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COMMUTATIVE GROUP ALGEBRAS OF DIRECT SUMS OF σ-SUMMABLE ABELIAN p-GROUPS

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Introduction

Throughout the rest in this paper, let $R[G]$ be the group ring (which is an R-algebra) of an abelian group G over the coefficient ring R which is abelian (i.e. in other terms is commutative) with identity of prime characteristic p. The maximal torsion subgroup of G is denoted by tG, and its p-component by G_p. We shall let $SR[G]$ denote the p-torsion part of the group $VR[G]$ of all invertible normed (i.e. of augmentation 1) elements in $R[G]$. For a subgroup A of G, $I(R[G]; A)$ denotes the relative augmentation ideal of $R[G]$ with respect to A. Put the set $V(R[G]; A) = 1 + I(R[G]; A)$. Besides, the letter F will denote a field of $\mathrm{char}(F) = p$. All other notations and the terminology to the abelian group theory and the commutative group algebra theory not explicitly defined herein are in agreement with Laszlo Fuchs [6] and Gregory Karpilovsky [9].

In the present work, a new criterion for totally projective abelian p-groups with length equal to the first uncountable ordinal Ω is found in the terms of direct sums of σ-summable groups. This group class contains the totally projective groups (of lengths cofinal with ω) as special cases. The main purpose here, however, is to study the commutative modular group algebras via their direct sums of σ-summable groups. Developing our technique in [4], we establish some results on the *isomorphism problem* and on the *direct factor problem*. They are a modern advance in this theme.

This research is organized as follows. In the first paragraph (i.e. here) we set up some notations and our main aims. In the second we investigate modular group algebras over a special class of abelian p-groups, called direct sums of σ-summables. The third section contains some applications which generalize and extend well-known and documented facts in this direction. We close the article with some left-open problems.

Well, we abstractly summarize our attainments thus: Suppose $R[G]$ is the group algebra of an abelian p-group G over a commutative ring R with 1 of prime characteristic p. As usual, $SR[G]$ is the normed p-component in $R[G]$. The first main result is that $SR[G]$ is a direct sum of σ-summables if and only if G is, provided R is a field. Besides, if G is a direct sum of σ-summable groups, then so is $SR[G]/G$.

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In the light of σ-summable abelian p-groups, a new criterion for total projectivity is also given, namely, when $\operatorname{length} G = \Omega$ it is fulfilled that G is totally projective if G/G^{p^α} is a direct sum of σ-summables for all $\alpha \leq \Omega$.

Moreover, if G is a direct sum of groups with countable lengths, then $SR[G]/G$ is totally projective, provided R is a perfect field. The last is a very strong expansion of almost all known results in this way.

So, we are in a position to begin with

Direct sums of σ-summable abelian p-groups and their group algebras

Now, we shall study here the class of direct sums of σ-summable abelian groups and moreover we shall give a convenient for us criterion for total projectivity. Following P. Hill [8], we shall say (analogous to the well-known Kulikov's criterion for direct sums of cyclic groups) that the reduced abelian p-group A is σ-summable if A is the union (group-theoretic or set-theoretic) of a countable number of subgroups A_n, where the heights of the elements of A_n computed in G are bounded by some ordinal $\lambda(n) < \lambda = \operatorname{length} A$, i.e. in the other words $A = \bigcup\limits_{n<\omega} A_n$, where $A_n \subseteq A_{n+1}$ eventually and for each $n < \omega$, there exists an ordinal $\lambda(n)$ strictly less than the length of A such that $A_n \cap A^{p^{\lambda(n)}} = 1$. Certainly, $\operatorname{length} A$ is limit cofinal with ω.

This definition may be restated thus:

Proposition. *The abelian p-group A is σ-summable if and only if $A = \bigcup\limits_{n<\omega} A_n$, where $A_n \subseteq A_{n+1}$, A_n are isotype in A and ${A_n}^{p^{\lambda(n)}} = 1$ whenever $\lambda(n) < \operatorname{length} A = \lambda$, namely A_n are precisely the $p^{\lambda(n)}$-high subgroups of A.*

Proof. The sufficiency is proved by P. Hill [8]. The necessity also holds automatically because if $A_n \cap A^{p^{\lambda(n)}} = 1$ with $\lambda(n) < \lambda$, then $A_n \subseteq B_n$ when B_n is $p^{\lambda(n)}$-high in A, whence isotype in A (by Irwin-Walker; [6]). So, we are done. $\square$

Commentary. Actually the last proposition is a modified variant of a definition for λ-summability in the terms of D. Cutler [1], where $\lambda = \lim\limits_{n<\omega} \lambda(n)$, i.e. equivalently $\lambda = \sup\limits_{n<\omega} \lambda(n)$. Moreover a criterion in the terms of Ulm factors and p^α-high subgroups when two λ-summable (in other words, σ-summable) groups are isomorphic is obtained in [1], too. Really, the socle is not enough to determine this group class; excellent examples for this are also given by P. Hill [8] and the author [5].

Clearly the present group class possesses the following properties: If A is σ-summable, so is any its subgroup of the same length and length A is cofinal with ω; every separable primary group is σ-summable if and only if it is an unbounded direct sum of cyclics. More generally, the following is true [4], [7].

