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RESULTS ON PRIME NEAR-RING WITH (σ, τ)-DERIVATION

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Abstract. Let N be a prime left near-ring with multiplicative center Z, and D be a (σ, τ)-derivation such that $\sigma D = D\sigma$ and $\tau D = D\tau$. (i) If $D(N) \subset Z$, or $[D(N), D(N)] = 0$ or $[D(N), D(N)]_{\sigma,\tau} = 0$, then $(N, +)$ is abelian. (ii) If N is 2-torsion free, d_1 is a (σ, τ)-derivation and d_2 is a derivation on N such that $d_1 d_2(N) = 0$, then $d_1 = 0$ or $d_2 = 0$.

1. Introduction

Recently, some results concerning commutativity in prime near-rings with derivation have been generalized in several ways. The primary purpose of this paper is to generalize some results obtained by H. E. Bell and G. Mason [1], and A. A. M. Kamal[2].

Throughout this paper, N will denote a zero-symetric left near-ring with multiplicative center Z. N is called a prime near-ring if $aNb = \{0\}$ implies that $a = 0$ or $b = 0$. Let σ and τ be two near-ring automorphisms of N. An additive mapping $D : N \to N$ is called a (σ, τ)-derivation if $D(xy) = \tau(x)D(y) + D(x)\sigma(y)$ holds for all $x, y \in N$. For $x, y \in N$, the symbol $[x, y]$ will denote $xy - yx$, while the symbol (x, y) will denote the additive-group commutator $x + y - x - y$. Given $x, y \in N$, we write $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$; in particular $[x, y]_{1,1} = [x, y]$, in the usual sense. As for terminologies used here without mention, we refer to G. Pilz [3].

2. Results

We begin with two quite general and useful lemmas.

Lemma 1. *Let D be a (σ, τ)-derivation of near ring N. Then $D(xy) = D(x)\sigma(y) + \tau(x)D(y)$ for all $x, y \in N$.*

Proof. Note that

$$\begin{aligned} D(x(y+y)) &= \tau(x)D(y+y) + D(x)\sigma(y+y) \\ &= \tau(x)D(y) + \tau(x)D(y) + D(x)\sigma(y) + D(x)\sigma(y), \end{aligned}$$

and

$$D(xy + xy) = \tau(x)D(y) + D(x)\sigma(y) + \tau(x)D(y) + D(x)\sigma(y).$$

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Comparing these two expressions, one can obtain

$$\tau(x)D(y) + D(x)\sigma(y) = D(x)\sigma(y) + \tau(x)D(y)$$

and so,

$$D(xy) = D(x)\sigma(y) + \tau(x)D(y), \text{ for all } x, y \in N.$$

□

Lemma 2. *Let D be a (σ,τ)-derivation on a near-ring N and $a \in N$. Then for all $x, y \in N$,*

$$(\tau(x)D(y) + D(x)\sigma(y))\sigma(a) = \tau(x)D(y)\sigma(a) + D(x)\sigma(y)\sigma(a).$$

Proof. For all $x, y \in N$, we get

$$\begin{aligned} D((xy)a) &= \tau(xy)D(a) + D(xy)\sigma(a) \\ &= \tau(x)\tau(y)D(a) + (\tau(x)D(y) + D(x)\sigma(y))\sigma(a). \end{aligned}$$

On the other hand,

$$\begin{aligned} D(x(ya)) &= \tau(x)D(ya) + D(x)\sigma(ya) \\ &= \tau(x)\tau(y)D(a) + \tau(x)D(y)\sigma(a) + D(x)\sigma(y)\sigma(a). \end{aligned}$$

For these two expressions of $D(xya)$, we obtain that, for all $x, y \in N$,

$$(\tau(x)D(y) + D(x)\sigma(y))\sigma(a) = \tau(x)D(y)\sigma(a) + D(x)\sigma(y)\sigma(a).$$

