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CERTAIN METRICS ON R^4_+ (III)

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1. PRELIMINARIES

We studied the following metrics on $R^4_+ = R^3 \times R_+$.

$$(1.1)\qquad ds^2 = \frac{1}{x_4x_4}\Big\{\sum_{b,c=1}^{3}\Big(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\Big)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dz_4\Big\}$$

and

$$(1.2)\quad ds^2 = \frac{1}{x_4x_4}\Big\{\sum_{b,c=1}^{3}\Big(\frac{8}{(x_3+3r)^2}(r^2\delta_{bc} - x_bx_c) + \frac{x_bx_c}{r^2(1+ar^2)}\Big)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\},$$

where $r^2 = \sum_{b=1}^{3} x_bx_b$, $a =$ constant, in [3], [4] and [5].

We proved that any geodesic of (1.1) is a plane curve in R^3 but thoses of (1.2) are not so in general. The curvature tensor ${R_j}^i{}_{hk}$ of both metrics $ds^2 = \sum_{i,j=1}^{4} g_{ij}dx_idx_j$ satisfies the equalities:

$$(1.3)\qquad {R_j}^i{}_{hk} = \delta^i_h g_{jk} - \delta^i_k g_{jh}, \quad i,j,h,k = 1,2,3,4.$$

They are derived as special ones from the metric in R^4_+:

$$ds^2 = \frac{1}{u_4u_4}\sum_{i,j=1}^{4} F_{ij}du_idu_j, \quad F_{ij} = F_{ji},$$

where $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, $u_4 = x_4$ and

$$x_1 = r\sin\theta\cos\phi,\ x_2 = r\sin\theta\sin\phi,\ x_3 = r\cos\theta,$$

and (r,θ,ϕ) are the polar coordinates of $R^3, g_{ij} = F_{ij}/u_4u_4$ satisfies the Einstein condition

$$R_{ij} = \frac{R}{4}g_{ij}, \quad R_{ij} = \sum_{k=1}^{4} {R_i}^k{}_{kj}, \quad R = \sum_{i,j=1}^{4} g^{ij}R_{ij},$$

$(g^{ij}) = (g_{ij})^{-1}$, and

$$F_{ij} = F_{ij}(u_1,u_2) \text{ except for } F_{44} = F_{44}(u_1,u_2,u_4)$$

and

$$F_{12} = F_{\alpha\lambda} = 0 \quad (\alpha = 1,2;\ \lambda = 3,4).$$

We proved in Proposition 1 in [4] that: Putting the restriction

$$F_{33}/F_{22} = \sin^2 u_2 \times \text{constant} \tag{1.4}$$

to obtain the above mentioned metric $g_{ij} = F_{ij}/u_4u_4$, it is necessary and sufficient to solve the system of the differential equations on $y = 1/F_{22}$ as

$$\frac{\partial^2 y}{\partial u_2 \partial u_1} - \frac{1}{y}\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_1} = -\sigma_2 \sin u_2, \tag{1.5}$$

$$\frac{\partial^2 y}{\partial u_2 \partial u_2} - \frac{1}{y}\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_2} + \frac{\cos u_2}{\sin u_2}\frac{\partial y}{\partial u_2} + 2y - \frac{1}{2}\sigma - 2c_1 = 0 \tag{1.6}$$

and

$$(\sigma + 4c_1)\frac{\partial y}{\partial u_1} + 2\sigma_2 \sin u_2 \frac{\partial y}{\partial u_2} - (\sigma' + 4\sigma_2 \cos u_2)y = 0, \tag{1.7}$$

where c_1 = integral constant and $\sigma = \sigma(u_1)$, $\sigma_2 = \sigma_2(u_1)$ are auxiliary functions of u_1 only.

And we obtained the main theorems, Theorem 3 and Theorem 4 in [4]: The solutions of the system of equations (1.5), (1.6) and (1.7) on y with $y(0,0) \neq 0$ and $y(0,0) = 0$ are given by

$$y = \frac{\sigma + 4c_1}{2}\Big\{\Big(b_0\rho^2 + \frac{1}{16b_0\rho^2}\Big)\Big(1 - \frac{1}{2}\sin^2 u_2\Big) + \frac{1}{4}\sin^2 u_2 + \Big(b_0\rho^2 - \frac{1}{16b_0\rho^2}\Big)\cos u_2\Big\} \tag{1.8}$$

and

$$y = (\sigma + 4c_1)\sin u_2 \Big\{\frac{1}{2}\sin u_2 + \frac{b_1\rho}{2}(1 + \cos u_2) + \frac{1}{8b_1\rho}(1 - \cos u_2)\Big\} \tag{1.9}$$

respectively, where $b_0 \neq 0$, $b_1 \neq 0$ and c_1 are constants and σ and ρ are auxiliary functions depending on u_1 only. For the above metrics, we see that

$$F_{11} = \frac{1}{\sigma}\Big(\frac{\partial \log F_{22}}{\partial u_1}\Big)^2$$

by (5.10) in [4] and for Case I:

$$F_{34} = bF_{33}, \quad b = \text{constant},$$

we have

$$F_{44} = b^2 F_{33} + \frac{1}{c_0 + c_1 u_4 u_4}, \quad c_0, c_1 = \text{constants}$$

by (5.2) in [4], and

$$\rho = \exp\Big(\int \frac{2\sigma_2}{\sigma + 4c_1} du_1\Big), \quad \text{with } \rho(0) = 1$$

by (6.4) in [4].

Now, we put

$$\sigma + 4c_1 = \frac{4}{u_1u_1} \quad \text{and} \quad \rho = 1 - mu_1,$$

$m =$ constant, then we obtain

$$\sigma_2 = \frac{\sigma - 2m}{u_1u_1(1 - mu_1)}. \tag{1.11}$$

Regarding (1.8) we have

$$\begin{aligned}
&\Big(b_0\rho^2 + \frac{1}{16b_0\rho^2}\Big)\Big(1 - \frac{1}{2}\sin^2 u_2\Big) + \frac{1}{4}\sin^2 u_2 + \Big(b_0\rho^2 - \frac{1}{16b_0\rho^2}\Big)\cos u_2 \\
&\quad = \Big(b_0\rho^2 + \frac{1}{16b_0\rho^2}\Big)\Big(\frac{1}{2} + \frac{1}{2}\cos^2 u_2\Big) + \frac{1}{4} - \frac{1}{4}\cos^2 u_2 \\
&\qquad\qquad + \Big(b_0\rho^2 - \frac{1}{16b_0\rho^2}\Big)\cos u_2 \\
&\quad = \Big(\frac{1}{2}\Big(b_0\rho^2 + \frac{1}{16b_0\rho^2}\Big) - \frac{1}{4}\Big)\cos^2 u_2 + \Big(b_0\rho^2 - \frac{1}{16b_0\rho^2}\Big)\cos u_2 \\
&\qquad\qquad + \frac{1}{2}\Big(b_0\rho^2 + \frac{1}{16b_0\rho^2}\Big) + \frac{1}{4} \\
&\quad = \frac{1}{2}b_0\Big(\rho - \frac{1}{4b_0\rho}\Big)^2\cos^2 u_2 + b_0\Big(\rho - \frac{1}{4b_0\rho}\Big)\Big(\rho + \frac{1}{4b_0\rho}\Big)\cos u_2 \\
&\qquad\qquad + \frac{1}{2}b_0\Big(\rho + \frac{1}{4b_0\rho}\Big)^2 \\
&\quad = \frac{1}{2}b_0\Big\{\Big(\rho - \frac{1}{4b_0\rho}\Big)\cos u_2 + \Big(\rho + \frac{1}{4b_0\rho}\Big)\Big\}^2
\end{aligned}$$

and so we obtain

$$\begin{aligned}
F_{22} = \frac{1}{y} &= \frac{2}{\sigma + 4c_1} \times \frac{2}{b_0\{(\rho - \frac{1}{4b_0\rho})\cos u_2 + (\rho + \frac{1}{4b_0\rho})\}^2} \\
&= \frac{4b_0\rho^2}{(\sigma + 4c_1)\{(b_0\rho^2 - \frac{1}{4})\cos u_2 + (b_0\rho^2 + \frac{1}{4})\}^2}
\end{aligned}$$

and substituting (1.10) into which we have

$$F_{22} = \frac{b_0u_1u_1(1 - mu_1)^2}{\{b_0(1 - mu_1)^2 - \frac{1}{4})\cos u_2 + b_0(1 - mu_1)^2 + \frac{1}{4}\}^2} \tag{1.12}$$

and we obtain

$$\begin{aligned}
F_{11} = \frac{1}{\sigma}\Big(\frac{\partial \log F_{22}}{\partial u_1}\Big)^2 &= \frac{1}{(1 - c_1u_1u_1)(1 - mu_1)^2} \\
&\times \Big\{1 - \frac{mu_1(1 - \cos u_2)}{2(b_0(1 - mu_1)^2 - \frac{1}{4})\cos u_2 + b_0(1 - mu_1)^2 + \frac{1}{4}}\Big\}^2.
\end{aligned} \tag{1.13}$$

And by Theorem 2 and Case I in [4] and setting $b = 0$, we obtain

$$F_{34} = 0, \tag{1.14}$$

$$F_{33} = c \sin^2 u_2 F_{22} \quad \text{and} \quad F_{44} = \frac{1}{c_0 + c_1 u_4 u_4} \tag{1.15}$$

where c, c_0, c_1 = constants.

In this work, we study the family of metrics given by (1.12) $\sim$ (1.15), especially with $m = 0$, and the metics (1.1) and (1.2) which belong to this family and show that the equalities (1.3) for the curvatures hold for any one of this family.

2. CURVATURES OF THE METRICS (1.12) $\sim$ (1,15)

Let us use an auxilliary function of u_2 for simplicity as

$$\begin{aligned} H &= (b_0 - \frac{1}{4}) \cos u_2 + b_0 + \frac{1}{4} \\ &= 2b_0 \cos^2 \frac{u_2}{2} + \frac{1}{2} \sin^2 \frac{u_2}{2}, \end{aligned} \tag{2.1}$$

and consider a metric:

$$ds^2 = \sum_{i,j} g_{ij} du_i du_j = \frac{1}{u_4 u_4} \sum_{i,j} F_{ij} du_i du_j,$$

$$g_{ij} = \frac{1}{u_4 u_4} F_{ij}, \quad F_{ij} = F_{ji}$$

given by (1.12) $\sim$ (1.15) as

$$\begin{aligned} &F_{11} = \frac{1}{1 + a u_1 u_1}, \quad F_{22} = \frac{b_0 u_1 u_1}{H^2}, \quad F_{33} = \frac{b_0 u_1 u_1 \sin^2 u_2}{H^2}, \\ &F_{34} = 0, \quad F_{44} = -\frac{1}{1 + a u_4 u_4}, \quad F_{12} = F_{\alpha\lambda} = 0, \end{aligned} \tag{2.2}$$

$\alpha = 1, 2$ and $\lambda = 3, 4$, where we set the constants as

$$m = 0, \; b = 0, \; c_0 = -1, \; c_1 = -a, \; c = 1.$$

In the following, we set as $\alpha, \beta, \gamma, \ldots = 1, 2$ and $\lambda, \mu, \nu, \ldots = 3, 4$

Proposition 1. *When $b_0 = \frac{1}{4}$, the metric (2.2) becomes the metric (1.1).*

