THE PERRON PROBLEM FOR C-SEMIGROUPS

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ABSTRACT. Characterizations of Perron-type for the exponential stability of exponentially bounded C-semigroups are given. Also, some applications for the asymptotic behavior of the integrated semigroups are obtained.

1. INTRODUCTION

Let X be a Banach space and V be a closed linear operator with the domain $D(V)$ and range $R(V)$ in X. The abstract Cauchy problem

$$\begin{cases} u'(t) = Vu(t), & t \geq 0 \\ u(0) = x \end{cases}$$

denoted by (V, x) is related to semigroup theory and, for this case, there are various results concerning the exponential stability of solutions.

However, the applications of strongly continuous semigroups to partial differential equations are limited since their generators must to have dense domains. In order to deal with the cases while the generators satisfy weaker conditions, it is necessary to have other classes of semigroups. So, in a natural way, C-semigroups arise.

The notion C-semigroups was introduced by Da Prato [4] some decades ago and by E.B. Davies [3] and M.M. Pang [3] independently.

The basic results of the theory of C-semigroups can be found in the book of R. deLaubenfels [8]. Also, notable results in this field were obtained by N. Tanaka [14,15,16,17,18] and I. Miyadera [12,15,16,17,18].

Generating a C-semigroup corresponds to the abstract Cauchy problem having an unique solution, whenever $x = Cy$ for some $y \in D(V)$, where C is a bounded injective linear operator on X. It is well known that the class of operators which generate C-semigroups is much larger than the class of operators which generate a strongly continuous semigroup.

An interesting characterization of exponential stability was given in 1930 by O. Perron [13], which state that, if X is finite dimensional and V is a matrix, then the solutions of the Cauchy problem (V, x) are exponentially stable if for every continuous and bounded function f from $\boldsymbol{R}_+$ into X the

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solution of the Cauchy problem

$$\begin{cases} x'(t) = Vx(t) + f(t) \\ x(0) = 0 \end{cases}$$

is bounded. This result is classical now and it was generalized in many directions. First R. Bellman [1] extend these results for the case when the space X is infinite dimensional. Also, after Bellman, similar results were obtained by M.G. Krein [2], J.L. Daleckij [2], J.L. Massera [10] and J.J. Schäffer [10], even for the non-autonomous case, where the operator V is replaced by a family $\{V(t)\}_{t\geq 0}$ of operators. In recent years this subject was reopened in the general case of evolutionary process by a large number of researchers. We can mention here names as Y. Latushkin [9], T. Randolph [9], R. Schnaubelt [9,11], F. Räbiger [11], N. van Minh [11] and many others. It is well-known that the C_0-semigroup is in fact a particular case of evolutionary processes, so the results above can be applied to obtain characterizations of the asymptotic behavior of strongly continuous semigroups. Using the Perron theorem, a number of long-standing open problems have recently been solved and the theory of Perron-type seems to have obtained a certain degree of maturity. Taking into account that C-semigroups are a generalization of strongly continuous semigroups, but are different in essence of these, the following question arise: Does there exists an analogue of the Perron theorem for C-semigroups? In this paper an answer is given to the above question and in this investigation we need to employ some new skills and thus we obtain some new results.

Also we note that a C-semigroup is not, in general, a particular case of a evolutionary process, so the results given by now for evolutionary processes cannot be applied in this area. So, the aim of this paper is to give characterizations of Perron's type for the exponential stability of the solutions of the some abstract Cauchy problem of (V, x) -type in the more general case where one must deal with the theory of C-semigroups.

Our methods are different from the usual methods applied in the case of evolutionary process because the C-semigroups does not posses a "lucrative" evolution property. Also as an application to our main result, using the connections between C-semigroups and integrated semigroups (see [15] for instance), we obtain some new results about the asymptotic behavior of the integrated semigroups.