□

Lemma 3. *Let N be a prime near-ring, D a nonzero (σ,τ)-derivation of N and $a \in N$.*

i) *If $D(N)\sigma(a) = 0$ then $a = 0$.*
ii) *If $aD(N) = 0$ then $a = 0$.*

Proof. *i*) For all $x, y \in N$, we get

$$0 = D(xy)\sigma(a) = \tau(x)D(y)\sigma(a) + D(x)\sigma(y)\sigma(a).$$

Using hypothesis and σ is an automorphism of N, we have

$$D(x)N\sigma(a) = 0.$$

Since N is prime near-ring and D is a nonzero (σ,τ)-derivation of N, we obtain $a = 0$.

ii) A similar argument works if $aD(N) = 0$. □

Lemma 4. *Let D be a (σ,τ)-derivation which commute σ and τ. If N is a 2-torsion free near-ring and $D^2 = 0$ then $D = 0$.*

Proof. For arbitrary $x, y \in N$, we have

$$\begin{aligned}0 &= D^2(xy) = D(D(xy)) = D(\tau(x)D(y) + D(x)\sigma(y)) \\ &= \tau^2(x)D^2(y) + D(\tau(x))\sigma(D(y)) + \tau(D(x))D(\sigma(y)) + D^2(x)\sigma^2(y).\end{aligned}$$

By hypothesis,

$$2D(\tau(x))D(\sigma(y)) = 0 \quad \text{for all } x, y \in N.$$

Since N is 2-torsion free near-ring and σ is an automorphism on N, we get

$$D(\tau(x))D(N) = 0.$$

It gives $D = 0$ by Lemma 3 (ii). □

Theorem 1. *Let N be a near-ring and D a nonzero (σ, τ)-derivation of N. If $u \in N$ is not a left zero divisor and $[D(u), u]_{\sigma,\tau} = 0$ then (x, u) is constant (that is, $D(x, u) = 0$) for every $x \in N$.*

Proof. Since $u(u + x) = u^2 + ux$, we have $D(u(u + x)) = D(u^2 + ux)$. Expanding this equation, we have

$$\tau(u)D(u + x) + D(u)\sigma(u + x) = D(u^2) + D(ux)$$

and so

$$\begin{aligned}&\tau(u)D(u) + \tau(u)D(x) + D(u)\sigma(u) + D(u)\sigma(x) \\ &\qquad = \tau(u)D(u) + D(u)\sigma(u) + \tau(u)D(x) + D(u)\sigma(x)\end{aligned}$$

which reduces to

$$\tau(u)D(x) + D(u)\sigma(u) - \tau(u)D(x) - D(u)\sigma(u) = 0.$$

Therefore

$$\tau(u)D(x, u) = 0$$

by using the assumption $[D(u), u]_{\sigma,\tau} = 0$. Since u is not a left zero divisor, we get $D(x, u) = 0$. Thus (x, u) is a constant for every $x \in N$. □

Theorem 2. *Let N be a prime near-ring with a nonzero (σ, τ)-derivation D such that $\sigma D = D\sigma$ and $\tau D = D\tau$. If $D(N) \subset Z$ then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof. Suppose that $a \in N$ such that $D(a) \neq 0$. So, $D(a) \in Z\backslash\{0\}$ and $D(a) + D(a) \in Z\backslash\{0\}$. For all $x, y \in N$, we have

$$(x + y)(D(a) + D(a)) = (D(a) + D(a))(x + y),$$

that is,

$$xD(a) + xD(a) + yD(a) + yD(a) = D(a)x + D(a)y + D(a)x + D(a)y.$$

Since $D(a) \in Z$, we get

$$D(a)x + D(a)y = D(a)y + D(a)x,$$

and so,

$$D(a)(x, y) = 0 \text{ for all } x, y \in N.$$

Since $D(a) \in Z\backslash\{0\}$ and N is a prime near-ring, it follows that $(x, y) = 0$, for all $x, y \in N$. Thus $(N, +)$ is abelian.

