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REMARK ON CUP-PRODUCTS

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1. Introduction.

Let L be finite dimensional CW complex and L_1 be a subcomplex of L given by

$$\text{(1.1)} \qquad \begin{cases} L = L_1 \cup_\alpha e^n & (\alpha \in \pi_{n-1}(L_1)) \\ L_1 = S^q \cup \big(\bigcup_{\lambda\in\Lambda} e_\lambda^{m(\lambda)} \big), \end{cases}$$

where $n-2 \geq \dim e_\lambda^{m(\lambda)} = m(\lambda) \geq q+2 \geq 4$ are integers ($\lambda \in \Lambda$). Let $i_q : S^q \to L_1$ and $\overline{\alpha} \in \pi_n(L, L_1)$ be an inclusion map and the characteristic map of the top cell e^n in L. Then it is known ([2], [4]) that

$$\begin{cases} \pi_k(L_1) = 0 \quad \text{if } k < q, \quad \pi_k(L, L_1) = 0 \quad \text{if } k < n, \\ \pi_q(L_1) = \mathbb{Z} \cdot i_q \cong \mathbb{Z}, \quad \pi_n(L, L_1) = \mathbb{Z} \cdot \overline{\alpha} \cong \mathbb{Z}, \quad \overline{\alpha}|S^{n-1} = \alpha, \\ \pi_{n+q-1}(L, L_1) = \mathbb{Z} \cdot [\overline{\alpha}, i_q]_r \oplus \overline{\alpha}_* \pi_{n+q-1}(D^n, S^{n-1}) \cong \mathbb{Z} \oplus \pi_{n+q-1}(S^n), \end{cases}$$

where $[\ ,\ ]_r$ denotes the relative Whitehead product.

In this paper we study the relation between relative Whitehead products and cup-products on certain finite dimensional complexes. In particular, we shall prove some generalization of the result obtained by I. M. James in [3] concerning these relations, and it is as follows:

Theorem 1.1. *Under the above assumption (1.1), let $\beta \in \pi_{n+q-1}(L)$ be an element such that*

$$\text{(1.2)} \qquad i_*(\beta) = m[\overline{\alpha}, i_q]_r + \overline{\alpha} \circ \rho \qquad (m \in \mathbb{Z}, \quad \rho \in \pi_{n+q-1}(D^n, S^{n-1})),$$

where $i_* : \pi_{n+q-1}(L) \to \pi_{n+q-1}(L, L_1)$ denotes the induced homomorphism.

Then if $K = L \cup_\beta e^{n+q}$ is the mapping cone of β and $e_k \in H^k(K, \mathbb{Z}) \cong \mathbb{Z}$ $(k = q, n, n+q)$ denotes the corresponding generator, the following relation holds:

$$\text{(1.3)} \qquad e_n \cdot e_q = m\ e_{n+q}.$$

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This kind of results was first obtained by I. M. James [3] for 3-complexes of the forms $K = S^p \cup e^q \cup e^{p+q}$. Although the above generalization should be well known on the basis of the result due to James, we could not find any literature and it may be worth-while to write this result. In fact, in the subsequent paper [5] we shall use this result to study the classifications of homotopy types of simply-connected 8 dimensional Poincaré complexes.

2. Coaction maps.

We take $M = L \vee S^n = (L_1 \cup_\alpha e^n) \vee S^n$ and let $\mu' : L \to M$ be the coaction map given by pinching the equator of the top cell e^n in L. Let

$$j_L : L \to M, \quad j_n : S^n \to M, \quad j_q : S^q \to M \quad j_M : M \to (M, L_1)$$

be corresponding natural inclusions, and let $r_L : M \to L$ be the retraction. Then we have

$$r_L \circ j_L = \mathrm{id}_L \simeq r_L \circ \mu'. \tag{2.1}$$

Lemma 2.1. *Let $\mu'_* : \pi_*(L, L_1) \to \pi_*(M, L_1)$ be the induced homomorphism.*

(i) $\mu'_*(\overline{\alpha}) = j_{L*}(\overline{\alpha}) + j_{M*}(j_n)$.
(ii) $\mu'_*([\overline{\alpha}, i_q]_r) = j_{L*}([\overline{\alpha}, i_q]_r) + [j_{M*}(j_n), i_q]_r$.
(iii) $\mu'_*(\overline{\alpha} \circ \rho) = j_{L*}(\overline{\alpha} \circ \rho) + j_M \circ j_n \circ \rho$.