2. PRELIMINARIES

In the beginning we will recall some standard notations. So, first we have:

- $C_b(\boldsymbol{R}_+, X)$ is the space of all bounded and continuous functions from $\boldsymbol{R}_+$ to X,

- $C_0(\mathbf{R}_+, X)$ is the space of all continuous functions f from $\mathbf{R}_+$ to X with $\lim_{t \to \infty} f(t) = 0$,
- $\mathrm{BUC}(\mathbf{R}_+, X)$ is the space of all bounded and uniformly continuous functions from $\mathbf{R}_+$ to X,
- $\mathrm{AP}(\mathbf{R}_+, X)$, is the linear closed hull in $C_b(\mathbf{R}_+, X)$ of the set of all functions

$$t \mapsto e^{i\lambda t} x : \ \mathbf{R}_+ \to X, \ \lambda \in \mathbf{R}, \ x \in X.$$

All these spaces are Banach spaces endowed with the supremum norm, denoted by $||| \cdot |||$.

Throughout in this paper, let $C \in B(X)$ be an injective operator.

Definition 2.1. *A family $\{S(t)\}_{t \geq 0}$ of bounded linear operators is called a C-semigroup if the following conditions hold:*

s_1) *$S(t+s)C = S(t)S(s)$ for all $t, s \geq 0$ and $S(0) = C$;*

s_2) *$S(\cdot)x : \mathbf{R}_+ \to X$ is continuous for all $x \in X$.*

If in addition $\{S(t)\}_{t \geq 0}$ satisfies the condition:

s_3) *there are $M, \omega > 0$ such that*

$$\|S(t)\| \leq M e^{\omega t}, \ \text{for all} \ t \geq 0.$$

Then it is called an exponentially bounded C-semigroup.

Remark 2.1. In [3] it is shown that there exist C-semigroups which are not exponentially bounded.

The generator A of a C-semigroup $\{S(t)\}_{t \geq 0}$ is defined by:

$$\begin{cases} D(A) = \left\{ x \in X : \text{ there exists } \lim_{t \to 0+} \dfrac{1}{t}(S(t)x - Cx) \in R(C) \right\} \\ Ax = C^{-1} \lim_{t \to 0+} \dfrac{1}{t}(S(t)x - Cx) \ \text{ for } x \in D(A). \end{cases}$$

It is known (see for example [6]) that the generator A of a C-semigroup $\{S(t)\}_{t \geq 0}$ has the following properties:

i) $S(t)x - Cx = \displaystyle\int_0^t S(s)Ax\,ds$ for $x \in D(A)$ and $t \geq 0$;

ii) $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for $x \in D(A)$ and $t \geq 0$.

For the generator A of a C-semigroup $\{S(t)\}_{t \geq 0}$ and a continuous function f from $\mathbf{R}_+$ to X, we will denote by (A, f) the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = 0. \end{cases}$$

By a classical solutions for (A, f) we understand that u is an application of C^1 class, $u(t) \in D(A)$ for all $t \geq 0$, and the two equalities of (A, f) are satisfied.

Proposition 2.1. *If u is a classical solution of (A, f), then*

$$Cu(t) = \int_0^t S(t-s)f(s)ds, \ \ for \ all \ t \geq 0.$$

Proof. For a fixed $t \geq 0$ we consider the function $v_t : [0, t] \to X$,

$$v_t(s) = S(t-s)u(s).$$

It is easy to see that v_t is a function of class C^1 and

$$v_t'(s) = -S(t-s)Au(s) + S(t-s)u'(s) = S(t-s)f(s)$$

for all $s \in [0, t]$. It follows that

$$Cu(t) = v_t(t) - v_t(0) = \int_0^t S(t-s)f(s)ds.$$

$\square$

This result suggest us to give the following definition:

Definition 2.2. *By the mild solution of the problem (A, f) we understand a continuous functions $u_f : \mathbf{R}_+ \to X$ with the property*

$$Cu_f(t) = \int_0^t S(t-s)f(s)ds, \ \ for \ all \ t \geq 0$$

Remark 2.2. The injectivity of C implies that for all $f \in C_b(\mathbf{R}_+, X)$ there is no more than one mild solution.