Theorem (Linton-Megibben, 1977; Hill, 1981). *Suppose that the p-primary abelian group A has length λ cofinal with ω. Then A is totally projective if and only if A is σ-summable and A/A^{p^α} is totally projective for all limit $\alpha < \lambda$.*

Since each countable limit ordinal is cofinal with ω, then following [4] (idea given by C. Megibben), from the above theorem we may conclude

Criterion. *Let the p-torsion abelian group A have countable length λ. Then A is totally projective if and only if A/A^{p^α} is σ-summable for all limit $\alpha \leq \lambda$. Besides, if A is with length $\Omega\rho$ where $\rho < \Omega$ is limit, then A is totally projective if and only if A/A^{p^α} is σ-summable for all cofinal with ω ordinals $\alpha \leq \Omega\rho$ and $A/A^{p^{\Omega n}}$ is totally projective for each natural n.*

That is why of a global interest is to establish a necessary and sufficient condition in the terms of σ-summability when the group length is not cofinal with ω; for example, Ω. This is made by the next

Criterion. *An abelian p-torsion reduced group A is a direct sum of countables with limit lengths if and only if* length $A \leq \Omega$ *and A/A^{p^α} is a direct sum of σ-summables for all limit $\alpha \leq \Omega$.*

Proof. Write $A = \coprod_{i\in I} A_i$, where A_i are countables. Hence length $A \leq \Omega$ by [6] and $A/A^{p^\alpha} \cong \coprod_{i\in I} A_i/{A_i}^{p^\alpha}$ is evidently a direct sum of σ-summables since all $A_i/{A_i}^{p^\alpha}$ are.

Conversely, write $A = \coprod_{i\in I} A_i$, where A_i are σ-summables. Because length $A \leq \Omega$ we derive length A_i is limit $< \Omega$, whence cofinal with ω. On the other hand A/A^{p^α} as a direct sum of σ-summables must be σ-summable from [8] since $\alpha < \Omega$ is cofinal with ω. Therefore, for each $i \in I$ and $\alpha \leq$ length $A_i \leq$ length A, $A_i/{A_i}^{p^\alpha}$ regarded as subgroups of A/A^{p^α} with equal lengths are σ-summables, too. Furthermore, the first criterion means that A_i are direct sums of countables, whence so is A. The proof is complete. $\square$

Since all countable limit ordinals are cofinal with ω and every summable group of countable limit length is σ-summable (more generally, each direct sum of length cofinal with ω of σ-summable abelian p-groups is σ-summable [8]), we are in position to prepare the following generalization of the classical Megibben's criterion for direct sums of countable groups [14, Theorem B], namely:

Criterion (Generalized Megibben criterion)**.** *Let A be a p-primary σ-summable abelian group of length λ that contains all p^{α}-high totally projective subgroups for each $\alpha < \lambda$. Then A is totally projective.*

Proof. The above proposition on σ-summability guarantees that $A = \bigcup_{n<\omega} A_n$, where $A_n \subseteq A_{n+1}$ and A_n are $p^{\lambda(n)}$-high subgroups of A with $\lambda(n) < \lambda$. Thus by hypothesis A_n are totally projective isotype subgroups of A and utilizing a paramount result of P. Hill [7], A must be totally projective as well. The proof is over. $\square$

We start now with series of lemmas on balanced (nice and isotype) subgroups necessary for our good presentation. The subgroup N of an abelian p-group A is said to be (by P. Hill; [6]) nice if $(A/N)^{p^{\alpha}} = (A^{p^{\alpha}}N)/N$ for every ordinal α. It is a simple exercise to verify that the last is equivalent to the equalities: $\bigcap_{\tau<\alpha}(A^{p^{\tau}}N) = (\bigcap_{\tau<\alpha} A^{p^{\tau}})N = A^{p^{\alpha}}N$ for all limit α. The next technical assertion is valuable.

Lemma 1. *Assume that N is nice in the p-group A. Then for each limit ordinal σ, $NA^{p^{\sigma}}$ is nice in A.*

Proof. Fix an arbitrary limit ordinal α and σ. As we see, it remains only to show that $\bigcap_{\tau<\alpha}(A^{p^{\tau}}NA^{p^{\sigma}}) = A^{p^{\alpha}}NA^{p^{\sigma}}$. In fact, for $\sigma \leq \alpha$ we have

$$\begin{aligned}
&\bigcap_{\tau<\alpha}(A^{p^{\tau}}NA^{p^{\sigma}}) = [\bigcap_{\tau<\sigma\leq\alpha}(A^{p^{\tau}}NA^{p^{\sigma}})] \cap [\bigcap_{\sigma<\tau<\alpha}(A^{p^{\tau}}NA^{p^{\sigma}})] \\
&= [\bigcap_{\tau<\sigma\leq\alpha}(A^{p^{\tau}}N)] \cap [\bigcap_{\sigma<\tau<\alpha}(A^{p^{\sigma}}N)] = (\bigcap_{\tau<\sigma}(A^{p^{\tau}}N)) \cap (A^{p^{\sigma}}N) \\
&= (A^{p^{\sigma}}N) \cap (A^{p^{\sigma}}N) = A^{p^{\sigma}}N = A^{p^{\alpha}}(NA^{p^{\sigma}}).
\end{aligned}$$

On the other hand, for $\alpha < \sigma$ we derive

$$\bigcap_{\tau<\alpha}(A^{p^{\tau}}NA^{p^{\sigma}}) = \bigcap_{\tau<\alpha}(A^{p^{\tau}}N) = A^{p^{\alpha}}N = A^{p^{\alpha}}(NA^{p^{\sigma}}).$$

After all $A^{p^{\sigma}}N$ is indeed nice in A, which finishes the proof. $\square$

Lemma 2 ([3])**.** *If H is isotype p-torsion in G, then for every ordinal α, $V^{p^{\alpha}}(R[G]; H) = V(R^{p^{\alpha}}[G^{p^{\alpha}}]; H^{p^{\alpha}})$.*

We hasten to the key lemma needed for the special theorem formulated below.