Proof. The metric (2.2) with $b_0 = \frac{1}{4}$ is written as

$$ds^2 = \frac{1}{u_4 u_4} \Big\{ \frac{1}{1 + a u_1 u_1} du_1 du_1 + u_1 u_1 (du_2 du_2 + \sin^2 u_2 du_3 du_3) \\ - \frac{1}{1 + a u_4 u_4} du_4 du_4 \Big\}$$

$$
\begin{aligned}
&= \frac{1}{x_4x_4}\Big\{\frac{1}{1+ar^2}drdr + r^2(d\theta d\theta + \sin^2\theta d\phi d\phi) - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\} \\
&= \frac{1}{x_4x_4}\Big\{\frac{1}{1+ar^2}drdr + \sum_b dx_bdx_b - (\sum_b \frac{x_b}{r}dx_b)^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\} \\
&= \frac{1}{x_4x_4}\Big\{\sum_b dx_bdx_b + \frac{1}{r^2(1+ar^2)}(\sum_b x_bdx_b)^2 - \frac{1}{r^2}(\sum_b x_bdx_b)^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\} \\
&= \frac{1}{x_4x_4}\Big\{\sum_b dx_bdx_b - \frac{a}{1+ar^2}(\sum_b x_bdx_b)^2 - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\} \\
&= \frac{1}{x_4x_4}\Big\{\sum_{b,c=1}^{3}\Big(\delta_{bc} - \frac{a}{1+ar^2}x_bx_c\Big)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\},
\end{aligned}
$$

since $H = \frac{1}{2}$, which is the metric (1.1). □

Proposition 2. *When $b_0 = \frac{1}{2}$, the metric (2.2) becomes the metric (1.2).*

Proof. Since we have

$$
H = \frac{1}{4}\cos u_2 + \frac{3}{4} = \frac{1}{4}(\cos u_2 + 3) = \frac{1}{4}(\frac{x_3}{r} + 3),
$$

the metric (2.2) is written as

$$
\begin{aligned}
ds^2 &= \frac{1}{u_4u_4}\Big\{\frac{1}{1+au_1u_1}du_1du_1 + \frac{b_0u_1u_1}{H^2}(du_2du_2 + \sin^2 u_2du_3du_3) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{1+au_4u_4}du_4du_4\Big\} \\
&= \frac{1}{x_4x_4}\Big\{\frac{1}{1+ar^2}drdr + \frac{8r^4}{(x_3+3r)^2}((d\theta d\theta + \sin^2 u_2d\phi d\phi) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\} \\
&= \frac{1}{x_4x_4}\Big\{\frac{(rdr)^2}{r^2(1+ar^2)} + \frac{8r^2}{(x_3+3r)^2}(\sum_b dx_bdx_b - (\sum_b \frac{x_b}{r}dx_b)^2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\} \\
&= \frac{1}{x_4x_4}\Big\{\frac{8}{(x_3+3r)^2}(r^2\sum_b dx_bdx_b - (\sum_b x_bdx_b)^2)
\end{aligned}
$$

$$+\frac{1}{r^2(1+ar^2)}(\sum_b x_b dx_b)^2 - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\}$$

$$=\frac{1}{x_4x_4}\Big\{\sum_{b,c=1}^{3}\Big(\frac{8}{(x_3+3r)^2}(r^2\delta_{bc}-x_bx_c) + \frac{x_bx_c}{r^2(1+ar^2)}\Big)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\},$$

which is the metric (1.2). □

In the following we shall compute the curvature tensor of the metric (2.2):

$$R_j{}^i{}_{hk} = \frac{\partial}{\partial u_h}\{{}_j{}^i{}_k\} - \frac{\partial}{\partial u_k}\{{}_j{}^i{}_h\} + \sum_\ell \{{}_\ell{}^i{}_h\}\}\{{}_j{}^\ell{}_k\} - \sum_\ell \{{}_\ell{}^i{}_k\}\{{}_j{}^\ell{}_h\}, \tag{2.3}$$

where $\{{}_j{}^i{}_h\}$ are the Christoffel symbols made by g_{ij}:

$$\{{}_j{}^i{}_h\} = \frac{1}{2}\sum_k g^{ik}\Big(\frac{\partial g_{kh}}{\partial u_j} + \frac{\partial g_{jk}}{\partial u_h} - \frac{\partial g_{jh}}{\partial u_k}\Big), \tag{2.4}$$

and $\Big(g^{ij}\Big) = \Big(g_{ij}\Big)^{-1}$, $i, j, h, k = 1, 2, 3, 4$.

Exactly, we obtain from (2.2) or by (1.4) in [4]

$$\{{}_\beta{}^\alpha{}_\gamma\} = \frac{1}{2}F^{\alpha\alpha}\Big(\delta^\alpha_\beta\frac{\partial F_{\alpha\alpha}}{\partial u_\gamma} + \delta^\alpha_\gamma\frac{\partial F_{\alpha\alpha}}{\partial u_\beta} - \delta_{\beta\gamma}\frac{\partial F_{\beta\beta}}{\partial u_\alpha}\Big),$$

that is

$$\{{}_1{}^1{}_1\} = \frac{1}{2}F^{11}\frac{\partial F_{11}}{\partial u_1} = -\frac{au_1}{1+au_1u_1}, \quad \{{}_1{}^2{}_1\} = -\frac{1}{2}F^{22}\frac{\partial F_{11}}{\partial u_2} = 0, \tag{2.5}$$

$$\{{}_1{}^1{}_2\} = \frac{1}{2}F^{11}\frac{\partial F_{11}}{\partial u_2} = 0, \quad \{{}_1{}^2{}_2\} - \frac{1}{2}F^{22}\frac{\partial F_{22}}{\partial u_1} = \frac{1}{u_1},$$

$$\{{}_2{}^1{}_2\} = -\frac{1}{2}F^{11}\frac{\partial F_{22}}{\partial u_2} = -\frac{1}{H^2}b_0u_1(1+au_1u_1),$$

$$\{{}_2{}^2{}_2\} = \frac{1}{2}F^{22}\frac{\partial F_{22}}{\partial u_2} = \frac{1}{H}(b_0 - \frac{1}{4})\sin u_2$$

and

$$\{{}_\beta{}^\lambda{}_\gamma\} = \frac{1}{u_4}F^{4\lambda}F_{\beta\gamma}, \quad \{{}_\beta{}^\alpha{}_\mu\} = -\frac{1}{u_4}\delta^\alpha_\beta\delta_{4\mu}, \quad \{{}_\beta{}^\lambda{}_\mu\} = \frac{1}{2}F^{\lambda\lambda}\frac{\partial F_{\lambda\mu}}{\partial u_\beta}, \tag{2.6}$$

$$\{{}_\mu{}^\alpha{}_\nu\} = -\frac{1}{2}F^{\alpha\alpha}\frac{\partial F_{\mu\nu}}{\partial u_\alpha}, \quad \{{}_\mu{}^\lambda{}_\nu\} = \{{}_\mu{}^\lambda{}_\nu\}_\Lambda - \frac{1}{u_4}(\delta^\lambda_\mu\delta^4_\nu + \delta^\lambda_\nu\delta^4_\mu - F^{4\lambda}F_{\mu\nu}),$$

where

$$\{{}_\mu{}^\lambda{}_\nu\}_\Lambda = \frac{1}{2}\sum_{\rho=3}^{4} F^{\lambda\rho}\Big(\frac{\partial F_{\rho\nu}}{\partial u_\mu} + \frac{\partial F_{\mu\rho}}{\partial u_\nu} - \frac{\partial F_{\mu\nu}}{\partial u_\rho}\Big), \quad \Big(F^{ij}\Big) = \Big(F_{ij}\Big)^{-1}$$

and

$$\{{}_3{}^\lambda{}_3\}_\Lambda = -\frac{1}{2}F^{\lambda 4}\frac{\partial F_{33}}{\partial u_4} = 0, \quad \{{}_3{}^\lambda{}_4\}_\Lambda = \frac{1}{2}F^{\lambda 3}\frac{\partial F_{33}}{\partial u_4} = 0,$$
$$\{{}_4{}^\lambda{}_4\}_\Lambda = \frac{1}{2}F^{\lambda 4}\frac{\partial F_{44}}{\partial u_4} = -\delta^\lambda_4\frac{a u_4}{1 + a u_4 u_4},$$

therefore we have

(2.7)
$$\{{}_3{}^\lambda{}_3\} = \frac{1}{u_4}F^{\lambda 4}F_{33}, \quad \{{}_3{}^\lambda{}_4\} = -\frac{1}{u_4}\delta^\lambda_3,$$
$$\{{}_4{}^\lambda{}_4\} = -\delta^\lambda_4\Big(\frac{1}{u_4} + \frac{a u_4}{1 + a u_4 u_4}\Big).$$

Now, using (2,5) $\sim$ (2,7), we compute the components $R_j{}^i{}_{hk}$. First, we have

$$R_\beta{}^\alpha{}_{12} = \frac{\partial}{\partial u_1}\{{}_\beta{}^\alpha{}_2\} - \frac{\partial}{\partial u_2}\{{}_\beta{}^\alpha{}_1\} + \sum_\ell \{{}_\ell{}^\alpha{}_1\}\{{}_\beta{}^\ell{}_2\} - \sum_\ell \{{}_\ell{}^\alpha{}_2\}\{{}_\beta{}^\ell{}_1\}$$
$$= \frac{1}{2}\frac{\partial}{\partial u_1}\Big(F^{\alpha\alpha}\Big(\frac{\partial F_{\beta\alpha}}{\partial u_2} + \frac{\partial F_{\alpha 2}}{\partial u_\beta} - \frac{\partial F_{\beta 2}}{\partial u_\alpha}\Big)\Big) - \frac{1}{2}\frac{\partial}{\partial u_2}\Big(F^{\alpha\alpha}\Big(\frac{\partial F_{\beta\alpha}}{\partial u_1}$$
$$+ \frac{\partial F_{\alpha 1}}{\partial u_\beta} - \frac{\partial F_{\beta 1}}{\partial u_\alpha}\Big)\Big) + \frac{1}{4}\sum_\gamma F^{\alpha\alpha}\Big(\frac{\partial F_{\gamma\alpha}}{\partial u_1} + \frac{\partial F_{\alpha 1}}{\partial u_\gamma} - \frac{\partial F_{\gamma 1}}{\partial u_\alpha}\Big)$$
$$\times F^{\gamma\gamma}\Big(\frac{\partial F_{\beta\gamma}}{\partial u_2} + \frac{\partial F_{\gamma 2}}{\partial u_\beta} - \frac{\partial F_{\beta 2}}{\partial u_\gamma}\Big) + \sum_\lambda\Big(-\frac{1}{u_4}\delta^\alpha_1\delta_{4\lambda}\Big)\Big(\frac{1}{u_4}F^{4\lambda}F_{\beta 2}\Big)$$
$$- \frac{1}{4}\sum_\gamma F^{\alpha\alpha}\Big(\frac{\partial F_{\gamma\alpha}}{\partial u_2} + \frac{\partial F_{\alpha 2}}{\partial u_\gamma} - \frac{\partial F_{\gamma 2}}{\partial u_\alpha}\Big)F^{\gamma\gamma}\Big(\frac{\partial F_{\beta\gamma}}{\partial u_1} + \frac{\partial F_{\gamma 1}}{\partial u_\beta} - \frac{\partial F_{\beta 1}}{\partial u_\gamma}\Big)$$
$$- \sum_\lambda\Big(-\frac{1}{u_4}\delta^\alpha_2\delta_{4\lambda}\Big)\Big(\frac{1}{u_4}F^{4\lambda}F_{\beta 1}\Big)$$
$$= \frac{1}{2}F^{\alpha\alpha}\Big(\frac{\partial^2 F_{\beta\alpha}}{\partial u_1\partial u_2} + \frac{\partial^2 F_{\alpha 2}}{\partial u_1\partial u_\beta} - \frac{\partial^2 F_{\beta 2}}{\partial u_1\partial u_\alpha} - \frac{\partial^2 F_{\beta\alpha}}{\partial u_2\partial u_1} - \frac{\partial^2 F_{\alpha 1}}{\partial u_2\partial u_\beta}$$
$$+ \frac{\partial^2 F_{\beta 1}}{\partial u_2\partial u_1}\Big) + \frac{1}{2}\frac{\partial F^{\alpha\alpha}}{\partial u_1}\Big(\frac{\partial F_{\beta\alpha}}{\partial u_2} + \frac{\partial F_{\alpha 2}}{\partial u_\beta} - \frac{\partial F_{\beta 2}}{\partial u_\alpha}\Big)$$
$$- \frac{1}{2}\frac{\partial F^{\alpha\alpha}}{\partial u_2}\Big(\frac{\partial F_{\beta\alpha}}{\partial u_1} + \frac{\partial F_{\alpha 1}}{\partial u_\beta} - \frac{\partial F_{\beta 1}}{\partial u_\alpha}\Big) + \frac{1}{4}\sum_\gamma F^{\alpha\alpha}F^{\gamma\gamma}\Big(\frac{\partial F_{\gamma\alpha}}{\partial u_1} + \frac{\partial F_{\alpha 1}}{\partial u_\gamma}$$