Definition 2.3. *A subspace E of $C_b(\mathbf{R}_+, X)$ is said to be continuously sectionable if for all $a > 0$ and all continuous functions f from $[0, a]$ to X there exists $g \in E$ with $|||g||| = \sup\limits_{t \in [0,a]} \|f(t)\|$ and $g_{|[0,a]} = f$.*

Example 2.1. $C_0(\mathbf{R}_+, X)$, $BUC(\mathbf{R}_+, X)$, $AP(\mathbf{R}_+, X)$ are continuously sectionable. For a given $a > 0$ and a continuous function f from $[0, a]$ to X we define $g : \mathbf{R}_+ \to X$ given by

$$g(t) = \begin{cases} f(t), & t \in [0, a] \\ (a+1-t)f(a), & t \in (a, a+1) \\ 0, & t \geq a+1 \end{cases}$$

Then $g \in C_0(\mathbf{R}_+, X) \subset BUC(\mathbf{R}_+, X)$, $g_{|[o,a]} = f$ and

$$|||g||| = \sup\limits_{t \in [0,a]} \|f(t)\|$$

In order to prove that $AP(\boldsymbol{R}_+, X)$ is also continuously sectionable we consider again $a > 0$ and a continuous function f from $[0, a]$ to X. We construct now $g : [0, 2a] \to X$ defined by

$$g(s) = \begin{cases} f(s), & s \in [0, a] \\ f(2a - s), & s \in (a, 2a] \end{cases}$$

Then it is easy to check that g is continuous, $g(0) = g(2a)$ and $\sup\limits_{s \in [0,2a]} \|g(s)\|$ = $\sup\limits_{t \in [0,a]} \|f(t)\|$. Now it is clear that there is h a continuous function from $\boldsymbol{R}$ to X, $2a$ periodic, such that $h_{|[0,2a]} = g$. By a classic Fourier theory result we have that $h \in AP(\boldsymbol{R}, X)$. But $h_{|[0,a]} = f$ and

$$\||h|\| = \sup_{s \in [0,2a]} \|h(s)\| = \sup_{s \in [0,2a]} \|g(s)\| = \sup_{t \in [0,a]} \|f(t)\|.$$

Definition 2.4. *A C-semigroup $\{S(t)\}_{t \geq 0}$ is said to be exponentially stable if there are $N, \nu > 0$ two constants such that*

$$\|S(t)\| \leq N e^{-\nu t}, \ \text{for all} \ t \geq 0.$$

Definition 2.5. *A subspace E of $C_b(\boldsymbol{R}_+, X)$ is admissible to a C-semigroup with the generator A if for all $f \in E$ the abstract Cauchy problem (A, f) has a mild solution which lies in $C_b(\boldsymbol{R}_+, X)$.*

Proposition 2.2. *If E is a closed subspace admissible to a C-semigroup $\{S(t)\}_{t \geq 0}$ with the generator A, then there exists $K > 0$ such that*

$$\||u_f|\| \leq K \||f|\|, \ \text{for all} \ f \in E.$$

Proof. Let us define the application $V_E : E \to C_b(\boldsymbol{R}_+, X)$ given by $V_E f = u_f$. Obviously V_E is a linear operator. Next, let $\{f_n\}_{n \in \boldsymbol{N}}$ be a sequence in E, $f \in E$, $g \in C_b(\boldsymbol{R}_+, X)$ with the properties

$$f_n \xrightarrow{E} f, \qquad V_E f_n \xrightarrow{C_b} g.$$

Then

$$\begin{aligned} \|C(V_E f_n)(t) - C(V_E f)(t)\| &= \left\| \int_0^t S(t - s)(f_n(s) - f(s)) ds \right\| \\ &\leq t \sup_{v \in [0,t]} \|S(v)\| \ \||f_n - f|\|, \end{aligned}$$

for all $t \geq 0$ and every $n \in \boldsymbol{N}$.