Proof. (i) The assertion (i) is clear.

$$\begin{aligned} (ii)\ \mu'_*([\overline{\alpha}, i_q]_r) &= [\mu'_*(\overline{\alpha}), i_q]_r = [j_{L*}(\overline{\alpha}) + j_{M*}(j_n), i_q]_r \quad \text{(by } (i)) \\ &= [j_{L*}(\overline{\alpha}), i_q]_r + [j_{M*}(j_n), i_q]_r \\ &= j_{L*}([\overline{\alpha}, i_q]_r) + [j_{M*}(j_n), i_q]_r. \end{aligned}$$

(iii) Since $2n-3 \geq n+q-1$, $\rho \in \pi_{n+q-1}(D^n, S^{n-1}) \cong \pi_{n+q-2}(S^{n-1}) = E\pi_{n+q-3}(S^{n-2})$ and we have

$$\begin{aligned} \mu'_*(\overline{\alpha} \circ \rho) &= \mu'_*(\overline{\alpha}) \circ \rho = (j_{L*}(\overline{\alpha}) + j_{M*}(j_n)) \circ \rho \quad \text{(by } (i)) \\ &= j_{L*}(\overline{\alpha} \circ \rho) + j_{M*}(j_n) \circ \rho. \end{aligned}$$

□

Lemma 2.2. *Under the same assumption as Theorem 1.1,*

$$\mu'_*(\beta) = j_{L*}(\beta) + m[j_n, j_q] + j_n \circ \rho \quad \text{in } \pi_{n+q-1}(M), \tag{2.2}$$

where $\mu'_* : \pi_{n+q-1}(L) \to \pi_{n+q-1}(M)$ denotes the induced homomorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\pi_r(L) \arrow[r, "\mu'_*"] \arrow[d, "i_*"'] & \pi_r(M) \arrow[r, "r_{L*}"] \arrow[d, "j_*"'] & \pi_r(L) \arrow[r, "j_{L*}"] \arrow[d, "i_*"'] & \pi_r(M) \arrow[d, "j_*"'] \\
\pi_r(L, L_1) \arrow[r, "\mu'_*"] & \pi_r(M, L_1) \arrow[r, "r_{L*}"] & \pi_r(L, L_1) \arrow[r, "j_{L*}"] & \pi_r(M, L_1).
\end{tikzcd} \tag{2.3}$$

We take $\theta = j_{L*}(\beta) + m[j_n, j_q] + j_n \circ \rho - \mu'_*(\beta)$. It suffices to show $\theta = 0$. For this purpose, it is sufficient to prove the following two equations:

$$j_*(\theta) = 0, \quad \text{and} \quad r_{L*}(\theta) = 0. \tag{2.4}$$

In fact, let $i_{L_1} : L_1 \to L$ be an inclusion and consider the exact sequence

$$\pi_{n+q-1}(L_1) \xrightarrow{(j_L \circ i_{L_1})_*} \pi_{n+q-1}(M) \xrightarrow{j_*} \pi_{n+q-1}(M, L_1).$$

Because $j_*(\theta) = 0$, there exists some element $\phi \in \pi_{n+q-1}(L_1)$ such that $\theta = (j_L \circ i_{L_1})_*(\phi)$. Hence, using $r_{L*}(\theta) = 0$, we have

$$\begin{aligned}
\theta &= (j_L \circ i_{L_1})_*(\phi) = j_L \circ (r_L \circ j_L) \circ i_{L_1 *}(\phi) \quad (\text{by } (2.1)) \\
&= j_L \circ r_L \circ (j_L \circ i_{L_1})_*(\phi) = j_L \circ r_{L*}(\theta) = 0.
\end{aligned}$$

So it remains to prove (2.4). First, we show $j_*(\theta) = 0$. We note that

$$\begin{aligned}
j_*\mu'_*(\beta) &= \mu'_*(i_*(\beta)) \qquad (\text{by } (2.3)) \\
&= \mu'_*(m[\overline{\alpha}, i_q]_r + \overline{\alpha} \circ \rho) \quad (\text{by } (1.2)) \\
&= m\mu'_*([\overline{\alpha}, i_q]_r) + \mu' \circ \overline{\alpha} \circ \rho \\
&= m(j_{L*}([\overline{\alpha}, i_q]_r + [j_M \circ j_n, i_q]_r)) \\
&\qquad + (j_L \circ \overline{\alpha} \circ \rho + j_M \circ j_n \circ \rho) \quad (\text{by Lemma 2.1}) \\
&= j_{L*}(m[\overline{\alpha}, i_q]_r + \overline{\alpha} \circ \rho) + m[j_M \circ j_n, i_q]_r + j_M \circ j_n \circ \rho \\
&= j_{L*}(i_*(\beta)) + m[j_M \circ j_n, i_q]_r + j_M \circ j_n \circ \rho \quad (\text{by } (1.2)).
\end{aligned}$$