$$
\begin{aligned}
&-\frac{\partial F_{\gamma 1}}{\partial u_\alpha}\Big)\Big(\frac{\partial F_{\beta\gamma}}{\partial u_2}+\frac{\partial F_{\gamma 2}}{\partial u_\beta}-\frac{\partial F_{\beta 2}}{\partial u_\gamma}\Big)-\frac{1}{4}\sum_\gamma F^{\alpha\alpha}F^{\gamma\gamma}\Big(\frac{\partial F_{\gamma\alpha}}{\partial u_2}+\frac{\partial F_{\alpha 2}}{\partial u_\gamma}\\
&-\frac{\partial F_{\gamma 2}}{\partial u_\alpha}\Big)\Big(\frac{\partial F_{\beta\gamma}}{\partial u_1}+\frac{\partial F_{\gamma 1}}{\partial u_\beta}-\frac{\partial F_{\beta 1}}{\partial u_\gamma}\Big)-\frac{1}{u_4u_4}\delta^\alpha_1 F^{44}F_{\beta 2}+\frac{1}{u_4u_4}\delta^\alpha_3 F^{44}F_{\beta 1}\\
=&\ \frac{1}{2}F^{\alpha\alpha}\Big(\delta^\alpha_2\frac{\partial^2 F_{22}}{\partial u_1\partial u_\beta}-\delta_{\beta 2}\frac{\partial^2 F_{22}}{\partial u_1\partial u_\alpha}\Big)+\frac{1}{2}\frac{\partial F^{\alpha\alpha}}{\partial u_1}\Big(\delta^\alpha_\beta\frac{\partial F_{\beta\beta}}{\partial u_2}+\delta^\alpha_2\frac{\partial F_{22}}{\partial u_\beta}\\
&-\delta_{\beta 2}\frac{\partial F_{22}}{\partial u_\alpha}\Big)-\frac{1}{2}\frac{\partial F^{\alpha\alpha}}{\partial u_2}\Big(\delta^\alpha_\beta\frac{\partial F_{\beta\beta}}{\partial u_1}+\delta^\alpha_1\frac{\partial F_{11}}{\partial u_\beta}-\delta_{\beta 1}\frac{\partial F_{11}}{\partial u_\alpha}\Big)\\
&+\frac{1}{4}\sum_\gamma F^{\alpha\alpha}F^{\gamma\gamma}\Big(\delta^\alpha_\gamma\frac{\partial F_{\alpha\alpha}}{\partial u_1}+\delta^\alpha_1\frac{\partial F_{11}}{\partial u_\gamma}-\delta_{\gamma 1}\frac{\partial F_{11}}{\partial u_\alpha}\Big)\Big(\delta_{\beta\gamma}\frac{\partial F_{\beta\beta}}{\partial u_2}\\
&+\delta_{\gamma 2}\frac{\partial F_{22}}{\partial u_\beta}-\delta_{\beta 2}\frac{\partial F_{22}}{\partial u_\gamma}\Big)-\frac{1}{4}\sum_\gamma F^{\alpha\alpha}F^{\gamma\gamma}\Big(\delta^\alpha_\gamma\frac{\partial F_{\alpha\alpha}}{\partial u_2}+\delta^\alpha_2\frac{\partial F_{22}}{\partial u_\gamma}\\
&-\delta_{\gamma 2}\frac{\partial F_{22}}{\partial u_\alpha}\Big)\Big(\delta_{\beta\gamma}\frac{\partial F_{\beta\beta}}{\partial u_1}+\delta_{\gamma 1}\frac{\partial F_{11}}{\partial u_\beta}-\delta_{\beta 1}\frac{\partial F_{11}}{\partial u_\gamma}\Big)\\
&-\frac{1}{u_4u_4}F^{44}(\delta^\alpha_1 F_{\beta 2}-\delta^\alpha_2 F_{\beta 1})\\
=&\ \frac{1}{2}\Big(\delta^\alpha_2 F^{22}\frac{\partial^2 F_{22}}{\partial u_\beta\partial u_1}-\delta_{\beta 2}F^{\alpha\alpha}\frac{\partial^2 F_{22}}{\partial u_\alpha\partial u_1}\Big)+\frac{1}{2}\Big(\delta^\alpha_\beta\frac{\partial F^{\beta\beta}}{\partial u_1}\frac{\partial F_{\beta\beta}}{\partial u_2}\\
&+\delta^\alpha_2\frac{\partial F^{22}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\beta}-\delta_{\beta 2}\frac{\partial F^{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\alpha}\Big)-\frac{1}{2}\Big(\delta^\alpha_\beta\frac{\partial F^{\beta\beta}}{\partial u_2}\frac{\partial F_{\beta\beta}}{\partial u_1}\\
&+\delta^\alpha_1\frac{\partial F^{11}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\beta}-\delta_{\beta 1}\frac{\partial F^{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\alpha}\Big)+\frac{1}{4}\Big\{F^{\alpha\alpha}F^{\alpha\alpha}\frac{\partial F_{\alpha\alpha}}{\partial u_1}\\
&\Big(\delta^\alpha_\beta\frac{\partial F_{\beta\beta}}{\partial u_2}+\delta^\alpha_2\frac{\partial F_{22}}{\partial u_\beta}-\delta_{\beta 2}\frac{\partial F_{22}}{\partial u_\alpha}\Big)+F^{\alpha\alpha}\delta^\alpha_1\Big(F^{\beta\beta}\frac{\partial F_{11}}{\partial u_\beta}\frac{\partial F_{\beta\beta}}{\partial u_2}\\
&+F^{22}\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{22}}{\partial u_\beta}-\delta_{\beta 2}F^{11}\frac{\partial F_{11}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big)\\
&-F^{\alpha\alpha}F^{11}\frac{\partial F_{11}}{\partial u_\alpha}\Big(\delta_{\beta 1}\frac{\partial F_{\beta\beta}}{\partial u_2}-\delta_{\beta 2}\frac{\partial F_{22}}{\partial u_1}\Big)\Big\}-\frac{1}{4}\Big\{F^{\alpha\alpha}F^{\alpha\alpha}\frac{\partial F_{\alpha\alpha}}{\partial u_2}\\
&\Big(\delta^\alpha_\beta\frac{\partial F_{\beta\beta}}{\partial u_1}+\delta^\alpha_1\frac{\partial F_{11}}{\partial u_\beta}-\delta_{\beta 1}\frac{\partial F_{11}}{\partial u_\alpha}\Big)+F^{\alpha\alpha}\delta^\alpha_2\Big(F^{\beta\beta}\frac{\partial F_{22}}{\partial u_\beta}\frac{\partial F_{\beta\beta}}{\partial u_1}\\
&+F^{11}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{11}}{\partial u_\beta}-\delta_{\beta 1}F^{11}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{11}}{\partial u_1}\Big)-F^{\alpha\alpha}F^{22}\Big(\frac{\partial F_{22}}{\partial u_\alpha}\delta_{\beta 2}\frac{\partial F_{\beta\beta}}{\partial u_1}\\
&-\frac{\partial F_{22}}{\partial u_\alpha}\delta_{\beta 1}\frac{\partial F_{11}}{\partial u_2}\Big)\Big\}+\frac{1+au_4u_4}{u_4u_4}(\delta^\alpha_1 F_{\beta 2}-\delta^\alpha_2 F_{\beta 1}),
\end{aligned}
$$

which is arranged as

$$
\begin{aligned}
&= \frac{1}{2}\delta_2^\alpha F^{22}\Big(-\frac{2}{u_1u_1}\delta_{1\beta}F_{22} + \frac{2}{u_1}\frac{\partial F_{22}}{\partial u_\beta}\Big) - \frac{1}{2}\delta_{\beta 2}F^{\alpha\alpha}\Big(-\frac{2}{u_1u_1}\delta_1^\alpha F_{22} \\
&\quad + \frac{2}{u_1}\frac{\partial F_{22}}{\partial u_\alpha}\Big) + \frac{1}{2}\Big(-\delta_2^\alpha\frac{1}{F_{22}F_{22}}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\beta} + \delta_{\beta 2}\frac{1}{F_{\alpha\alpha}F_{\alpha\alpha}}\frac{\partial F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\alpha} \\
&\quad - \delta_{\beta 1}\frac{1}{F_{\alpha\alpha}F_{\alpha\alpha}}\frac{\partial F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\alpha}\Big) + \frac{1}{4}\Big\{F^{\alpha\alpha}F^{\alpha\alpha}\Big(\delta_2^\alpha\frac{\partial F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\beta} \\
&\quad - \delta_1^\alpha\frac{\partial F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\beta} - \delta_{\beta 2}\frac{\partial F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\alpha} + \delta_{\beta 1}\frac{\partial F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\alpha}\Big) \\
&\quad + F^{\alpha\alpha}F^{\beta\beta}\Big(\delta_1^\alpha\frac{\partial F_{11}}{\partial u_\beta}\frac{\partial F_{\beta\beta}}{\partial u_2} - \delta_2^\alpha\frac{\partial F_{22}}{\partial u_\beta}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big) - F^{\alpha\alpha}\delta_2^\alpha F^{11}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{11}}{\partial u_\beta} \\
&\quad - F^{\alpha\alpha}F^{11}\Big(\delta_1^\alpha\delta_{\beta 2}\frac{\partial F_{11}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1} - \delta_2^\alpha\delta_{\beta 1}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{11}}{\partial u_1} - \delta_{\beta 2}\frac{\partial F_{11}}{\partial u_\alpha}\frac{\partial F_{22}}{\partial u_1}\Big) \\
&\quad + F^{\alpha\alpha}F^{22}\delta_{\beta 2}\frac{\partial F_{22}}{\partial u_\alpha}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big\} + a(\delta_1^\alpha F_{\beta 2} - \delta_2^\alpha F_{\beta 1}) + \frac{1}{u_4u_4}(\delta_1^\alpha F_{\beta 2} - \delta_2^\alpha F_{\beta 1}) \\
&= \delta_1^\alpha\Big[\frac{1}{u_1u_1}\delta_{\beta 2}F^{\alpha\alpha}F_{22} - \frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\beta} + \frac{1}{4}F^{\alpha\alpha}\frac{\partial F_{11}}{\partial u_\beta}\frac{\partial\log F_{\beta\beta}}{\partial u_2} \\
&\quad - \frac{1}{4}F^{\alpha\alpha}\delta_{\beta 2}\frac{\partial\log F_{11}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1} + aF_{\beta 2} + \frac{1}{u_4u_4}F_{\beta 2}\Big] \\
&\quad - \delta_2^\alpha\Big[\frac{1}{u_1u_1}\delta_{1\beta} - \frac{1}{u_1}\frac{\partial\log F_{22}}{\partial u_\beta} + \frac{1}{2}\frac{\partial\log F_{22}}{\partial u_1}\frac{\partial\log F_{22}}{\partial u_\beta} \\
&\quad - \frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\beta} + \frac{1}{4}F^{\alpha\alpha}\frac{\partial F_{22}}{\partial u_\beta}\frac{\partial\log F_{\beta\beta}}{\partial u_1} + \frac{1}{4}F^{\alpha\alpha}\frac{\partial F_{22}}{\partial u_1}\frac{\partial\log F_{11}}{\partial u_\beta} \\
&\quad - \frac{1}{4}\delta_{\beta 1}F^{\alpha\alpha}\frac{\partial F_{22}}{\partial u_1}\frac{\partial\log F_{11}}{\partial u_1} + aF_{\beta 1} + \frac{1}{u_4u_4}F_{\beta 1}\Big] \\
&\quad + \delta_{\beta 1}\Big[-\frac{1}{2}F^{\alpha\alpha}\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\alpha} + \frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\alpha}\Big] \\
&\quad - \delta_{\beta 2}\Big[\frac{1}{u_1}F^{\alpha\alpha}\frac{\partial F_{22}}{\partial u_\alpha} - \frac{1}{2}F^{\alpha\alpha}\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\alpha} + \frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\alpha} \\
&\qquad - \frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{11}}{\partial u_\alpha}\frac{\partial F_{22}}{\partial u_1} - \frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{22}}{\partial u_\alpha}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big] \\
&= \delta_1^\alpha\Big[F^{\alpha\alpha}\Big(\frac{1}{u_1u_1}\delta_{\beta 2}F_{22} - \frac{1}{4}\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\beta} - \frac{1}{2u_1}\delta_{\beta 2}\frac{\partial\log F_{11}}{\partial u_1}F_{22}\Big) \\
&\quad + aF_{\beta 2}\Big] - \delta_2^\alpha\Big[\frac{1}{u_1u_1}\delta_{1\beta} + F^{\alpha\alpha}\Big(-\frac{1}{4}\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\beta}
\end{aligned}
$$