Lemma 3. *Suppose H is an isotype p-subgroup of G. Then H is balanced in $V(R[G];H)$.*

Proof. The isotypity is trivial (it also follows by Lemma 2). Next owing to the above observation on niceness and to Lemma 2, it is sufficient to prove only that $\bigcap_{\tau<\alpha}(V(R^{p^\tau}[G^{p^\tau}];H^{p^\tau})H) = V(R^{p^\alpha}[G^{p^\alpha}];H^{p^\alpha})H$ for each limit α. Really, choose x in the left-hand side. Then

$$x = h_1 \sum_{g\in G^{p^\tau}} r_g g = h_2 \sum_{g'\in G^{p^\sigma}} r_{g'} g' (= \cdots),$$

where $h_1, h_2 \in H$ ($h_1 \neq h_2$, or otherwise there is nothing to prove); $r_g \in R^{p^\tau}$, $r_{g'} \in R^{p^\sigma}$ so that $\tau < \sigma \leq \alpha$. But on the other hand,

$$\sum_{g\in \bar{g}H^{p^\tau}} r_g = \begin{cases} 1, & \bar{g} \in H^{p^\tau} \\ 0, & \bar{g} \notin H^{p^\tau} \end{cases} \text{ for any } \bar{g} \in G^{p^\tau}$$

$$\text{and} \quad \sum_{g'\in \bar{g}'H^{p^\sigma}} r_{g'} = \begin{cases} 1, & \bar{g}' \in H^{p^\sigma} \\ 0, & \bar{g}' \notin H^{p^\sigma} \end{cases} \text{ for any } \bar{g}' \in G^{p^\sigma}.$$

Besides $r_g = r_{g'}$ and

$$(*) \qquad h_1 g = h_2 g'.$$

Hence elementary $x = h_1 h \sum_{g\in G^{p^\tau}} r_g g h^{-1}$, where $h \in H^{p^\tau}$ is a member of the sum $\sum_{g\in G^{p^\tau}} r_g g$ (we omit the details). Apparently $h_1 h \in H$, $gh^{-1} \in G^{p^\tau}$ (by $(*)$ it lies in G^{p^σ}) and

$$\sum_{gh^{-1}\in \bar{g}h^{-1}H^{p^\sigma}} r_{gh^{-1}} = \begin{cases} 1, & \bar{g}h^{-1} \in H^{p^\sigma}, \\ 0, & \bar{g}h^{-1} \notin H^{p^\sigma}, \end{cases}$$

because if $\bar{g}h^{-1} \notin H^{p^\tau}$, then $\bar{g}h^{-1} \notin H^{p^\sigma}$; but if $\bar{g}h^{-1} \in H^{p^\tau}$, then by $(*)$, $\bar{g}h^{-1} \in H \cap G^{p^\sigma} = H^{p^\sigma}$. That is why, as a final, $x \in HV(R^{p^\alpha}[G^{p^\alpha}];H^{p^\alpha})$, as required. The proof is finished. □

Remark 4. The same statement was proved by W. May [13], but when R is a field. Our scheme of proof and the used technique are different to that of May.

Corollary 5. *Assume reduced H is an isotype p-subgroup of G. Then* $\operatorname{length} V(R[G];H) = \operatorname{length} H$. *Moreover if H has limit length, then* $\operatorname{length}(V(R[G];H)/H) = \operatorname{length} H$.

Proof. The first claim follows immediately by virtue of Lemma 2 and the definition for a length, since $V^{p^\varepsilon}(R[G];H) = V(R^{p^\varepsilon}[G^{p^\varepsilon}];H^{p^\varepsilon}) = 1$ if and only if $H^{p^\varepsilon} = 1$ for any ordinal ε.

Let us now $\operatorname{length} H = \lambda$ be a limit ordinal. Since $H^{p^\lambda} = 1$, Lemmas 2 and 3 lead to $(V(R[G]; H)/H)^{p^\lambda} = V(R^{p^\lambda}[G^{p^\lambda}]; H^{p^\lambda})H/H = 1$, i.e. in the other words, $\operatorname{length}(V(R[G]; H)/H) \leq \lambda$. We claim that $\operatorname{length}(V(R[G]; H)/H) = \lambda$. If not, then owing again to Lemmas 2 and 3 we conclude, $V(R^{p^\delta}[G^{p^\delta}]; H^{p^\delta}) \subseteq H$ for some $\delta < \lambda$. Then either $H^{p^\delta} = 1$, which is a contradiction, or if $H^{p^\delta} \neq 1$ we can deduce $|H^{p^\delta}| = 2$. But then $H^{p^{\delta+1}} = 1$, since $H^{p^\delta} \neq H^{p^{\delta+1}}$ (if $H^{p^\delta} = H^{p^{\delta+1}}$, then $H^{p^\delta} = 1$ as a reduced divisible group). In this light $\delta + 1 < \lambda$ because λ is limit, thus contrary to the fact that $\lambda = \operatorname{length} H$. This concludes the proof. □