It results that $C(V_E f)(t) = Cg(t)$, for all $t \geq 0$, and hence $V_E f = g$. This implies that V_E is also bounded and so

$$\||u_f|\| = \||V_E f|\| \leq \|V_E\| \ \||f|\|, \ \text{for all} \ f \in E.$$

$\square$

3. The main result

Theorem 3.1. *If E is a closed, continuously sectionable subspace of $C_b(\mathbf{R}_+, X)$ and E is admissible to an exponentially bounded C-semigroup $\{S(t)\}_{t\geq 0}$, then $\{S(t)\}_{t\geq 0}$ is exponentially stable.*

Proof. Step 1. In this section of the proof we will prove that S is bounded. Consider $t \geq 0$, $x \in X$ and the function $f : [0,t] \to X$ defined by $f(s) = e^{-\omega s} S(s)x$. Then there is $g \in E$ such that

$$g_{|[0,t]} = f \quad \text{and} \quad |||g||| = \sup_{s \in [0,t]} \|f(s)\| \leq M\|x\|.$$

$$C(u_g)(t) = \int_0^t S(t-s)g(s)ds$$
$$= \int_0^t e^{-\omega s} S(t-s)S(s)x\,ds = \frac{1}{\omega}(1 - e^{-\omega t})CS(t)x.$$

Using Proposition 2.2, we obtain that

$$\frac{1}{\omega}(1 - e^{-\omega t})\|S(t)x\| = \|(u_g)(t)\| \leq |||u_g||| \leq K|||g||| \leq MK\|x\|$$

and so

$$\|S(t)\| \leq L, \quad \text{for all } t \geq 0,$$

where $L = M(K\omega + 1)$.

Step 2. In this section we will prove inductively that

$$\frac{t^n}{n!}\|S(t)\| \leq LK^n, \quad \text{for all } t \geq 0 \text{ and all } n \in \mathbf{N}.$$

From step 1 it follows that the inequality above is true for $n = 0$. Next, if we assume that the inequality above is true for a fixed $n \in \mathbf{N}$ and if we define $f_n : [0,t] \to X$, $f_n(s) = \dfrac{s^n}{n!}S(s)x$, where $t \geq 0$ and $x \in X$ are arbitrarily chosen, then there exists $g_n \in E$ such that $g_{n|[0,t]} = f_n$ and $|||g_n||| = \sup\limits_{s \in [0,t]} \|f_n(s)\| \leq LK^n\|x\|$.

It results that

$$C(u_{g_n})(t) = \int_0^t S(t-s)\left(\frac{s^n}{n!}S(s)x\right)ds$$
$$= \int_0^t \frac{s^n}{n!}ds\, CS(t)x = \frac{t^{n+1}}{(n+1)!}CS(t)x,$$

and using again the Proposition 2.2 we have that

$$\frac{t^{n+1}}{(n+1)!}\|S(t)x\| = \|(u_{g_n})(t)\| \leq |||u_{g_n}||| \leq K|||g_n||| \leq LK^{n+1}\|x\|.$$

which implies that

$$\frac{t^{n+1}}{(n+1)!}\|S(t)\| \leq LK^{n+1}, \ \ \text{for all} \ \ t \geq 0.$$

With this it is now clear that

$$\frac{t^n}{n!}\|S(t)\| \leq LK^n, \ \ \text{for all} \ \ t \geq 0 \ \ \text{and every} \ \ n \in \boldsymbol{N}.$$