$$+\frac{1}{4}\frac{\partial F_{22}}{\partial u_\beta}\frac{\partial \log F_{\beta\beta}}{\partial u_1}\Big)+aF_{\beta 1}\Big]+\frac{1}{u_4u_4}(\delta_1^\alpha F_{\beta 2}-\delta_2^\alpha F_{\beta 1})$$
$$-\frac{1}{4}\delta_{\beta 1}F^{\alpha\alpha}\frac{\partial \log F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{11}}{\partial u_\alpha}-\delta_{\beta 2}\Big[\frac{1}{u_1}F^{\alpha\alpha}\frac{\partial F_{22}}{\partial u_\alpha}-\frac{1}{4}F^{\alpha\alpha}\frac{\partial \log F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{22}}{\partial u_\alpha}$$
$$-\frac{1}{2u_1}F^{\alpha\alpha}\frac{\partial \log F_{11}}{\partial u_\alpha}F_{22}-\frac{1}{4}F^{\alpha\alpha}\frac{\partial \log F_{22}}{\partial u_\alpha}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big],$$

here we compute the above expression by dividing in the two cases $\alpha = 1$ and 2 as follows.

$$\begin{aligned}
R_\beta{}^1{}_{12} &= F^{11}\Big(\frac{1}{u_1u_1}\delta_{\beta 2}F_{22}-\frac{1}{2u_1}\delta_{\beta 2}\frac{-2au_1}{1+au_1u_1}F_{22}\Big)+aF_{\beta 2}\\
&\quad+\frac{1}{u_4u_4}F_{\beta 2}-\delta_{\beta 2}\Big[\frac{F^{11}}{u_1}\frac{2F_{22}}{u_1}-\frac{1}{4}(1+au_1u_1)\frac{-2au_1}{1+au_1u_1}\frac{2F_{22}}{u_1}\\
&\quad-\frac{1}{2u_1}(1+au_1u_1)\frac{-2au_1}{1+au_1u_1}F_{22}-\frac{1}{4}F^{11}\frac{2}{u_1}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big]\\
&= F^{11}F_{22}\delta_{\beta 2}\Big(\frac{1}{u_1u_1}+\frac{a}{1+au_1u_1}\Big)+aF_{\beta 2}+\frac{1}{u_4u_4}F_{\beta 2}\\
&\quad-\delta_{\beta 2}\Big[F^{11}\frac{2}{u_1u_1}F_{22}+aF_{22}+aF_{22}-\frac{1}{2u_1}F^{11}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big]\\
&= \delta_{\beta 2}F_{22}\Big(\frac{1+au_1u_1}{u_1u_1}+a\Big)+a\delta_{\beta 2}F_{22}+\frac{1}{u_4u_4}F_{\beta 2}\\
&\quad-\delta_{\beta 2}\Big[\frac{2(1+au_1u_1)}{u_1u_1}F_{22}+2aF_{22}-\frac{1+au_1u_1}{2u_1}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big]\\
&= \delta_{\beta 2}F_{22}\Big(-\frac{1+au_1u_1}{u_1u_1}+a\Big)-a\delta_{\beta 2}F_{22}+\delta_{\beta 2}\frac{1+au_1u_1}{2u_1}\frac{\partial F_{\beta\beta}}{\partial u_1}\\
&\quad+\frac{1}{u_4u_4}F_{\beta 2}\\
&= -\frac{1+au_1u_1}{u_1u_1}\delta_{\beta 2}F_{22}+\delta_{\beta 2}\frac{1+au_1u_1}{2u_1}\frac{2}{u_1}F_{22}+\frac{1}{u_4u_4}F_{\beta 2}\\
&= \frac{1}{u_4u_4}F_{\beta 2}.
\end{aligned}$$

And

$$\begin{aligned}
R_\beta{}^2{}_{12} &= -\Big[\frac{1}{u_1u_1}\delta_{\beta 1}+F^{22}\Big(-\frac{1}{4}\frac{2}{u_1}\frac{\partial F_{22}}{\partial u_\beta}+\frac{1}{4}\frac{\partial F_{22}}{\partial u_\beta}\frac{\partial \log F_{\beta\beta}}{\partial u_1}\Big)+aF_{\beta 1}\\
&\quad+\frac{1}{u_4u_4}F_{\beta 1}\Big]-\delta_{\beta 2}\Big[\frac{1}{u_1}\frac{\partial \log F_{22}}{\partial u_2}-\frac{1}{4}F^{22}\frac{2}{u_1}\frac{\partial F_{22}}{\partial u_2}\\
&\quad-\frac{1}{4}F^{22}\frac{\partial \log F_{22}}{\partial u_2}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big]
\end{aligned}$$

$$= -\Big[\frac{1}{u_1u_1}\delta_{\beta 1} - \frac{1}{2u_1}\frac{\partial \log F_{22}}{\partial u_\beta} + \frac{1}{4}\frac{\partial \log F_{22}}{\partial u_\beta}\frac{\partial \log F_{\beta\beta}}{\partial u_1} + aF_{\beta 1} + \frac{1}{u_4u_4}F_{\beta 1}\Big] - \delta_{\beta 2}\Big[\frac{1}{u_1}\frac{\partial \log F_{22}}{\partial u_2} - \frac{1}{2u_1}\frac{\partial \log F_{22}}{\partial u_2} - \frac{1}{4}F^{22}\frac{\partial \log F_{22}}{\partial u_2}\frac{\partial F_{22}}{\partial u_1}\Big],$$

from which we obtain

$$\begin{aligned} R_1{}^2{}_{12} &= -\Big[\frac{1}{u_1u_1} - \frac{1}{2u_1}\frac{2}{u_1} + \frac{1}{4}\frac{2}{u_1}\frac{\partial \log F_{11}}{\partial u_1} + aF_{11} + \frac{1}{u_4u_4}F_{11}\Big] \\ &= -\Big[\frac{1}{2u_1}\frac{\partial \log F_{11}}{\partial u_1} + aF_{11} + \frac{1}{u_4u_4}F_{11}\Big] \\ &= -\frac{1}{2u_1}\frac{-2au_1}{1+au_1u_1} - \frac{a}{1+au_1u_1} - \frac{1}{u_4u_4}F_{11} \\ &= -\frac{1}{u_4u_4}F_{11} \end{aligned}$$

and

$$R_2{}^2{}_{12} = -\Big[-\frac{1}{2u_1}\frac{\partial \log F_{22}}{\partial u_2} + \frac{1}{4}\frac{\partial \log F_{22}}{\partial u_2}\frac{\partial \log F_{22}}{\partial u_1}\Big] - \Big[\frac{1}{u_1}\frac{\partial \log F_{22}}{\partial u_2} - \frac{1}{2u_1}\frac{\partial \log F_{22}}{\partial u_2} - \frac{1}{2u_1}\frac{\partial \log F_{22}}{\partial u_2}\Big] = 0.$$

These results can be written as

$$R_\beta{}^\alpha{}_{12} = \frac{1}{u_4u_4}(\delta_1^\alpha F_{\beta 2} - \delta_2^\alpha F_{\beta 1}) = \delta_1^\alpha g_{\beta 2} - \delta_2^\alpha g_{\beta 1}. \tag{2.8}$$

Next, we obtain by (1.6) in [4]

$$\begin{aligned} R_\beta{}^\lambda{}_{12} = {} & -\frac{1}{u_4}\delta_{\beta 1}\Big\{\frac{\partial}{\partial u_2}(F^{\lambda 4}F_{11}) + \frac{1}{2}F_{11}\sum_{\nu,\mu}F^{\lambda\nu}F^{\mu 4}\frac{\partial F_{\nu\mu}}{\partial u_2} \\ & - \frac{1}{2}F_{11}F^{\lambda 4}F^{\beta\beta}\frac{\partial F_{\beta\beta}}{\partial u_2} - \frac{1}{2}F^{\lambda 4}\frac{\partial F_{\beta\beta}}{\partial u_2}\Big\} + \frac{1}{u_4}\delta_{\beta 2}\Big\{\frac{\partial}{\partial u_1}(F^{\lambda 4}F_{22}) \\ & + \frac{1}{2}F_{22}\sum_{\nu,\mu}F^{\lambda\nu}F^{\mu 4}\frac{\partial F_{\nu\mu}}{\partial u_1} - \frac{1}{2}F_{22}F^{\lambda 4}F^{\beta\beta}\frac{\partial F_{\beta\beta}}{\partial u_1} - \frac{1}{2}F^{\lambda 4}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big\} \\ = {} & -\frac{1}{u_4}\delta_{\beta 1}\Big\{\frac{\partial F^{\lambda 4}}{\partial u_2}F_{11} + \frac{1}{2}F_{11}F^{\lambda 4}F^{44}\frac{\partial F_{44}}{\partial u_2} - \frac{1}{2}F_{11}F^{\lambda 4}F^{11}\frac{\partial F_{11}}{\partial u_2} \\ & - \frac{1}{2}F^{\lambda 4}\frac{\partial F_{11}}{\partial u_2}\Big\} + \frac{1}{u_4}\delta_{\beta 2}\Big\{\frac{\partial F^{\lambda 4}}{\partial u_1}F_{22} + F^{\lambda 4}\frac{\partial F_{22}}{\partial u_1} \\ & + \frac{1}{2}F_{22}F^{\lambda 4}F^{44}\frac{\partial F_{44}}{\partial u_1} - \frac{1}{2}F_{22}F^{\lambda 4}F^{22}\frac{\partial F_{22}}{\partial u_1} - \frac{1}{2}F^{\lambda 4}\frac{\partial F_{22}}{\partial u_1}\Big\} = 0, \end{aligned}$$

that is

$$R_{\beta}{}^{\lambda}{}_{12} = 0. \tag{2.9}$$

Next, we have by (1.7) in [4]

$$R_{\lambda}{}^{\alpha}{}_{12} = -\frac{1}{2u_4}\Big\{F^{34}\Big(\delta_1^{\alpha}\frac{\partial F_{3\lambda}}{\partial u_2} - \delta_2^{\alpha}\frac{\partial F_{3\lambda}}{\partial u_1}\Big) + F^{44}\Big(\delta_1^{\alpha}\frac{\partial F_{4\lambda}}{\partial u_2} - \delta_2^{\alpha}\frac{\partial F_{4\lambda}}{\partial u_1}\Big)\Big\} = 0,$$

that is

$$R_{\lambda}{}^{\alpha}{}_{12} = 0. \tag{2.10}$$

Then we have by (1.8) in [4]