Corollary 6. *Presume H is an isotype p-subgroup of G. If H is separable, then $V(R[G]; H)/H$ is separable. Moreover $V(R[G]; H)$ is separable if and only if H is separable.*

Proof. Follows directly applying Corollary 5. The proof is complete. □

Proposition 7 ([3], [4])**.** *Presume that R is without nilpotents. Then $SR[G] = V(R[G]; G_p)$. If R is arbitrary, but G is p-primary, then $SR[G] = VR[G] = V(R[G]; G)$.*

Corollary 8. $\operatorname{length} SR[G] = \operatorname{length} G_p$ *assuming R is with no nilpotents. Moreover* $\operatorname{length} VR[G] = \operatorname{length} G$ *assuming G is a p-group and R is arbitrary.*

Proof. Follows by a direct application of Corollary 5 and Proposition 7. □

The next technical matters are crucial.

Proposition 9. *Assume $1 \in L \leq R$, $A \leq B \leq G$ and $C \leq G$. Then*

(i) $V(R[G]; A) \cap V(L[C]; C) = V(L[C]; C \cap A)$,

(ii) $BV(R[G]; A) \cap V(L[C]; C) \subseteq BV(L[C]; C \cap A)$,

(iii) $(BVR[A]) \cap VL[C] \subseteq BVL[A \cap C]$.

Proof. (i) Given $x \in V(R[G]; A) \cap V(L[C]; C)$, hence $x = \sum_{c \in C} \alpha_c c$, where $\alpha_c \in L$ and

$$\sum_{c \in \bar{c}A} \alpha_c = \begin{cases} 0, & \bar{c} \notin A \\ 1, & \bar{c} \in A \end{cases}$$

for every $\bar{c} \in C$. But $\bar{c}A \cap C = \bar{c}(A \cap C)$ since $\bar{c} \in C$ and so

$$\sum_{c \in \bar{c}(A \cap C)} \alpha_c = \begin{cases} 0, & \bar{c} \notin (A \cap C), \\ 1, & \bar{c} \in (A \cap C). \end{cases}$$

Consequently $x \in V(L[C]; C \cap A)$ and we are done.

(ii) Choose x to belong in the left-hand side. Furthermore $x = \sum_{c\in C}\alpha_c c = b\sum_{g\in G} r_g g$, where $\alpha_c \in L$, $r_g \in R$ and $b \in B$, and moreover

$$\sum_{g\in\bar{g}A} r_g = \begin{cases} 0, & \bar{g} \notin A \\ 1, & \bar{g} \in A \end{cases}$$

for any $\bar{g} \in G$. Clearly $\alpha_c = r_g$ and $g = b^{-1}c_g$ where $c_g \in C$. But since $\bar{c}A \cap C = \bar{c}(A \cap C)$ for each $\bar{c} \in C$ holds, then as in the proof of Lemma 3 it is a routine matter to verify that $x \in BV(L[C]; C \cap A)$, as required.

(iii) Take x in the left-hand side. Hence $x = b\sum_i r_i a_i = \sum_i \alpha_i c_i$ $(1 \leq i \leq n)$, where $b \in B$; $r_i \in R$, $\alpha_i \in L$; $a_i \in A$, $c_i \in C$. The canonical forms yield $r_i = \alpha_i$ and $ba_i = c_i$. Thus we can write

$$x = ba_1(r_1 + \cdots + r_n a_n a_1^{-1}) \in BVL[A \cap C],$$

because $ba_1 \in B$ and

$$r_1 + \cdots + r_n a_n a_1^{-1} = a_1^{-1}(r_1 a_1 + \cdots + r_n a_n) \in VR[A] \cap VL[C] = VL[A \cap C].$$

The proposition is verified. $\square$

We now come to one of our central claims for this section. The used below technique is a nontrivial development and generalization to that given by us in [4].

Theorem 10. (i) *Suppose H is an isotype p-subgroup of G. Then $V(R[G]; H)$ is σ-summable if and only if H is σ-summable. Moreover if H is σ-summable, then $V(R[G]; H)/H$ is σ-summable.*

(ii) *Let H be a pure p-subgroup of G. Then $V(R[G]; H)$ is a direct sum of cyclic groups if and only if H is a direct sum of cyclic groups. Moreover if H is a direct sum of cyclics, then $V(R[G]; H)/H$ is a direct sum of cyclics and thus H is a direct factor of $V(R[G]; H)$ with a complement which is a direct sum of cyclics.*

Proof. (i) If $V(R[G]; H)$ is σ-summable, then H is the same according to Corollary 5. Now we treat the more difficult converse question. Really, we can write $H = \bigcup_{n<\omega} H_n$, where $H_n \subseteq H_{n+1}$ and $H_n \cap H^{p^{\lambda(n)}} = 1$ for each $n < \omega$ and some $\lambda(n)$ with $\lambda(n) < \lambda = \mathrm{length}\, H$. Therefore $V(R[G]; H) = \bigcup_{n<\omega} V(R[G]; H_n)$ and $V(R[G]; H)/H = \bigcup_{n<\omega} [V(R[G]; H_n)H/H]$. Further by virtue of Lemma 2 and Proposition 9 we compute

$$\begin{aligned} V(R[G]; H_n) \cap V^{p^{\lambda(n)}}(R[G]; H) &= V(R[G]; H_n) \cap V(R^{p^{\lambda(n)}}[G^{p^{\lambda(n)}}]; H^{p^{\lambda(n)}}) \\ &\subseteq V(R^{p^{\lambda(n)}}[G^{p^{\lambda(n)}}]; G^{p^{\lambda(n)}} \cap H_n) = 1, \end{aligned}$$

since $1 = H_n \cap H^{p^{\lambda(n)}} = H_n \cap G^{p^{\lambda(n)}}$. On the other hand Corollary 5 guarantees $\lambda(n) < \lambda = \operatorname{length} V(R[G]; H)$ and so by definition, $V(R[G]; H)$ is σ-summable, as claimed.