Let now $\nu = \dfrac{1}{2K}$ and $N = 2L$. One can easily verify that

$$e^{\nu t}\|S(t)\| = \sum_{n=0}^{\infty} \frac{\nu^n t^n}{n!}\|S(t)\|$$

$$\leq \sum_{n=0}^{\infty} L(\nu K)^n = L\sum_{n=0}^{\infty}\frac{1}{2^n} = 2L = N, \ \ \text{for all} \ \ t \geq 0$$

The proof is now complete. $\hfill\square$

Theorem 3.2. *If $\{S(t)\}_{t\geq 0}$ is an exponentially bounded C-semigroup then it is exponentially stable if one of the following conditions hold*

1) $C_0(\boldsymbol{R}_+, X)$ *is admissible to* $\{S(t)\}_{t\geq 0}$;
2) $BUC(\boldsymbol{R}_+, X)$ *is admissible to* $\{S(t)\}_{t\geq 0}$;
3) $AP(\boldsymbol{R}_+, X)$ *is admissible to* $\{S(t)\}_{t\geq 0}$.

Proof. Follows easily from Theorem 3.1. and Example 2.1. $\hfill\square$

In what follows we will apply the above results to obtain some properties of the asymptotic behavior of the so-called integrated semigroups. We recall that a family of bounded linear operators $\{U(t)\}_{t\geq 0}$ acting on a Banach space X is called a n-times integrated semigroup if the following statements hold:

i_1) $U(\cdot)x : \boldsymbol{R}_+ \to X$ is continuous for all $x \in X$;
i_2) $U(t)U(s)x = \frac{1}{(n-1)!}[\int_t^{t+s}(t+s-r)^{n-1}U(r)xdr - \int_0^s(t+s-r)^{n-1}U(r)dr,$
   for all $t, s \geq 0$, $x \in X$, and $U(0) = 0$.
i_3) $U(t)x = 0$, for all $t > 0$ implies that $x = 0$.
i_4) there are $M > 0$ and $\omega \in \boldsymbol{R}$ such that $\|U(t)\| \leq Me^{\omega t}$, for all $t \geq 0$.

The generator of a n-times integrated semigroup is defined as the unique closed linear operator A which satisfy the following conditions:

1) $(\omega, \infty) \subset \rho(A)$
2) $R(\lambda, A)x = \int_0^{\infty} \lambda^n e^{-\lambda t}U(t)xdt$, for all $x \in X$, $\lambda \geq \omega$.

In [15] it is proved that for a densely defined, closed A with $\rho(A) \neq \emptyset, c \in \rho(A), n \in \mathbf{N}^*$ the following conditions are equivalent:

1) A is the generator of a n-times integrated semigroup $\{U(t)\}_{t\geq 0}$;
2) A is the generator of a C-semigroup $\{S(t)\}_{t\geq 0}$ with $C = R(c, A)^n$;
3) there exist $M > 0$, $a \in \mathbf{R}$ such that $(a, \infty) \subset \rho(A)$ and

$$\|R(\lambda, A)^m R(c, A)^n\| \leq \frac{M}{(\lambda-a)^m} \quad \text{for all} \quad m \in \mathbf{N}^*, \ \lambda > a.$$

In this case we have that:

$$U(t)x = (cI - A)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} S(t_n)x \, dt_n \ldots dt_1, \quad \text{for all} \ t \geq 0, \ x \in X.$$

In the next example we show that there exist closed linear operators which generates a C- semigroup exponentially stable and a n-times integrated semigroup which don't have a limit when t tends to ∞ and $n \geq 2$.

Example 3.1. Let $X = \mathbf{R}, Ax = -x$. Then A generates the C-semigroup $T(t) = e^{-t}$ (here $Cx = x$) and the 2-times integrated semigroup $U(t) = \int_0^t \int_0^{t_1} e^{-t_2} dt_2 dt_1 = t + e^{-t} - 1$. It is easy to see that $\lim_{t \to \infty} U(t)x$ does not exist except the case when $x = 0$.