$$\begin{aligned} R_{\lambda}{}^{\mu}{}_{12} &= \frac{1}{4}\sum_{\nu}\Big(\frac{\partial F^{\mu\nu}}{\partial u_1}\frac{\partial F_{\nu\lambda}}{\partial u_2} - \frac{\partial F^{\mu\nu}}{\partial u_2}\frac{\partial F_{\nu\lambda}}{\partial u_1}\Big) \\ &= \frac{1}{4}\Big\{\frac{\partial F^{\mu 3}}{\partial u_1}\frac{\partial F_{3\lambda}}{\partial u_2} + \frac{\partial F^{\mu 4}}{\partial u_1}\frac{\partial F_{4\lambda}}{\partial u_2} - \frac{\partial F^{\mu 3}}{\partial u_2}\frac{\partial F_{3\lambda}}{\partial u_1} - \frac{\partial F^{\mu 4}}{\partial u_2}\frac{\partial F_{4\lambda}}{\partial u_1}\Big\} \\ &= \frac{1}{4}\Big\{\delta^{\mu 3}\frac{\partial F^{33}}{\partial u_1}\delta_{3\lambda}\frac{\partial F_{33}}{\partial u_2} + \delta^{\mu 4}\frac{\partial F^{44}}{\partial u_1}\delta_{4\lambda}\frac{\partial F_{44}}{\partial u_2} - \delta^{\mu 3}\frac{\partial F^{33}}{\partial u_2}\delta_{3\lambda}\frac{\partial F_{33}}{\partial u_1} \\ &\qquad - \delta^{\mu 4}\frac{\partial F^{44}}{\partial u_2}\delta_{4\lambda}\frac{\partial F_{44}}{\partial u_1}\Big\} \\ &= \frac{1}{4}\delta^{\mu 3}\delta_{3\lambda}\Big(\frac{\partial F^{33}}{\partial u_1}\frac{\partial F_{33}}{\partial u_2} - \frac{\partial F^{33}}{\partial u_2}\frac{\partial F_{33}}{\partial u_1}\Big) = 0, \end{aligned}$$

that is

$$R_{\lambda}{}^{\mu}{}_{12} = 0, \tag{2.11}$$

Next, we have by (1.9) in [4]

$$\begin{aligned} R_{\beta}{}^{\alpha}{}_{\nu\gamma} &= \frac{1}{2u_4}\Big(\delta_{\gamma}^{\alpha}\sum_{\sigma}F^{4\sigma}\frac{\partial F_{\sigma\nu}}{\partial u_{\beta}} - F^{\alpha\alpha}F_{\beta\gamma}\sum_{\sigma}F^{4\sigma}\frac{\partial F_{\sigma\nu}}{\partial u_{\alpha}}\Big) \\ &= \frac{1}{2u_4}\Big\{\delta_{\gamma}^{\alpha}\Big(F^{43}\frac{\partial F_{3\nu}}{\partial u_{\beta}} + F^{44}\frac{\partial F_{4\nu}}{\partial u_{\beta}}\Big) - F^{\alpha\alpha}F_{\beta\gamma}F^{44}\frac{\partial F_{4\nu}}{\partial u_{\alpha}}\Big\} = 0, \end{aligned}$$

that is

$$R_{\beta}{}^{\alpha}{}_{\nu\gamma} = 0. \tag{2.12}$$

And we have by (1.10) in [4]

$$\begin{aligned} R_{\beta}{}^{\lambda}{}_{\nu\gamma} = &\Big(\frac{1}{2u_4}\frac{\partial F^{4\lambda}}{\partial u_{\nu}} - \frac{1}{u_4 u_4}\delta_{\nu}^{\lambda}F^{44}\Big)F_{\beta\gamma} - \frac{1}{2}\sum_{\sigma}F^{\lambda\sigma}\frac{\partial^2 F_{\sigma\nu}}{\partial u_{\beta}\partial u_{\gamma}} \\ &- \frac{1}{2}\sum_{\sigma}\frac{\partial F^{\lambda\sigma}}{\partial u_{\gamma}}\frac{\partial F_{\sigma\nu}}{\partial u_{\beta}} + \frac{1}{4}\sum_{\sigma}F^{\lambda\sigma}\Big(\frac{\partial F_{\sigma\nu}}{\partial u_{\beta}}F^{\beta\beta}\frac{\partial F_{\beta\beta}}{\partial u_{\gamma}} + \frac{\partial F_{\sigma\nu}}{\partial u_{\gamma}}F^{\gamma\gamma}\frac{\partial F_{\gamma\gamma}}{\partial u_{\beta}} \end{aligned}$$

$$- \sum_{\alpha} \frac{\partial F_{\nu\sigma}}{\partial u_\alpha} F^{\alpha\alpha} \frac{\partial F_{\beta\beta}}{\partial u_\alpha} \delta_{\beta\gamma} - \sum_{\mu,\rho} F^{\mu\rho} \frac{\partial F_{\sigma\mu}}{\partial u_\gamma} \frac{\partial F_{\rho\nu}}{\partial u_\beta} \Big)$$

$$= \frac{1}{2u_4} \Big(\delta^{\lambda 4} \frac{\partial F^{44}}{\partial u_4} \delta^4_\nu + \frac{2}{u_4}(1 + a u_4 u_4)\delta^\lambda_\nu \Big) F_{\beta\gamma} - \frac{1}{2} F^{\lambda 3} \frac{\partial^2 F_{3\nu}}{\partial u_\beta \partial u_\gamma}$$

$$- \frac{1}{2} F^{\lambda 4} \frac{\partial^2 F_{4\nu}}{\partial u_\beta \partial u_\gamma} - \frac{1}{2} \frac{\partial F^{\lambda 3}}{\partial u_\gamma} \frac{\partial F_{3\nu}}{\partial u_\beta} - \frac{1}{2} \frac{\partial F^{\lambda 4}}{\partial u_\gamma} \frac{\partial F_{4\nu}}{\partial u_\beta}$$

$$+ \frac{1}{4} F^{\lambda\lambda} \Big(\frac{\partial F_{\lambda\nu}}{\partial u_\beta} F^{\beta\beta} \frac{\partial F_{\beta\beta}}{\partial u_\gamma} + \frac{\partial F_{\lambda\nu}}{\partial u_\gamma} F^{\gamma\gamma} \frac{\partial F_{\gamma\gamma}}{\partial u_\beta}$$

$$- \sum_{\alpha} \frac{\partial F_{\nu\lambda}}{\partial u_\alpha} F^{\alpha\alpha} \frac{\partial F_{\beta\beta}}{\partial u_\alpha} \delta_{\beta\gamma} - \sum_{\nu,\rho} F^{\mu\rho} \frac{\partial F_{\lambda\mu}}{\partial u_\gamma} \frac{\partial F_{\rho\nu}}{\partial u_\beta} \Big),$$

which is arranged as follows:

$$R_\beta{}^\lambda{}_{\nu\gamma} = \frac{1}{u_4 u_4} \delta^\lambda_\nu F_{\beta\gamma} + \delta^\lambda_3 \delta^3_\nu A_{\beta\gamma} + \frac{1}{4} \delta^\lambda_\nu B_\beta{}^\lambda{}_{\nu\gamma}, \tag{2.13}$$

where we set

$$A_{\beta\gamma} = a F_{\beta\gamma} - \frac{1}{2} F^{33} \frac{\partial^2 F_{33}}{\partial u_\beta \partial u_\gamma} - \frac{1}{2} \frac{\partial F^{33}}{\partial u_\gamma} \frac{\partial F_{33}}{\partial u_\beta} \tag{2.14}$$

and

$$B_\beta{}^\lambda{}_{\nu\gamma} = F^{\lambda\lambda} \frac{\partial F_{\lambda\lambda}}{\partial u_\beta} \frac{\partial \log F_{\beta\beta}}{\partial u_\gamma} + F^{\lambda\lambda} \frac{\partial F_{\lambda\lambda}}{\partial u_\gamma} \frac{\partial \log F_{\gamma\gamma}}{\partial u_\beta} \tag{2.15}$$
$$- \sum_\alpha F^{\lambda\lambda} \frac{\partial F_{\lambda\lambda}}{\partial u_\alpha} F^{\alpha\alpha} \frac{\partial F_{\beta\beta}}{\partial u_\alpha} \delta_{\beta\gamma} - F^{\lambda\lambda} F^{\lambda\lambda} \frac{\partial F_{\lambda\lambda}}{\partial u_\gamma} \frac{\partial F_{\nu\nu}}{\partial u_\beta}.$$

Regarding these auxiliary expressions $A_{\beta\gamma}$, $B_\beta{}^\lambda{}_{\nu\gamma}$ we show the following. First we set

$$H^* = b_0 - \frac{1}{4} + (b_0 + \frac{1}{4}) \cos u_2 \tag{2.16}$$

which is similar to $H = (b_0 - \frac{1}{4}) \cos u_2 + b_0 + \frac{1}{4}$. We have from (2.14)

$$A_{\beta\gamma} = a F_{\beta\gamma} - \frac{1}{2} \Big(\frac{\partial^2 \log F_{33}}{\partial u_\beta \partial u_\gamma} + \frac{\partial \log F_{33}}{\partial u_\beta} \frac{\partial \log F_{33}}{\partial u_\gamma} \Big) + \frac{1}{2} \frac{\partial \log F_{33}}{\partial u_\gamma} \frac{\partial \log F_{33}}{\partial u_\beta}$$
$$= a F_{\beta\gamma} - \frac{1}{2} \frac{\partial^2 \log F_{33}}{\partial u_\beta \partial u_\gamma}$$
$$= a F_{\beta\gamma} - \frac{\partial^2}{\partial u_\beta \partial u_\gamma} \log \Big(\frac{u_1 \sin u_2}{H} \Big)$$
$$= a F_{\beta\gamma} - \frac{\partial}{\partial u_\beta} \Big(\frac{1}{u_1} \delta^1_\gamma + \Big(\frac{\cos u_2}{\sin u_2} + \frac{1}{H}(b_0 - \frac{1}{4}) \sin u_2 \Big) \delta^2_\gamma \Big)$$

$$= aF_{\beta\gamma} + \frac{1}{u_1u_1}\delta^1_\beta\delta^1_\gamma - \frac{\partial}{\partial u_2}\Big(\frac{H^*}{H\sin u_2}\Big)\delta^2_\beta\delta^2_\gamma$$
$$= aF_{\beta\gamma} + \frac{1}{u_1u_1}\delta^1_\beta\delta^1_\gamma - \frac{H^*}{H\sin u_2}\Big(-\frac{1}{H^*}(b_0+\frac{1}{4})\sin u_2 + \frac{1}{H}(b_0-\frac{1}{4})\sin u_2 - \frac{\cos u_2}{\sin u_2}\Big)\delta^2_\beta\delta^2_\gamma,$$

that is

$$A_{\beta\gamma} = aF_{\beta\gamma} + \frac{1}{u_1u_1}\delta^1_\beta\delta^1_\gamma + \Big(\frac{b_0}{H^2} + \frac{H^*\cos u_2}{H\sin^2 u_2}\Big)\delta^2_\beta\delta^2_\gamma. \tag{2.14'}$$