For the second part invoking Lemmas 2, 3 and Proposition 9 we calculate

$$\begin{aligned}
&(V(R[G]; H_n)H/H) \cap [V(R[G]; H)/H]^{p^{\lambda(n)}} \\
&= (V(R[G]; H_n)H/H) \cap (V^{p^{\lambda(n)}}(R[G]; H)H/H) \\
&= [(V(R[G]; H_n)H) \cap (V^{p^{\lambda(n)}}(R[G]; H)H)]/H \\
&= [(V(R[G]; H_n)H) \cap (V(R^{p^{\lambda(n)}}[G^{p^{\lambda(n)}}]; H^{p^{\lambda(n)}})H)]/H \\
&\subseteq H[(V(R[G]; H_n)H) \cap V(R^{p^{\lambda(n)}}[G^{p^{\lambda(n)}}]; G^{p^{\lambda(n)}}))]/H \\
&= HV(R^{p^{\lambda(n)}}[G^{p^{\lambda(n)}}]; G^{p^{\lambda(n)}} \cap H_n)/H = 1,
\end{aligned}$$

because as we see elementary $G^{p^{\lambda(n)}} \cap H_n = 1$. On the other hand Corollary 5 gives $\lambda(n) < \lambda = \operatorname{length}(V(R[G]; H)/H)$. Finally $V(R[G]; H)/H$ is σ-summable, as stated. The proof is completed.

(ii) Follows directly by application of point (i), Corollary 6 and in view of ([6], p. 143, Theorem 28.2 of L. Kulikov) together with Lemma 3. The proof is finished. $\square$

Remark 11. The point (ii) of the above theorem is announced in [2].

Corollary 12 ([4])**.** *$SR[G]$ is σ-summable if and only if G_p is, provided R is without nilpotents. If G is a p-group and R is arbitrary, $VR[G]$ is σ-summable if and only if G is.*

Proof. Follows by a direct application of Theorem 10 and Proposition 7 at $H = G_p$ and $H = G$, respectively. $\square$

Proposition 13. *Suppose $M \leq G$ and $B \leq G$, where B is p-torsion. Then $VF[G] = VF[M] \times V(F[G]; B)$ if and only if $G = M \times B$.*

Proof. "**necessity**". Indeed, $M \cap B \subseteq VF[M] \cap V(F[G]; B) = 1$ and so $M \cap B = 1$. Now, for given $x \in G \subseteq VF[G]$ we write

$$x = (\sum_i r_i m_i)(1 + \sum_{i,j} \alpha_{ij} g_{ij}(1 - b_i)) = \sum_k r_k m_k + \sum_{i,j,k} r_k \alpha_{ij} m_k g_{ij}(1 - b_i),$$

where $r_k, \alpha_{ij} \in F$; $m_k \in M$, $g_{ij} \in G$, $b_i \in B$. Hence $x = mgb$ or eventually $x = m'g'$ for some (fixed) $m \in M$, $m' \in M$; $g \in G$, $g' \in G$; $b \in B$. Moreover we observe that $r_k \alpha_{ij} \neq 0$ for all k, i, j. Since in the canonical form of $1 + \sum_{i,j} \alpha_{ij} g_{ij}(1 - b_i)$ there is an element with nonzero coefficient which lies in B, in the support of x there exists a member that belongs to MB, and whence that $g \in MB$ or eventually $g' \in MB$. Finally $x \in MB = M \times B$ and this finishes the proof.

"**sufficiency**". Because $G = M \times B$ we conclude that $F[G] = (F[M])[B]$. Therefore for every $x \in VF[G]$ we have $x = \sum_{a \in B} x_a a$, where $x_a \in F[M]$. Choose $\bar{x} = \sum_{a \in B} x_a \in F[M]$. Apparently $x = \bar{x} + \sum_{a \in B \setminus \{1\}} x_a(a-1)$. But B is p-torsion and obviously $x^{p^k} = \bar{x}^{p^k}$ for some natural k. Thus it is not difficult to verify that $\bar{x} \in VF[G]$ and consequently $\bar{x} \in VF[G] \cap F[M] = VF[M]$. Moreover select $v = 1 + \bar{x}^{-1} \cdot \sum_{a \in B \setminus \{1\}} x_a(a-1)$. Evidently $v \in V(F[G]; B)$, and $x = \bar{x}v$. So $VF[G] \subseteq VF[M] \cdot V(F[G]; B)$. On the other hand, using Proposition 9 we conclude that

$$VF[M] \cap V(F[G]; B) = V(F[M]; M \cap B) = 1$$

since $M \cap B = 1$ by hypothesis. As a final, $VF[G] = VF[M] \times V(F[G]; B)$ as desired. This gives the equality. The proposition is verified. $\square$

Theorem 14. *Suppose G is a direct sum of σ-summable abelian p-groups. Then the same holds for $VR[G]$ and $VR[G]/G$. Conversely, if $VF[G]$ is a direct sum of σ-summable p-groups, then so is G. Moreover, $FH \cong FG$ as F-algebras for some group H and G a direct sum of σ-summable p-groups yield that so is H.*

Proof. Write down $G = \coprod_{\alpha < \lambda} G_\alpha$, where all G_α are σ-summables. Therefore Proposition 13 means that $VR[G] = \coprod_{\beta < \lambda} V(R[\coprod_{\alpha \leq \beta} G_\alpha]; G_\beta)$ and consequently

$$VR[G]/G \cong \coprod_{\beta < \lambda} [V(R[\coprod_{\alpha \leq \beta} G_\alpha]; G_\beta)/G_\beta].$$

But G_β is isotype in $\coprod_{\alpha \leq \beta} G_\alpha$ and thus Theorem 10 is applicable to finish the first two claims.