We recall that for a strongly continuous family of bounded linear operators $\{W(t)\}_{t\geq 0}$ and for a continuous function $f : \mathbf{R}_+ \to X$ we denote by $W * f$ the map defined by:

$$(W * f)(t) = \int_0^t W(t - s) f(s) ds.$$

Lemma 3.1. *If* $\{U(t)\}_{t\geq 0}$ *is a 1-times integrated semigroup, with the generator* $A, c \in \rho(A), C = R(c, A)$, *the C-semigroup* $\{S(t)\}_{t\geq 0}$ *having also the generator* $A, f \in \mathcal{C}_b(\mathbf{R}_+, X)$, *then:*

$$\int_0^t (S * f)(s) ds = C(U * f)(t) \quad \text{for all} \ t \geq 0.$$

Proof. $\int_0^t (S * f)(s) ds = \int_0^t \int_0^s S(s - \tau) f(\tau) d\tau ds = \int_0^t \int_\tau^t S(s - \tau) f(\tau) ds d\tau = \int_0^t \int_0^{t-\tau} S(v) f(\tau) dv d\tau = \int_0^t CU(t - \tau) f(\tau) d\tau = C(U * f)(t)$, for all $t \geq 0$. $\square$

Theorem 3.3. *If $\{U(t)\}_{t\geq0}$ is a 1-times integrated semigroup, E is a closed continuously sectionable subspace of $\mathcal{C}_b(\mathbf{R}_+, X)$ with the property that $U * f$ is of C^1 class and $(U * f)' \in \mathcal{C}_b(\mathbf{R}_+, X)$ for all $f \in E$, then there exists $\lim\limits_{t\to\infty} U(t)x$, for all $x \in X$.*

Proof. Let A be the generator of $\{U(t)\}_{t\geq0}, c \in \rho(A), C = R(c, A), \{S(t)\}_{t\geq0}$ the C-semigroup which has the generator A. If $f \in E$ and if we set $u_f = (U * f)' \in \mathcal{C}_b(\mathbf{R}_+, X)$, then by Lemma 3.1 we have that:

$$\int\limits_0^t Cu_f(s)ds = C\int\limits_0^t (U * f)'(s)ds$$

$$= C(U * f)(t) = \int\limits_0^t (S * f)(s)ds, \text{ for all } t \geq 0,$$

which implies that:

$$Cu_f(t) = (S * f)(t) = \int\limits_0^t S(t - s)f(s)ds, \text{ for all } t \geq 0.$$

This shows that E is admissible to $\{S(t)\}_{t\geq0}$ and hence by Theorem 3.1 we obtain that $\{S(t)\}_{t\geq0}$ is exponentially stable.

It follows that there exists

$$\lim\limits_{t\to\infty} \int\limits_0^t S(s)xds, \text{ for all } x \in X.$$

Having in mind that

$$U(t)x = (cI - A)\int\limits_0^t S(s)xds$$

$$= c\int\limits_0^t S(s)xds - S(t)x + Cx, \text{ for all } t \geq 0, \ x \in X,$$

it results what is to prove. □

Theorem 3.4. *If $\{U(t)\}_{t\geq0}$ is a 1-times integrated semigroup, then there exists $\lim\limits_{t\to\infty} U(t)x$, for all $x \in X$, if one of the following conditions hold:*

1) *$U * f$ has a derivative in $\mathcal{C}_b(\mathbf{R}_+, X)$, for all $f \in \mathcal{C}_0(\mathbf{R}_+, X)$.*
2) *$U * f$ has a derivative in $\mathcal{C}_b(\mathbf{R}_+, X)$, for all $f \in BUC(\mathbf{R}_+, X)$.*
3) *$U * f$ has a derivative in $\mathcal{C}_b(\mathbf{R}_+, X)$, for all $f \in AP(\mathbf{R}_+, X)$.*

Proof. It follows from Theorem 3.3 and Example 2.1. □

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