Then, from (2.15) we have

$$B_\beta{}^3{}_{3\gamma} = F^{33}\frac{\partial F_{33}}{\partial u_\beta}\frac{\partial \log F_{\beta\beta}}{\partial u_\gamma} + F^{33}\frac{\partial F_{33}}{\partial u_\gamma}\frac{\partial \log F_{\gamma\gamma}}{\partial u_\beta} - \sum_\alpha F^{33}\frac{\partial F_{33}}{\partial u_\alpha}F^{\alpha\alpha}\frac{\partial F_{\beta\beta}}{\partial u_\alpha}\delta_{\beta\gamma} - F^{33}F^{33}\frac{\partial F_{33}}{\partial u_\gamma}\frac{\partial F_{33}}{\partial u_\beta}$$
$$= \frac{\partial \log F_{33}}{\partial u_\beta}\frac{\partial \log F_{\beta\beta}}{\partial u_\gamma} + \frac{\partial \log F_{33}}{\partial u_\gamma}\frac{\partial \log F_{\gamma\gamma}}{\partial u_\beta} - \Big(\frac{\partial \log F_{33}}{\partial u_1}F^{11}\frac{\partial F_{\beta\beta}}{\partial u_1} + \frac{\partial \log F_{33}}{\partial u_2}F^{22}\frac{\partial F_{\beta\beta}}{\partial u_2}\Big)\delta_{\beta\gamma} - \frac{\partial \log F_{33}}{\partial u_\gamma}\frac{\partial \log F_{33}}{\partial u_\beta},$$

using the equality

$$\frac{\partial \log F_{33}}{\partial u_\beta} = \frac{2}{u_1}\delta^1_\beta + \frac{2H^*}{H\sin u_2}\delta^2_\beta$$

the above expression becomes

$$= 2\Big(\frac{1}{u_1}\delta^1_\beta + \frac{H^*}{H\sin u_2}\delta^2_\beta\Big)\frac{\partial \log F_{\beta\beta}}{\partial u_\gamma} + 2\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\frac{\partial \log F_{\gamma\gamma}}{\partial u_\beta} - 2\Big(\frac{1}{u_1}F^{11}\frac{\partial F_{\beta\beta}}{\partial u_1} + \frac{H^*}{H\sin u_2}F^{22}\frac{\partial F_{\beta\beta}}{\partial u_2}\Big)\delta_{\beta\gamma} - 4\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\Big(\frac{1}{u_1}\delta^1_\beta + \frac{H^*}{H\sin u_2}\delta^2_\beta\Big)$$
$$= 2\Big(\frac{1}{u_1}\delta^1_\beta\frac{\partial \log F_{11}}{\partial u_\gamma} + \frac{H^*}{H\sin u_2}\delta^2_\beta\frac{\partial \log F_{22}}{\partial u_\gamma}\Big) + 2\Big(\frac{1}{u_1}\delta^1_\gamma\frac{\partial \log F_{11}}{\partial u_\beta} + \frac{H^*}{H\sin u_2}\delta^2_\gamma\frac{\partial \log F_{22}}{\partial u_\beta}\Big) - 2\Big(\frac{1}{u_1}F^{11}\frac{\partial F_{\beta\beta}}{\partial u_1} + \frac{H^*}{H\sin u_2}F^{22}\frac{\partial F_{\beta\beta}}{\partial u_2}\Big)\delta_{\beta\gamma}$$

$$- 4\Big(\frac{1}{u_1u_1}\delta^1_\beta\delta^1_\gamma + \frac{H^*}{u_1H\sin u_2}(\delta^1_\gamma\delta^2_\beta + \delta^1_\beta\delta^2_\gamma) + \frac{H^*H^*}{H^2\sin^2 u_2}\delta^2_\gamma\delta^2_\beta\Big).$$

From these equalities we obtain

$$\begin{aligned}
R_1{}^3{}_{\nu 1} &= \frac{1}{u_4u_4}\delta^3_\nu F_{11} + \delta^3_\nu A_{11} + \frac{1}{4}\delta^3_\nu B_1{}^3{}_{31} \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{11} + \delta^3_\nu\Big\{aF_{11} + \frac{1}{u_1u_1} + \frac{1}{4}\Big(\frac{4}{u_1}\frac{\partial \log F_{11}}{\partial u_1} \\
&\qquad - 2\Big(\frac{1}{u_1}F^{11}\frac{\partial F_{11}}{\partial u_1} + \frac{H^*}{H\sin u_2}F^{22}\frac{\partial F_{11}}{\partial u_2}\Big) - \frac{4}{u_1u_1}\Big)\Big\} \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{11} + \delta^3_\nu\Big\{aF_{11} + \frac{1}{2u_1}\frac{\partial \log F_{11}}{\partial u_1}\Big\} \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{11} + \delta^3_\nu\Big\{\frac{a}{1+au_1u_1} + \frac{1}{2u_1}\frac{-2au_1}{1+au_1u_1}\Big\} \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{11}, \\
R_1{}^3{}_{\nu 2} &= \frac{1}{u_4u_4}\delta^3_\nu F_{12} + \delta^3_\nu A_{12} + \frac{1}{4}\delta^3_\nu B_1{}^3{}_{32} \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{12} + \delta^3_\nu aF_{12} + \frac{1}{4}\delta^3_\nu\Big(\frac{2}{u_1}\frac{\partial \log F_{11}}{\partial u_2} + \frac{2H^*}{H\sin u_2}\frac{\partial \log F_{22}}{\partial u_1} \\
&\qquad - 4\frac{H^*}{u_1H\sin u_2}\Big) \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{12} + \delta^3_\nu\Big(aF_{12} + \frac{1}{4}\Big(\frac{2H^*}{H\sin u_2}\frac{2}{u_1} - \frac{4H^*}{u_1H\sin U_2}\Big)\Big) \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{12} = 0, \\
R_2{}^3{}_{\nu 1} &= \frac{1}{u_4u_4}\delta^3_\nu F_{21} + \delta^3_\nu A_{21} + \frac{1}{4}\delta^3_\nu B_2{}^3{}_{31} \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{21} + \delta^3_\nu aF_{21} + \frac{1}{4}\delta^3_\nu\Big(\frac{2H^*}{H\sin u_2}\frac{\partial \log F_{22}}{\partial u_1} + \frac{2}{u_1}\frac{\partial \log F_{11}}{\partial u_2} \\
&\qquad - \frac{4H^*}{u_1H\sin u_2}\Big) \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{21} + \delta^3_\nu aF_{21} = 0
\end{aligned}$$

and

$$\begin{aligned}
R_2{}^3{}_{\nu 2} &= \frac{1}{u_4u_4}\delta^3_\nu F_{22} + \delta^3_\nu A_{22} + \frac{1}{4}\delta^3_\nu B_2{}^3{}_{32} \\
&= \frac{1}{u_4u_4}\delta^3_\nu F_{22} + \delta^3_\nu\Big(aF_{22} + \frac{b_0}{H^2} + \frac{\cos u_2}{\sin^2 u_2}\frac{H^*}{H}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\delta^3_\nu\Big(\frac{4H^*}{H\sin u_2}\frac{\partial\log F_{22}}{\partial u_2}-2\Big(\frac{1}{u_1}F^{11}\frac{\partial F_{22}}{\partial u_1}\\
&\qquad+\frac{H^*}{H\sin u_2}\frac{\partial\log F_{22}}{\partial u_2}\Big)-4\Big(\frac{H^*}{H\sin u_2}\Big)^2\Big)\\
&=\frac{1}{u_4u_4}\delta^3_\nu F_{22}+\delta^3_\nu\Big(aF_{22}+\frac{b_0}{H^2}+\frac{H^*}{H}\frac{\cos u_2}{\sin^2 u_2}\\
&\qquad+\frac{H^*}{H\sin u_2}\frac{2}{H}(b_0-\frac{1}{4})\sin u_2-\frac{1}{2}\Big(\frac{1}{u_1}F^{11}\frac{2F_{22}}{u_1}\\
&\qquad+\frac{H^*}{H\sin u_2}\frac{2}{H}(b_0-\frac{1}{4})\sin u_2\Big)-\Big(\frac{H^*}{H\sin u_2}\Big)^2\Big)\\
&=\frac{1}{u_4u_4}\delta^3_\nu F_{22}+\delta^3_\nu\Big(aF_{22}+\frac{b_0}{H^2}+\frac{H^*}{H}\frac{\cos u_2}{\sin^2 u_2}+\frac{2H^*}{H^2}(b_0-\frac{1}{4})\\
&\qquad-\frac{1+au_1u_1}{u_1u_1}F_{22}-\frac{H^*}{H^2}(b_0-\frac{1}{4})-\Big(\frac{H^*}{H\sin u_2}\Big)^2\Big)\\
&=\frac{1}{u_4u_4}\delta^3_\nu F_{22}+\delta^3_\nu\Big(\frac{b_0}{H^2}+\frac{H^*}{H}\frac{\cos u_2}{\sin^2 u_2}+\frac{H^*}{H^2}(b_0-\frac{1}{4})\\
&\qquad-\frac{1}{u_1u_1}F_{22}-\Big(\frac{H^*}{H\sin u_2}\Big)^2\Big)\\
&=\frac{1}{u_4u_4}\delta^3_\nu F_{22}+\delta^3_\nu\Big(\frac{b_0}{H^2}+\frac{H^*H^*}{H^2\sin^2 u_2}-\frac{b_0}{H^2}-\Big(\frac{H^*}{H\sin u_2}\Big)^2\Big)\\
&=\frac{1}{u_4u_4}\delta^3_\nu F_{22}.
\end{aligned}
$$

These results can be written as

$$R_\beta{}^3{}_{\nu\gamma}=\frac{1}{u_4u_4}\delta^3_\nu F_{\beta\gamma}.$$

Then, we get from (2.15') by setting $\lambda=4$

$$
\begin{aligned}
R_\beta{}^4{}_{\nu\gamma}&=\frac{1}{u_4u_4}\delta^4_\nu F_{\beta\gamma}+\frac{1}{4}\delta^4_\nu B_\beta{}^4{}_{4\gamma}\\
&=\frac{1}{u_4u_4}\delta^4_\nu F_{\beta\gamma}+\frac{1}{4}\delta^4_\nu\Big(F^{44}\frac{\partial F_{44}}{\partial u_\beta}\frac{\partial\log F_{\beta\beta}}{\partial u_\gamma}+F^{44}\frac{\partial F_{44}}{\partial u_\gamma}\frac{\partial\log F_{\gamma\gamma}}{\partial u_\beta}\\
&\qquad-\sum_\alpha F^{44}\frac{\partial F_{44}}{\partial u_\alpha}F^{\alpha\alpha}\frac{\partial F_{\beta\beta}}{\partial u_\alpha}\delta_{\beta\gamma}-F^{44}F^{44}\frac{\partial F_{44}}{\partial u_\gamma}\frac{\partial F_{\nu\nu}}{\partial u_\beta}\Big)\\
&=\frac{1}{u_4u_4}\delta^4_\nu F_{\beta\gamma}.
\end{aligned}
$$

From these results we obtain the formilas

$$R_\beta{}^\lambda{}_{\nu\gamma}=\frac{1}{u_4u_4}\delta^\lambda_\nu F_{\beta\gamma}. \tag{2.17}$$

Next, we have by (1.11) in [4]

$$\begin{aligned}
R_\lambda{}^\alpha{}_{\nu\gamma} = \delta^\alpha_\gamma\Big\{&\frac{1}{4}F^{\alpha\alpha}\Big(\frac{\partial F_{\alpha\alpha}}{\partial u_1}F^{11}\frac{\partial F_{\lambda\nu}}{\partial u_1}+\frac{\partial F_{\alpha\alpha}}{\partial u_2}F^{22}\frac{\partial F_{\lambda\nu}}{\partial u_2}\Big)\\
&+\frac{1}{2u_4}F^{34}\Big(\frac{\partial F_{\lambda 3}}{\partial u_\nu}+\frac{\partial F_{3\nu}}{\partial u_\lambda}\Big)+\frac{1}{2u_4}F^{44}\Big(\frac{\partial F_{\lambda 4}}{\partial u_\nu}+\frac{\partial F_{4\nu}}{\partial u_\lambda}-\frac{\partial F_{\lambda\nu}}{\partial u_4}\Big)\\
&+\frac{1}{u_4u_4}F_{\lambda\nu}\Big\}+\frac{1}{4}\frac{\partial F^{\alpha\alpha}}{\partial u_\gamma}\frac{\partial F_{\lambda\nu}}{\partial u_\alpha}+\frac{1}{2}F^{\alpha\alpha}\frac{\partial^2 F_{\lambda\nu}}{\partial u_\gamma\partial u_\alpha}\\
&-\frac{1}{4}F^{\alpha\alpha}\sum_{\rho,\sigma}F^{\rho\sigma}\frac{\partial F_{\rho\nu}}{\partial u_\alpha}\frac{\partial F_{\sigma\lambda}}{\partial u_\gamma}-\frac{1}{4}F^{\alpha\alpha}F^{\gamma\gamma}\frac{\partial F_{\gamma\gamma}}{\partial u_\alpha}\frac{\partial F_{\lambda\nu}}{\partial u_\gamma}\\
= \delta^\alpha_\gamma\Big\{&\frac{1}{4}F^{\alpha\alpha}\Big(\delta_{\lambda\nu}\frac{\partial F_{\alpha\alpha}}{\partial u_1}F^{11}\frac{\partial F_{\lambda\lambda}}{\partial u_1}+\delta_{\lambda\nu}\frac{\partial F_{\alpha\alpha}}{\partial u_2}F^{22}\frac{\partial F_{\lambda\lambda}}{\partial u_2}\Big)\\
&+\frac{1}{2u_4}F^{44}\Big(\delta^4_\lambda\frac{\partial F_{44}}{\partial u_\nu}+\delta^4_\nu\frac{\partial F_{44}}{\partial u_\lambda}-\delta_{\lambda\nu}\frac{\partial F_{\lambda\lambda}}{\partial u_4}\Big)+\frac{1}{u_4u_4}\delta_{\lambda\nu}F^{44}F_{\lambda\lambda}\Big\}\\
&+\frac{1}{4}\delta_{\lambda\nu}\frac{\partial F^{\alpha\alpha}}{\partial u_\gamma}\frac{\partial F_{\lambda\lambda}}{\partial u_\alpha}+\frac{1}{2}\delta_{\lambda\nu}F^{\alpha\alpha}\frac{\partial^2 F_{\lambda\lambda}}{\partial u_\gamma\partial u_\alpha}\\
&-\frac{1}{4}F^{\alpha\alpha}\delta_{\lambda\nu}F^{\lambda\lambda}\frac{\partial F_{\nu\nu}}{\partial u_\alpha}\frac{\partial F_{\lambda\lambda}}{\partial u_\gamma}-\frac{1}{4}F^{\alpha\alpha}\delta_{\lambda\nu}\frac{\partial \log F_{\gamma\gamma}}{\partial u_\alpha}\frac{\partial F_{\lambda\lambda}}{\partial u_\gamma},
\end{aligned}$$