Let now $VF[G] = \coprod_{\alpha < \lambda} V_\alpha$, where all V_α are direct sums of σ-summables p-primary. Without harm of generality we may presume that $V_\alpha = \coprod_{\gamma < \alpha} V(F[\coprod_{\delta \leq \gamma} G_\delta]; G_\gamma)$ for some special selected subgroups G_γ of G. Therefore

$$VF[G] = \coprod_{\gamma < \lambda} V(F[\coprod_{\delta \leq \gamma} G_\delta]; G_\gamma),$$

where by hypothesis $V(F[\coprod_{\delta \leq \gamma} G_\delta]; G_\gamma)$ is σ-summable, hence Theorem 10 ensures that G_γ is one also. On the other hand Proposition 13 guarantees that $G = \coprod_{\gamma<\lambda} G_\gamma$, which completes this situation. The final part is a routine back-and-forth consequence. The proof is verified. □

Question 15. Of a major interest is the more general version of whether $SR[G]$ is a direct sum of σ-summable groups if and only if G_p is?

Applications

The first main application, however, is on the total projectivity.

Definition. The abelian p-group A in length λ is said to be an S_λ-group for some ordinal λ provided A/A^{p^α} is σ-summable for all limit ordinals $\alpha < \lambda$.

Well, we can attack

Proposition 16. *Suppose G is a reduced p-group and $R^p = R^{p^2}$. Then $VR[G]$ is an S_λ-group if and only if G is. Moreover if G is an S_λ-group, the same is true for $VR[G]/G$.*

Proof. We will use the definition. For the necessity of the first half we observe that $G/G^{p^\alpha} \cong GV^{p^\alpha}R[G]/V^{p^\alpha}R[G]$ is a subgroup of $VR[G]/V^{p^\alpha}R[G]$ with the same length.

In the other two ways, foremost we observe that $\text{length}\, VR[G] = \text{length}\, G$, according to Corollary 8. Further $G/G^{p^\alpha} = \bigcup_{n<\omega}(G_n/G^{p^\alpha})$ for all limit $\alpha < \text{length}\, G = \lambda$ such that $G^{p^\alpha} \subseteq G_n = {G_n}^{(\alpha)}$. Moreover

$$[G_n/G^{p^\alpha}] \cap (G/G^{p^\alpha})^{p^{\beta_n(\alpha)}} = [G_n \cap G^{p^{\beta_n(\alpha)}}]/G^{p^\alpha} = 1$$

for each $n < \omega$ and arbitrary, but a fixed α with $n \leq \beta_n(\alpha) < \alpha$. It is not difficult to see that $G/G^{p^\alpha} = (\bigcup_{n<\omega} G_n)/G^{p^\alpha}$, i.e. $G = \bigcup_{n<\omega} G_n$. Thus $VR[G] = \bigcup_{n<\omega} VR[G_n]$ and whence

$$VR[G]/V^{p^\alpha}R[G] = \bigcup_{n<\omega}(VR[G_n]V^{p^\alpha}R[G]/V^{p^\alpha}R[G])$$

and

$$VR[G]/GV^{p^\alpha}R[G] = \bigcup_{n<\omega}(VR[G_n]GV^{p^\alpha}R[G]/GV^{p^\alpha}R[G]).$$

Apparently

$$VR[G]/GV^{p^\alpha}R[G] \cong VR[G]/G/(VR[G]/G)^{p^\alpha},$$

owing to Lemma 3 and Proposition 7 which mean that G is balanced in $VR[G]$. Further we compute

$$\begin{aligned}&(VR[G_n]V^{p^\alpha}R[G]/V^{p^\alpha}R[G]) \cap (VR[G]/V^{p^\alpha}R[G])^{\beta_n(\alpha)}\\ &= [(VR[G_n]V^{p^\alpha}R[G]) \cap V^{p^{\beta_n(\alpha)}}R[G]]/V^{p^\alpha}R[G]\\ &= V^{p^\alpha}R[G](VR[G_n] \cap V^{p^{\beta_n(\alpha)}}R[G])/V^{p^\alpha}R[G]\\ &= V^{p^\alpha}R[G]VR^{p^\alpha}[G_n \cap G^{p^{\beta_n(\alpha)}}]/V^{p^\alpha}R[G] = 1.\end{aligned}$$