Because $[j_M \circ j_n, i_q]_r = j_*([j_n, j_q])$ and $j_M \circ j_n \circ \rho = j_*(j_n \circ \rho)$, we can rewrite

$$j_*\mu'_*(\beta) = j_*(j_{L*}(\beta)) + mj_*([j_n, j_q]) + j_*(j_n \circ \rho).$$

Hence,

$$j_*(\theta) = j_*(j_{L*}(\beta) + m[j_n, j_q] + j_n \circ \rho - \mu'_*(\beta)) = 0.$$

Next we prove $r_{L*}(\theta) = 0$. Since $r_L : M \to L$ is a retraction, $r_L \circ j_n = 0$ and we have

$$\begin{cases} r_{L*}(j_n \circ \rho) = (r_L \circ j_n) \circ \rho = 0, \\ r_{L*}([j_n, j_q]) = [r_L \circ j_n, r_L \circ j_q] = 0. \end{cases} \tag{2.5}$$

Hence,

$$\begin{aligned} r_{L*}(\theta) &= r_{L*}(j_{L*}(\beta) + m[j_n, j_q] + j_n \circ \rho - \mu'_*(\beta)) \\ &= r_{L*}j_{L*}(\beta) + m \cdot r_{L*}([j_n, j_q]) + r_{L*}(j_n \circ \rho) - r_{L*}\mu'_*(\beta) \\ &= r_{L*}j_{L*}(\beta) - r_{L*}\mu'_*(\beta) \quad \text{(by (2.5))} \\ &= \beta - \beta \quad \text{(by (2.1))} \\ &= 0. \end{aligned}$$

□

3. Proof of Theorem 1.1.

First, recall the function first defined by I. M. James [3].

Definition. Let $p, q \geq 2$ be integers and let X be a finite dimensional CW complex with $\dim X \leq p+q-1$ such that $H^{p+q-1}(X, \mathbb{Z})$ is a finite group.

We fix the elements $0 \neq e_k \in H^k(X, \mathbb{Z})$ $(k = p, q)$. Since $\dim X \leq p+q-1$, $e_p \cdot e_q = 0 \in H^{p+q}(X, \mathbb{Z})$. We define the function $h : \pi_{p+q-1}(X) \to \mathbb{Z}$ as follows. For any element $\lambda \in \pi_{p+q-1}(X)$, we take $X^* = X \cup_\lambda e^{p+q}$. We denote by e'_{p+q} the generator of $H^{p+q}(X^*, X; \mathbb{Z}) \cong \mathbb{Z}$ corresponding to the top cell in X^*. Let $\tilde{i} : X \to X^*$ be an inclusion and let $e'_k \in H^k(X^*, \mathbb{Z})$ $(k = p, q)$ be corresponding elements such that $\tilde{i}^*(e'_k) = e_k$.

In this situation, consider the exact sequence

$$H^{p+q}(X^*, X; \mathbb{Z}) \xrightarrow{\tilde{j}^*} H^{p+q}(X^*, \mathbb{Z}) \xrightarrow{\tilde{i}^*} H^{p+q}(X, \mathbb{Z}).$$

Since $0 = e_p \cdot e_q = \tilde{i}^*(e'_p) \cdot \tilde{i}^*(e'_q) = \tilde{i}^*(e'_p \cdot e'_q)$, there exists an integer $m' \in \mathbb{Z}$ such that $e'_p \cdot e'_q = m' e'_{p+q}$. Then we define the function $h : \pi_{p+q-1}(X) \to \mathbb{Z}$ by $h(\lambda) = m'$ (cf. [3], page 378).