that is

(2.18)

$$\begin{aligned}
R_\lambda{}^\alpha{}_{\nu\gamma} = \delta^\alpha_\gamma\delta_{\lambda\nu}\Big\{&\frac{1}{4}\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}F^{11}\frac{\partial F_{\lambda\lambda}}{\partial u_1}+\frac{1}{4}\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}F^{22}\frac{\partial F_{\lambda\lambda}}{\partial u_2}\\
&-\frac{1}{2u_4}F^{44}\frac{\partial F_{\lambda\lambda}}{\partial u_4}+\frac{1}{u_4u_4}F^{44}F_{\lambda\lambda}\Big\}+\delta_{\lambda\nu}\Big\{\frac{1}{4}\frac{\partial F^{\alpha\alpha}}{\partial u_\gamma}\frac{\partial F_{\lambda\lambda}}{\partial u_\alpha}\\
&+\frac{1}{2}F^{\alpha\alpha}\frac{\partial^2 F_{\lambda\lambda}}{\partial u_\gamma\partial u_\alpha}-\frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{\lambda\lambda}}{\partial u_\gamma}\frac{\partial F_{\nu\nu}}{\partial u_\alpha}\\
&-\frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{\gamma\gamma}}{\partial u_\alpha}\frac{\partial F_{\lambda\lambda}}{\partial u_\gamma}\Big\}+\delta^\alpha_\gamma\frac{F^{44}}{2u_4}\Big\{\delta^4_\lambda\frac{\partial F_{44}}{\partial u_\nu}+\delta^4_\nu\frac{\partial F_{44}}{\partial u_\lambda}\Big\}.
\end{aligned}$$

Now, from

(2.19) $$\frac{\partial\log F_{33}}{\partial u_\alpha}=\frac{2}{u_1}\delta^1_\alpha+\frac{2H^*}{H\sin u_2}\delta^2_\alpha,$$

we obtain

$$\frac{\partial^2\log F_{33}}{\partial u_\gamma\partial u_\alpha}=-\frac{2}{u_1u_1}\delta^1_\gamma\delta^1_\alpha+2\frac{\partial}{\partial u_2}\Big(\frac{H^*}{H\sin u_2}\Big)\delta^2_\gamma\delta^2_\alpha,$$

and

$$\frac{\partial}{\partial u_2}\frac{H^*}{H\sin u_2}=\frac{H^*}{H\sin u_2}\Big\{-\frac{1}{H^*}(b_0+\frac{1}{4})\sin u_2+\frac{1}{H}(b_0-\frac{1}{4})\sin u_2-\frac{\cos u_2}{\sin u_2}\Big\}$$

$$= \frac{H^*}{H\sin u_2}\Big\{\frac{\sin u_2}{HH^*}\Big(-(b_0+\frac{1}{4})H+(b_0-\frac{1}{4})H^*\Big)-\frac{\cos u_2}{\sin u_2}\Big\}$$
$$= \frac{H^*}{H\sin u_2}\Big\{\frac{\sin u_2}{HH^*}(-b_0)-\frac{\cos u_2}{\sin u_2}\Big\}$$
$$= -\Big(\frac{b_0}{H^2}+\frac{H^*\cos u_2}{H\sin^2 u_2}\Big),$$

hence we have

$$\frac{\partial^2\log F_{33}}{\partial u_\gamma\partial u_\alpha} = -\frac{2}{u_1u_1}\delta^1_\gamma\delta^1_\alpha - 2\Big(\frac{b_0}{H^2}+\frac{H^*}{H}\frac{\cos u_2}{\sin^2 u_2}\Big)\delta^2_\gamma\delta^2_\alpha. \tag{2.20}$$

Now, from (2.18) we have

$$\begin{aligned}
R_3{}^\alpha{}_{\nu\gamma} &= \delta^\alpha_\gamma\delta_{3\nu}\Big\{\frac{1}{4}\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}F^{11}\frac{\partial F_{33}}{\partial u_1}+\frac{1}{4}\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}F^{33}\sin^2 u_2\frac{\partial F_{33}}{\partial u_2}\\
&\quad -\frac{1}{2u_4}F^{44}\frac{\partial F_{33}}{\partial u_4}+\frac{1}{u_4u_4}F^{44}F_{33}\Big\}\\
&\quad +\delta_{3\nu}\Big\{\frac{1}{4}\frac{\partial F^{\alpha\alpha}}{\partial u_\gamma}\frac{\partial F_{33}}{\partial u_\alpha}+\frac{1}{2}F^{\alpha\alpha}\frac{\partial^2 F_{33}}{\partial u_\gamma\partial u_\alpha}-\frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{33}}{\partial u_\gamma}\frac{\partial F_{\nu\nu}}{\partial u_\alpha}\\
&\quad -\frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{\gamma\gamma}}{\partial u_\alpha}\frac{\partial F_{33}}{\partial u_\gamma}\Big\}+\delta^\alpha_\gamma\frac{F^{44}}{2u_4}\delta^4_\nu\frac{\partial F_{44}}{\partial u_3}\\
&= \delta^\alpha_\gamma\delta_{3\nu}\Big\{\frac{1}{2u_1}\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}(1+au_1u_1)F_{33}+\frac{1}{2}\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}\sin^2 u_2\frac{H^*}{H\sin u_2}\\
&\quad -\frac{1+au_4u_4}{u_4u_4}F_{33}\Big\}+\delta_{3\nu}\Big\{\frac{1}{2}\frac{\partial F^{\alpha\alpha}}{\partial u_\gamma}F_{33}\Big(\frac{1}{u_1}\delta^1_\alpha+\frac{H^*}{H\sin u_2}\delta^2_\alpha\Big)\\
&\quad +\frac{1}{2}F^{\alpha\alpha}F_{33}\Big(\frac{\partial^2\log F_{33}}{\partial u_\gamma\partial u_\alpha}+\frac{\partial\log F_{33}}{\partial u_\gamma}\frac{\partial\log F_{33}}{\partial u_\alpha}\Big)\\
&\quad -\frac{1}{4}F^{\alpha\alpha}\frac{\partial\log F_{33}}{\partial u_\gamma}\frac{\partial F_{\nu\nu}}{\partial u_\alpha}-\frac{1}{4}F_{33}F^{\alpha\alpha}\frac{\partial\log F_{\gamma\gamma}}{\partial u_\alpha}\frac{\partial\log F_{33}}{\partial u_\gamma}\Big\},
\end{aligned}$$

that is

(2.18')

$$\begin{aligned}
R_3{}^\alpha{}_{\nu\gamma} &= \delta^\alpha_\gamma\delta_{3\nu}\Big\{\frac{1+au_1u_1}{2u_1}\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}F_{33}+\frac{1}{2}\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}\frac{H^*}{H}\sin u_2\\
&\quad -\frac{1}{u_4u_4}F_{33}-aF_{33}\Big\}+\delta_{3\nu}\Big\{\frac{1}{2}\frac{\partial F^{\alpha\alpha}}{\partial u_\gamma}F_{33}\Big(\frac{1}{u_1}\delta^1_\alpha+\frac{H^*}{H\sin u_2}\delta^2_\alpha\Big)\\
&\quad +F^{\alpha\alpha}F_{33}\Big(-\frac{1}{u_1u_1}\delta^1_\gamma\delta^1_\alpha-\Big(\frac{b_0}{H^2}+\frac{H^*}{H}\frac{\cos u_2}{\sin^2 u_2}\Big)\delta^2_\gamma\delta^2_\alpha\\
&\quad +2\Big(\frac{1}{u_1}\delta^1_\gamma+\frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\Big(\frac{1}{u_1}\delta^1_\alpha+\frac{H^*}{H\sin u_2}\delta^2_\alpha\Big)\Big)
\end{aligned}$$

$$- \frac{1}{2} F^{\alpha\alpha} \Big(\frac{1}{u_1} \delta^1_\gamma + \frac{H^*}{H \sin u_2} \delta^2_\gamma \Big) \frac{\partial F_{\nu\nu}}{\partial u_\alpha}$$
$$- \frac{1}{2} F^{\alpha\alpha} F_{33} \frac{\partial \log F_{\gamma\gamma}}{\partial u_\alpha} \Big(\frac{1}{u_1} \delta^1_\gamma + \frac{H^*}{H \sin u_2} \delta^2_\gamma \Big) \Big\}.$$

When $\alpha = 1$, from this equality we have

$$\begin{aligned} R_3{}^1{}_{\nu\gamma} &= \delta^1_\gamma \delta_{3\nu} \Big\{ \frac{1}{2u_1} \frac{-2au_1}{1 + au_1u_1} (1 + au_1u_1) F_{33} - \frac{1}{u_4u_4} F_{33} - aF_{33} \Big\} \\ &\quad + \delta_{3\nu} \Big\{ \frac{1}{2} 2au_1 \delta^1_\gamma F_{33} \frac{1}{u_1} + (1 + au_1u_1) F_{33} \Big(- \frac{1}{u_1u_1} \delta^1_\gamma + 2 \Big(\frac{1}{u_1} \delta^1_\gamma \\ &\quad + \frac{H^*}{H \sin u_2} \delta^2_\gamma \Big) \frac{1}{u_1} \Big) - \frac{1 + au_1u_1}{2} \Big(\frac{1}{u_1} \delta^1_\gamma + \frac{H^*}{H \sin u_2} \delta^2_\gamma \Big) \frac{\partial F_{\nu\nu}}{\partial u_1} \\ &\quad - \frac{1}{2} (1 + au_1u_1) F_{33} \frac{\partial \log F_{\gamma\gamma}}{\partial u_1} \Big(\frac{1}{u_1} \delta^1_\gamma + \frac{H^*}{H \sin u_2} \delta^2_\gamma \Big) \Big\}, \end{aligned}$$

which is arranged as follows

(2.18")

$$\begin{aligned} R_3{}^1{}_{\nu\gamma} &= -\delta^1_\gamma \frac{F_{3\nu}}{u_4u_4} + \delta_{3\nu} \Big\{ \frac{1}{u_1u_1} \delta^1_\gamma F_{33} + \frac{2(1 + au_1u_1)}{u_1} F_{33} \frac{H^*}{H \sin u_2} \delta^2_\gamma \\ &\quad - \frac{1}{2} (1 + au_1u_1) \Big(\frac{1}{u_1} \delta^1_\gamma + \frac{H^*}{H \sin u_2} \delta^2_\gamma \Big) \Big(\frac{\partial F_{\nu\nu}}{\partial u_1} + F_{33} \frac{\partial \log F_{\gamma\gamma}}{\partial u_1} \Big) \Big\}. \end{aligned}$$