Using hypothesis, for any $b, c \in N$,

$$\sigma(c)D(ab) = D(ab)\sigma(c).$$

By Lemma 2, we have

$$\sigma(c)\tau(a)D(b) + \sigma(c)D(a)\sigma(b) = \tau(a)D(b)\sigma(c) + D(a)\sigma(b)\sigma(c).$$

Comparing these two expressions, using $D(N) \subset Z$ and $(N, +)$ is abelian, we obtain that

$$\sigma(c)\tau(a)D(b) + D(a)\sigma(c)\sigma(b) = \tau(a)D(b)\sigma(c) + D(a)\sigma(b)\sigma(c)$$

so we have

$$D(b)[\tau(a), \sigma(c)] = D(a)\sigma([c, b]) \text{ for all } b, c \in N.$$

Suppose now that N is not commutative. Choosing $b, c \in N$ such that $[b, c] \neq 0$ and $a = D(x) \in Z$, we get

$$D^2(x)\sigma([c, b]) = 0 \quad \text{for all } x \in N.$$

Since the central element $D^2(x)$ can not be a nonzero divisor of zero, we conclude $D^2(x) = 0$ for all $x \in N$. By Lemma 4, this cannot happen for nontrivial D. □

Theorem 3. *Let N be a prime near-ring admitting a nonzero (σ, τ)-derivation D such that $\sigma D = D\sigma$ and $\tau D = D\tau$. If $[D(N), D(N)] = 0$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof. The argument used in the proof of Theorem 2 shows that if both z and $z + z$ commute elementwise with $D(N)$,then we have

$$zD(x, y) = 0 \text{ for all } x, y \in N. \tag{2.1}$$

Substituting $D(t), t \in N$ for z in (2.1), we get $D(t)D(x, y) = 0$. Since σ is an automorphism of N, we have $\sigma(D(t))\sigma(D(x, y)) = 0$. Using $\sigma D = D\sigma$, we get

$$D(\sigma(t))\sigma(D(x, y)) = 0 \text{ for all } x, y, t \in N.$$

By Lemma 3 (i), we obtain that $D(x, y) = 0$ for all $x, y \in N$. For $w \in N$, we have $0 = D(wx, wy) = D(w(x, y))$ and so we obtain

$$D(w)\sigma((x, y)) = 0.$$

Again, applying Lemma 3 (i), we get $(x, y) = 0$ for all $x, y \in N$.

Now, assume that N is 2-torsion free. By the assumption $[D(N), D(N)] = 0$,

$$D(\sigma(z))D(D(x)y) = D(D(x)y)D(\sigma(z)) \text{ for all } x, y, z \in N.$$

Hence, we get

$$\begin{aligned} &D(\sigma(z))\tau(D(x))D(y) + D(\sigma(z))D^2(x)\sigma(y) \\ &\qquad = \tau(D(x))D(y)D(\sigma(z)) + D^2(x)\sigma(y)D(\sigma(z)) \end{aligned}$$

by Lemma 2. Using $D(\tau(x))D(\sigma(z)) = D(\sigma(z))\ D(\tau(x))$, $\sigma D = D\sigma$ and $\tau D = D\tau$, we have

$$\begin{aligned} &D(\tau(x))D(\sigma(z))D(y) + D(\sigma(z))D^2(x)\sigma(y) \\ &\qquad = D(\tau(x))D(y)D(\sigma(z)) + D^2(x)\sigma(y)D(\sigma(z)) \end{aligned}$$

Since $(N, +)$ is abelian, we conclude that

$$D(\tau(x))[D(\sigma(z)), D(y)] = D^2(x)\sigma([D(z), y]) \text{ for all } x, y, z \in N.$$

The left term of this equation is zero by the hypothesis, so we get

$$D^2(x)\sigma(D(z))\sigma(y) = D^2(x)\sigma(y)\sigma(D(z)) \text{ for all } x, y, z \in N. \tag{2.2}$$

Replacing y by $yt, (t \in N)$ in (2.2) and using (2.2), we have

$$\begin{aligned} D^2(x)\sigma(y)\sigma(t)\sigma(D(z)) &= D^2(x)\sigma(D(z))\sigma(y)\sigma(t) \\ &= D^2(x)\sigma(y)\sigma(D(z))\sigma(t) \end{aligned}$$

and so,

$$D^2(x)N\sigma([t, D(z)]) = 0 \text{ for all } x, t, z \in N. \tag{2.3}$$

Since N is a prime near-ring, we have

$$D^2(N) = 0 \text{ or } D(N) \subset Z$$

by Brauers's Trick. If $D^2(N) = 0$, then it contradicts that D is a nonzero (σ, τ)-derivation of N by Lemma 4. So, $D(N) \subset Z$. Thus, N is a commutative ring by Theorem 2. □

