1 - q)(x; q)_\infty \Phi(x, \alpha).$$

Hence the Theorem 3(i). $\qquad \square$

Proof of (ii). Taking $a = \frac{x}{q}$, $b = q$, $p = q$ and $\alpha_n = \frac{(x,q^2;q^2)_n q^{n(\alpha+1)}}{(q^2;q)_n(q^3;q^2)_n}$ in (8.2) and after a little simplification, we have the Theorem 3(ii). $\qquad \square$

Proof of (iii). Taking $a = 0$, $b = q^2$, $p = q$ and $\alpha_n = \frac{(x)_n(q^3;q^3)_n(-q^\alpha)^n}{(q^3)_n(-q)_{2n}}$ in (8.3), we have the Theorem 3(iii). $\qquad \square$

Proof of (iv). Taking $a = 0$, $b = q$, $p = q$ and $\alpha_n = \frac{(x)_n(q^3;q^3)_n(-q^{\alpha+1})^n}{(-q^2)_{2n}(q^2)_n}$ in (8.3), we have the Theorem 3(iv). $\qquad \square$

For $x = 0$, $\alpha = 1$, we have the multibasic expansion of the tenth order mock theta functions.

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