But besides Lemma 1 occurs that $GV^{p^\alpha}R[G]$ is nice in $VR[G]$ and so

$$\begin{aligned}&(VR[G_n]GV^{p^\alpha}R[G]/GV^{p^\alpha}R[G]) \cap (VR[G]/GV^{p^\alpha}R[G])^{p^{\beta_n(\alpha)}}\\ &= [(VR[G_n]GV^{p^\alpha}R[G]) \cap (V^{p^{\beta_n(\alpha)}}R[G]G)]/GV^{p^\alpha}R[G]\\ &= GV^{p^\alpha}R[G](VR[G_n] \cap (VR^{p^{\beta_n(\alpha)}}[G^{p^{\beta_n(\alpha)}}]G))/GV^{p^\alpha}R[G]\\ &= GV^{p^\alpha}R[G]VR^{p^{\beta_n(\alpha)}}[G^{p^{\beta_n(\alpha)}} \cap G_n]/GV^{p^\alpha}R[G]\\ &= GV^{p^\alpha}R[G]VR^p[G^{p^\alpha}]/GV^{p^\alpha}R[G] = 1,\end{aligned}$$

owing to Proposition 9 and the fact that $R^p = R^{p^\alpha}$. As a final, the definition leads us to this that $VR[G]/GV^{p^\alpha}R[G]$ is σ-summable, as well. The claim is shown. □

Using the preceding our technique, we are in a position to attack one classical old-standing conjecture, namely that $VF[G]/G$ is totally projective. In the abstract of his paper [8], Hill has claimed that the σ-summability plays essentially no role in regard to the question of whether or not $VF[G]/G$ is totally projective. This refutes by the following significant facts, starting with

Theorem 17. *Suppose G is a p-group and R is a perfect ring with no zero divisors. If G is a direct sum of groups of countable lengths, then $VR[G]/G$ is totally projective of length $\leq \Omega$. Thus G is a direct factor of $VR[G]$ with a totally projective complement.*

Proof. Foremost presume that $\operatorname{length} G < \Omega$. In view of [8, Proposition 3] and our first criterion in the form of Linton-Megibben-Hill, it is sufficient to prove that if G is σ-summable, then $VR[G]/GV^{p^\alpha}R[G]$ is one also for each limit $\alpha \leq \lambda = \operatorname{length} G$. When $\alpha = \lambda$, the claim follows immediately from [4].

Now we consider the case $\alpha < \lambda$. So, write down $G = \bigcup_{n<\omega} G_n$, $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^{\alpha_n}} = 1$ for $\alpha_n < \lambda$. Therefore

$$V = VR[G]/GV^{p^\alpha}R[G] = \bigcup_{n<\omega} (VR[G_n]GV^{p^\alpha}R[G]/GV^{p^\alpha}R[G]).$$

Since α is countable limit whence cofinal with ω, there is an ascending sequence of ordinals $\beta_1 < \cdots < \beta_n < \cdots$ with $\sup\limits_{n<\omega} \beta_n = \alpha$. Next, we construct the subgroups

$$V_n = \langle r_{1n} + r_{2n}g_{2n} + \cdots + r_{s_n n}g_{s_n n} \in VR[G_n] \mid 0 \neq r_{in} \in R,\ \sum_{i=1}^{s_n} r_{in} = 1;$$
$$1 \neq g_{in} \in G_n \text{ with all possible products of } {g_{in}}^{\varepsilon_n} \notin G^{p^{\beta_n}} \text{ or } {g_{in}}^{\varepsilon_n} \in G^{p^\alpha}$$
$$\text{for each } 1 \leq \varepsilon_n < \text{order}(g_{in}) \text{ and } 2 \leq i \leq s_n\rangle.$$

Clearly $V = \bigcup\limits_{n<\omega} (V_n GV^{p^\alpha} R[G]/GV^{p^\alpha} R[G])$ and $V_n \subseteq V_{n+1}$. Besides, we shall calculate that the nonidentity elements of $V_n GV^{p^\alpha} R[G]/GV^{p^\alpha} R[G]$ have heights $< \beta_n$ computed in V, i.e. equivalently $V_n \cap (GV^{p^{\beta_n}} R[G]) \subseteq GV^{p^\alpha} R[G]$. In fact, it is not difficult to be seen that every element in V_n can be written as gy where $g \in G$ and y is an element of $VR[G_n]$ with the properties: $y \notin V^{p^{\beta_n}} R[G]$ or $y \in V^{p^\alpha} R[G]$, and 1 is a basis member of the canonical sum of y (y can not be a generating element of V_n). So, if $y \in GV^{p^{\beta_n}} R[G]$ and hence $y \in V^{p^{\beta_n}} R[G]$, we have a contradiction. Finally in this case $y = 1$, completing the first half. In the general case for direct sums, the proof follows by making use of Proposition 13 and a standard transfinite induction. The proof is over. $\square$

We may (almost) completely solve the Generalized Direct Factor Problem (**GDFP**) for p-groups, that asks whether or not $VR[G]/G$ is totally projective, exploiting the transfinite induction and the following scheme of conclusions listed in

Algorithm. Suppose G is a σ-summable p-group and the GDFP holds for every abelian p-group A in length so that $\text{length}\, A < \text{length}\, G$. Then the GDFP holds for G and if it is fulfilled and for any such G, the GDFP is valid for all arbitrary abelian p-groups.