Lemma 3.1 ([3]). *$h : \pi_{p+q-1}(X) \to \mathbb{Z}$ is a homomorphism.*

Proof. This follows from Theorem 4.1 of [3]. □

Lemma 3.2. *Under the same assumption as Theorem 1.1, we denote by $h : \pi_{p+q-1}(M) \to \mathbb{Z}$ the linear function given as above. Then we have*

(i) $h(j_{L*}(\beta)) = h(j_n \circ \rho) = 0$

(ii) $h([j_n, j_q]) = 1$

Proof. (*i*) Let us consider the mapping cone $M_1^* = M \cup_{j_{L*}(\beta)} e^{n+q}$. Since $M_1^* \cong (L \cup_\beta e^{n+q}) \vee S^n$, clearly $a' \cdot b' = 0$, where $a' \in H^n(M_1^*, \mathbb{Z}) \cong \mathbb{Z}$ and $b' \in H^q(M_1^*, \mathbb{Z}) \cong \mathbb{Z}$ denote the generators corresponding to S^n and S^q. Hence, $h(j_{L*}(\beta)) = 0$. Similar method also shows $h(j_n \circ \rho) = 0$.

(*ii*) Consider the mapping cone $M_2^* = M \cup_{[j_n, j_q]} e^{n+q}$. Then $M_2^* = (L \vee S^n) \cup_{[j_n, j_q]} e^{n+q}$ contains the subcomplex $S^q \vee S^n \cup_{[i_q, i_n]} e^{n+q} = S^q \times S^n$.

So if $a' \in H^n(M_2^*, \mathbb{Z}) \cong \mathbb{Z}$ and $b' \in H^q(M_2^*, \mathbb{Z}) \cong \mathbb{Z}$ denote the generators corresponding to S^n and S^q, the product $a' \cdot b'$ represents the generator of $H^{n+q}(M_2^*, \mathbb{Z})$ and we have $h([j_n, j_q]) = 1$. $\square$

Proof of Theorem 1.1. Let N be a mapping cone of $\mu'_*(\beta)$ given by $N = M \cup_{\mu'_*(\beta)} e^{n+q} = (L \vee S^n) \cup_{\mu'_*(\beta)} e^{n+q}$. Define the map $f : K = L \cup_\beta e^{n+q} \to M \cup_{\mu'_*(\beta)} e^{n+q} = N$ by

$$\begin{cases} f|L = \mu' : L \to M \\ f|e^{n+q} = \text{degree one map on the top cell } e^{n+q}. \end{cases}$$

If $f^* : H^*(N, \mathbb{Z}) \to H^*(K, \mathbb{Z})$ denotes the induced homomorphism and let $e'_k \in H^k(N, \mathbb{Z}) \cong \mathbb{Z}$ $(k = q, n, n+q)$ be the generators corresponding to cells S^q, S^n or e^{n+q}, then $f^*(e'_k) = e_k$ for $k = q, n$ or $n+q$. Now consider the homomorphism $h : \pi_{n+q-1}(M) \to \mathbb{Z}$. Then we have

$$e'_n \cdot e'_q = h(\mu'_*(\beta))e'_{n+q}. \tag{3.1}$$

Hence,

$$\begin{aligned} e_n \cdot e_q &= f^*(e'_n) \cdot f^*(e'_q) = f^*(e'_n \cdot e'_q) = f^*(h(\mu'_*(\beta))e'_{n+q}) \\ &= h(\mu'_*(\beta))f^*(e'_{n+q}) = h(\mu'_*(\beta))e_{n+q}. \end{aligned}$$

So it remains to show that $h(\mu'_*(\beta)) = m$. Since h is a homomorphism, it follows from Lemma 2.2 and 3.2 that

$$\begin{aligned} h(\mu'_*(\beta)) &= h(j_{L*}(\beta) + m[j_n, j_q] + j_n \circ \rho) \\ &= h(j_{L*}(\beta)) + m \cdot h([j_n, j_q]) + h(j_n \circ \rho) \\ &= 0 + m \cdot 1 + 0 = m \end{aligned}$$

and this completes the proof of Theorem 1.1. $\square$

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References

[1] A. L. Blakers and W. S. Massey, *Products in homotopy theory*, Annals of Math. **58**(1953) 295–324.

[2] I. M. James, *On the homotopy groups of certain pairs and triads*, Quart. J. Math. Oxford **5**(1954) 260–270.

[3] I. M. James, *Note on cup-products*, Proc. Amer. Math. Soc. **8**(1957) 374–383.

[4] W. S. Massey, *Exact couples in algebraic topology (Parts I and II)*, Annals of Math. **56**(1952) 363–396.

[5] J. Mukai and K. Yamaguchi, *Homotopy classification of twisted complex projective spaces of dimension* 4, to appear in J. Math. Soc. Japan.

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