Theorem 4. *Let N be a 2-torsion free prime near-ring, d_1 a (σ, τ)-derivation of N and d_2 a derivation of N. If $d_1d_2(N) = 0$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. For $x, y \in N$, we have

$$\begin{aligned} 0 &= d_1d_2(xy) = d_1(xd_2(y) + d_2(x)y) \\ &= \tau(x)d_1d_2(y) + d_1(x)\sigma(d_2(y)) + \tau(d_2(x))d_1(y) + d_1d_2(x)\sigma(y). \end{aligned}$$

That is,

$$d_1(x)\sigma(d_2(y)) + \tau(d_2(x))d_1(y) = 0 \text{ for all } x, y \in N. \tag{2.4}$$

If we take $d_2(x)$ instead of x in (2.4), then

$$\tau(d_2^2(x))d_1(y) = 0 \text{ for all } x, y \in N.$$

Using Lemma 3 (ii) one can obtain $d_1 = 0$ or $d_2^2 = 0$. If $d_2^2 = 0$, we have $d_2 = 0$ by Lemma 4. This completes the proof of theorem. □

Theorem 5. *Let N be a 2-torsion free prime near-ring, d_1 a derivation and d_2 be a (σ, τ)-derivation of N such that $\tau d_2 = d_2\tau$ and $\tau d_1 = d_1\tau$. If $d_1 d_2(N) = 0$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. The same argument in the proof of Theorem 4, we can write

$$d_1(\tau(x))d_2(y) + d_2(x)d_1(\sigma(y) = 0 \text{ for all } x, y \in N. \tag{2.5}$$

Replacing x by $d_2(x)$ in (2.5) and using $\tau d_2 = d_2\tau$ and $\tau d_1 = d_1\tau$, we have

$$d_2^2(x)d_1(\sigma(y) = 0 \text{ for all } x, y \in N.$$

Applying [1, Lemma 3 (ii)], we obtain $d_1 = 0$ or $d_2^2 = 0$. If $d_2^2 = 0$, then $d_2 = 0$ by Lemma 4. □

Theorem 6. *Let D be a nonzero (σ, τ)-derivation of a prime near-ring N and $a \in N$. If $[D(N), a]_{\sigma,\tau} = 0$ then $D(a) = 0$ or $a \in Z$.*

Proof. By hypothesis,

$$D(ax)\sigma(a) = \tau(a)D(ax) \text{ for all } x \in N$$

and so,

$$(\tau(a)D(x) + D(a)\sigma(x))\sigma(a) = \tau(a)(\tau(a)D(x) + D(a)\sigma(x)).$$

Since N satisfies the partial distributive law by Lemma 2, we get

$$\tau(a)D(x)\sigma(a) + D(a)\sigma(x)\sigma(a) = \tau(a)\tau(a)D(x) + \tau(a)D(a)\sigma(x).$$

Using the hypothesis, we have

$$\tau(a)\tau(a)D(x) + D(a)\sigma(x)\sigma(a) = \tau(a)\tau(a)D(x) + D(a)\sigma(a)\sigma(x),$$

that is,

$$D(a)\sigma([x, a]) = 0 \text{ for all } x \in N. \tag{2.6}$$

Substituting $xy, (y \in N)$ for x and using (2.6), we have

$$D(a)\sigma(x)\sigma([y, a]) = 0 \text{ for all } x, y \in N.$$

Since σ is automorphism of prime near-ring of N, we get $D(a) = 0$ or $a \in Z$. This completes the proof. □

Theorem 7. *Let D be a nonzero (σ, τ)-derivation of a prime near-ring N such that $\sigma D = D\sigma$ and $\tau D = D\tau$. If $[D(N), D(N)]_{\sigma,\tau} = 0$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion free then N is a commutative ring.*

Proof. By Theorem 6, we have

$$N = \{x \in N \mid D^2(x) = 0\} \cup \{x \in N \mid D(x) \in Z\}.$$

By Brauer's Trick, we get $D^2(N) = 0$ or $D(N) \subset Z$. Since D is a nonzero (σ, τ)-derivation of N, we get $D(N) \subset Z$. By Theorem 2, we prove the theorem. □

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