Proof. By what we have just argued above, $G = \bigcup\limits_{n<\omega} G_n$, $G_n \subseteq G_{n+1}$ are isotype in G and $\text{length}\, G_n < \text{length}\, G$. Therefore

$$VR[G]/G = \bigcup_{n<\omega} (VR[G_n])G/G.$$

The application of Proposition 9 (iii) means that $(VR[G_n])G/G$ are isotype in $VR[G]/G$. On the other hand $(VR[G_n])G/G \cong VR[G_n]/G_n$ are totally projective by an hypothesis. Consequently the above mentioned result of Hill [7] gives that $VR[G]/G$ must be totally projective, too. Furthermore [8] finishes our methodology after all, because if $VR[A]/A$ is totally projective for length α then this factor-group is totally projective for lengths $\alpha + n$,

$n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Thus remains the question for total projectivity of lengths not cofinal with ω, namely $VR[G]/G$ is totally projective if and only if $VR[A]/A$ is so for all abelian p-groups A in lengths not cofinal with ω. $\square$

The following is important.

Proposition 18. *Suppose G is a p-group of countable length. Then an F-isomorphism $FH \cong FG$ for any group H implies the existence of a totally projective p-group T with countable length such that $H \times T \cong G \times T$.*

Proof. Certainly H is p-torsion of the same countable length. Hence applying Theorem 17 we derive $H \times T_1 \cong G \times T_2$ for some totally projective p-groups T_1 and T_2. After selecting of a totally projective p-group T so that $T_1 \times T \cong T \cong T_2 \times T$, we finish the proof. $\square$

Remark 19. The last completely encompasses almost all contemporary results due in this aspect.

Proposition 20. *Let F be a field in* $\mathrm{char}(F) = p \neq 0$. *Then*

(i) *$VF[G]$ is a torsion-complete p-group if and only if G is a bounded p-group*;
(ii) *$VF[G]$ is a direct sum of torsion-complete p-groups if and only if G is.*

Proof. Follows complying with the main Direct Factor Theorem 17 plus the corresponding group facts appeared in [6].

In fact, since in both cases length $G \leq \omega$, exploiting Theorem 17 we detect $VF[G] \cong G \times VF[G]/G$ where the factor-group is totally projective. But $\mathrm{length}(VF[G]/G) \leq \omega$, so [6] leads us to the fact that $VF[G]/G$ would be a direct sum of cyclics. Thus the restriction on F being perfect may be ignored.

Now, we consider (i). By what we have just shown, together with [6], $VF[G]$ torsion-complete yields $VF[G]/G$ is bounded. Furthermore in view of the proof of the second half of Corollary 5, G is bounded. The converse is simple to check.

Next, we treat (ii). This that $VF[G]$ as a direct sum of torsion-complete p-groups implies the same property for G holds true via the above given decomposition Direct Factor formula along with [6]. The reverse holds valid by the same arguments. This extracts our claim. $\square$

Now, as a final, one well-known and documented classical fact due to May ([11], [12]) also will be confirmed.

Theorem (W. May, 1979–1988)**.** *Suppose G is a direct sum of countable abelian p-groups. Then $F[H] \cong F[G]$ as F-algebras for any group H yields that $H \cong G$.*

Proof. Apparently $F[G/G^{p^\alpha}] \cong F[H/H^{p^\alpha}]$ for each $\alpha \leq \Omega$. Therefore we need only apply back-and-forth our second criterion for total projectivity and Theorem 14 to end the proof. □

Concluding discussion

Here is a major problem which immediately arises. In fact, suppose H is an isotype p-subgroup of G. If H is σ-summable is then $V(R[G]; H)/H$ totally projective? If the above holds, then H will be a direct factor of $V(R[G]; H)$ with a totally projective complement. Special consequences of some interest are these at $H = G$ or $H = G_p$.

Moreover, whether or not $F[G]$ determines G when tG is a direct sum of σ-summable abelian p-groups.

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References

[1] D. O. Cutler, *On the structure of primary abelian groups of countable Ulm type*, Trans. Amer. Math. Soc. **152**(1970), 503–518.
[2] P. Danchev, *Units in abelian group rings of prime characteristic*, Compt. Rend. Acad. Bulg. Sci. (8) **48**(1995), 5–8.
[3] P. Danchev, *Topologically pure and basis subgroups in commutative group rings*, Compt. Rend. Acad. Bulg. Sci. (9-10) **48**(1995), 7–10.
[4] P. Danchev, *Commutative group algebras of σ-summable abelian groups*, Proc. Amer. Math. Soc. (9) **125**(1997), 2559–2564.
[5] P. Danchev, *Commutative group algebras of summable abelian p-groups*, Commun. in Algebra (to appear).
[6] L. Fuchs, *Infinite abelian groups **I** and **II***, Mir, Moscow, 1974 and 1977.
[7] P. Hill, *Criteria for total projectivity*, Can. J. Math. (4) **33**(1981), 817–825.
[8] P. Hill, *A note on σ-summable groups*, Proc. Amer. Math. Soc. (11) **126**(1998), 3133–3135.
[9] G. Karpilovsky, *Unit groups of group rings*, North-Holland, Amsterdam, 1989.
[10] W. May, *Commutative group algebras*, Trans. Amer. Math. Soc. **136**(1969), 139–149.
[11] W. May, *Modular group algebras of totally projective p-primary groups*, Proc. Amer. Math. Soc. (1) **76**(1979), 31–34.
[12] W. May, *Modular group algebras of simply presented abelian groups*, Proc. Amer. Math. Soc. (2) **104**(1988), 403–409.
[13] W. May, *The direct factor problem for modular abelian group algebras*, Contemp. Math. **93**(1989), 303–308.
[14] C. Megibben, *The generalized Kulikov criterion*, Can. J. Math. **21**(1969), 1192–1205.

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