Hence we get from this equality

$$\begin{aligned} R_3{}^1{}_{3\gamma} &= -\delta^1_\gamma \frac{F_{33}}{u_4u_4} + \frac{1}{u_1u_1} \delta^1_\gamma F_{33} + \frac{2(1 + au_1u_1)}{u_1} F_{33} \frac{H^*}{H \sin u_2} \delta^2_\gamma \\ &\quad - \frac{1}{2} (1 + au_1u_1) \Big(\frac{1}{u_1} \delta^1_\gamma + \frac{H^*}{H \sin u_2} \delta^2_\gamma \Big) \Big(\frac{2F_{33}}{u_1} + F_{33} \frac{\partial \log F_{\gamma\gamma}}{\partial u_1} \Big). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} R_3{}^1{}_{31} &= - \frac{1}{u_4u_4} F_{33} + \frac{1}{u_1u_1} F_{33} - \frac{1 + au_1u_1}{2u_1} \Big(\frac{2F_{33}}{u_1} + F_{33} \frac{-2au_1}{1 + au_1u_1} \Big) \\ &= - \frac{1}{u_4u_4} F_{33} \end{aligned}$$

and

$$\begin{aligned} R_3{}^1{}_{32} &= \frac{2(1 + au_1u_1)}{u_1} F_{33} \frac{H^*}{H \sin u_2} - \frac{1}{2} (1 + au_1u_1) \frac{H^*}{H \sin u_2} \Big(\frac{2F_{33}}{u_1} + F_{33} \frac{2}{u_1} \Big) \\ &= 0. \end{aligned}$$

Furthermore, we get easily from (2.18")

$$R_3{}^1{}_{4\gamma} = -\delta^1_\gamma \frac{F_{34}}{u_4u_4} = 0.$$

These results can be written as

$$R_3{}^1{}_{\nu\gamma} = -\frac{1}{u_4u_4}\delta^1_\gamma F_{3\nu}.$$

When $\alpha = 2$, from (2.18') we obtain

$$\begin{aligned}
R_3{}^2{}_{\nu\gamma} = \delta^2_\gamma\delta_{3\nu}\Big\{ &\frac{1+au_1u_1}{2u_1}\frac{\partial \log F_{22}}{\partial u_1}F_{33} + \frac{1}{2}\frac{\partial \log F_{22}}{\partial u_2}\frac{H^*}{H}\sin u_2 \\
& - \frac{1}{u_4u_4}F_{33} - aF_{33}\Big\} + \delta_{3\nu}\Big\{\frac{1}{2}\frac{\partial F^{22}}{\partial u_\gamma}F_{33}\frac{H^*}{H\sin u_2} \\
& + F^{22}F_{33}\Big(-\Big(\frac{b_0}{H^2} + \frac{H^*}{H}\frac{\cos u_2}{\sin^2 u_2}\Big)\delta^2_\gamma \\
& + 2\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\frac{H^*}{H\sin u_2}\Big) \\
& - \frac{1}{2}F^{22}\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\frac{\partial F_{\nu\nu}}{\partial u_2} \\
& - \frac{1}{2}F^{22}F_{33}\frac{\partial \log F_{\gamma\gamma}}{\partial u_2}\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\Big\} \\
= \delta^2_\gamma\delta_{3\nu}\Big\{ &\frac{1+au_1u_1}{u_1u_1}F_{33} + \frac{H^*}{H^2}(b_0 - \frac{1}{4})\sin^2 u_2 - \frac{1}{u_4u_4}F_{33} - aF_{33}\Big\} \\
& + \delta_{3\nu}\Big\{\sin^2 u_2\Big(-\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{1}{H}(b_0 - \frac{1}{4})\sin u_2\delta^2_\gamma\Big)\frac{H^*}{H\sin u_2} \\
& - \Big(\frac{b_0}{H^2} + \frac{H^*\cos u_2}{H\sin^2 u_2}\Big)\delta^2_\gamma + 2\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\frac{H^*}{H\sin u_2}\Big) \\
& - \frac{1}{2}\frac{H^2}{b_0u_1u_1}\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\frac{\partial F_{\nu\nu}}{\partial u_2} \\
& - \frac{1}{2}\sin^2 u_2\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\frac{\partial \log F_{\gamma\gamma}}{\partial u_2}\Big\}.
\end{aligned}$$

Since we have the equality

$$-(b_0 - \frac{1}{4})H^*\sin^2 u_2 - b_0\sin^2 u_2 - HH^*\cos u_2 + 2H^*H^* = -b_0\sin^2 u_2 + H^*H^*,$$

the above equality become

(2.18''')

$$\begin{aligned}
R_3{}^2{}_{\nu\gamma} = \delta^2_\gamma\delta_{3\nu}&\Big\{\frac{\sin^2 u_2}{H^2}(b_0 + (b_0 - \frac{1}{4})H^*) - \frac{1}{u_4u_4}F_{33}\Big\} \\
& + \delta_{3\nu}\Big\{\delta^1_\gamma\frac{H^*\sin u_2}{u_1H} + \delta^2_\gamma\frac{-b_0\sin^2 u_2 + H^*H^*}{H^2} \\
& - \frac{1}{2}\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\Big(\frac{H^2}{b_0u_1u_1}\frac{\partial F_{\nu\nu}}{\partial u_2} + \sin^2 u_2\frac{\partial \log F_{\gamma\gamma}}{\partial u_2}\Big)\Big\},
\end{aligned}$$

from which we get

$$R_3{}^2{}_{3\gamma} = \delta^2_\gamma\Big\{\frac{\sin^2 u_2}{H^2}(b_9 + (b_0 - \frac{1}{4}))H^*) - \frac{1}{u_4u_4}F_{33}\Big\} + \delta^1_\gamma\frac{H^*\sin u_2}{u_1H} + \delta^2_\gamma\frac{-b_0\sin^2 u_2 + H^*H^*}{H^2} - \frac{1}{2}\Big(\frac{1}{u_1}\delta^1_\gamma + \frac{H^*}{H\sin u_2}\delta^2_\gamma\Big)\Big(\frac{2H^*\sin u_2}{H} + \sin^2 u_2\frac{\partial\log F_{\gamma\gamma}}{\partial u_2}\Big).$$

Therefore we obtain

$$R_3{}^2{}_{31} = 0$$

and

$$R_3{}^2{}_{32} = \frac{\sin^2 u_2}{H^2}(b_0 + (b_0 - \frac{1}{4})H^*) - \frac{1}{u_4u_4}F_{33} + \frac{-b_0\sin^2 u_2 + H^*H^*}{H^2} - \frac{H^*}{H}\Big(\frac{H^*}{H} + \frac{\sin^2 u_2}{H}(b_0 - \frac{1}{4})\Big) = -\frac{1}{u_4u_4}F_{33}.$$

Since we get easily from (2.18") $R_3{}^2{}_{4\gamma} = 0$, we can explain these results as

$$R_3{}^2{}_{\nu\gamma} = -\frac{1}{u_4u_4}\delta^2_\gamma F_{3\nu}.$$

Then we get from (2.18)

$$\begin{aligned} R_4{}^\alpha{}_{\nu\gamma} &= \delta^\alpha_\gamma\delta_{4\nu}\Big\{-\frac{1}{2u_4}F^{44}\frac{\partial F_{44}}{\partial u_4} + \frac{F^{44}}{u_4u_4}F_{44}\Big\} + \delta^\alpha_\gamma\frac{F^{44}}{2u_4}\Big\{\frac{\partial F_{44}}{\partial u_\nu} + \delta^4_\nu\frac{\partial F_{44}}{\partial u_4}\Big\} \\ &= \delta^\alpha_\gamma\delta_{4\nu}\Big\{\frac{1}{2u_4}\frac{2au_4}{1+au_4u_4} + \frac{1}{u_4u_4}\Big\} + \delta^\alpha_\gamma\delta^4_\nu\frac{-2a}{1+au_4u_4} \\ &= \delta^\alpha_\gamma\delta_{4\nu}\Big\{-\frac{a}{1+au_4u_4} + \frac{1}{u_4u_4}\Big\} = \delta^\alpha_\gamma\delta_{4\nu}\frac{1}{u_4u_4(1+au_4u_4)} \\ &= -\frac{1}{u_4u_4}\delta^\alpha_\gamma\delta_{4\nu}F_{44} = -\frac{1}{u_4u_4}\delta^\alpha_\gamma F_{4\nu}. \end{aligned}$$

These results can be arranged as

$$(2.18^*) \qquad R_\lambda{}^\alpha{}_{\nu\gamma} = -\frac{1}{u_4u_4}\delta^\alpha_\gamma F_{\lambda\nu}.$$

Next, we obtain by (1.12) in [4]

$$R_\lambda{}^\mu{}_{\nu\gamma} = -\frac{\partial}{\partial u_\gamma}\{\lambda{}^\mu{}_\nu\}_\Lambda - \frac{1}{2}\sum_{\rho,\sigma}\{\rho{}^\sigma{}_\nu\}_\Lambda F^{\mu\rho}\frac{\partial F_{\sigma\lambda}}{\partial u_\gamma} - \frac{1}{2}\sum_{\rho,\sigma}\{\lambda{}^\rho{}_\nu\}_\Lambda F^{\mu\sigma}\frac{\partial F_{\sigma\rho}}{\partial u_\gamma} + \frac{1}{2}\sum_\sigma F^{\mu\sigma}\frac{\partial^2 F_{\sigma\lambda}}{\partial u_\nu\partial u_\gamma} - \frac{1}{2u_4}\Big(\frac{\partial F^{4\mu}}{\partial u_\gamma}F_{\lambda\nu} + F^{4\mu}\frac{\partial F_{\lambda\nu}}{\partial u_\gamma}\Big)$$

$$- \frac{1}{2u_4}\delta^{\mu}_{\nu}\sum_{\sigma} F^{4\sigma}\frac{\partial F_{\sigma\lambda}}{\partial u_\gamma} + \frac{1}{2u_4}F^{4\mu}\frac{\partial F_{\lambda\nu}}{\partial u_\gamma}$$

$$= -\frac{1}{2}\delta_{4\nu}\{{}_4{}^4{}_4\}_\Lambda F^{\mu 4}\frac{\partial F_{4\lambda}}{\partial u_\gamma} - \frac{1}{2}\{{}_\lambda{}^4{}_\nu\}_\Lambda F^{\mu 4}\frac{\partial F_{44}}{\partial u_\gamma} + \frac{1}{2}F^{\mu\mu}\frac{\partial^2 F_{\mu\lambda}}{\partial u_\nu \partial u_\gamma}$$

$$- \frac{1}{2u_4}F^{4\mu}\frac{\partial F_{\lambda\nu}}{\partial u_\gamma} - \frac{1}{2u_4}\delta^{\mu}_{\nu}F^{44}\frac{\partial F_{4\lambda}}{\partial u_\gamma} + \frac{1}{2u_4}F^{4\mu}\frac{\partial F_{\lambda\nu}}{\partial u_\gamma} = 0,$$

that is

(2.21) $$R_\lambda{}^\mu{}_{\nu\gamma} = 0.$$

Next, we obtain by (1.13) in [4]

$$R_\beta{}^\alpha{}_{34} = \frac{1}{4}F^{\alpha\alpha}\sum_{\rho,\sigma}F^{\rho\sigma}\Big(\frac{\partial F_{\rho 4}}{\partial u_\alpha}\frac{\partial F_{3\sigma}}{\partial u_\beta} - \frac{\partial F_{\rho 3}}{\partial u_\alpha}\frac{\partial F_{4\sigma}}{\partial u_\beta}\Big) = 0,$$

that is

(2.22) $$R_\beta{}^\alpha{}_{34} = 0.$$

Next, we obtain by (1.14) in [4]

$$R_\beta{}^\lambda{}_{34} = -\frac{1}{2}\sum_{\sigma}\frac{\partial F^{\lambda\sigma}}{\partial u_4}\frac{\partial F_{\sigma 3}}{\partial u_\beta} - \frac{1}{2}\sum_{\sigma}F^{\lambda\sigma}\frac{\partial^2 F_{\sigma 3}}{\partial u_4 \partial u_\beta}$$

$$+ \frac{1}{2}\sum_{\rho,\sigma}F^{\rho\sigma}\Big(\{{}_\rho{}^\lambda{}_3\}_\Lambda\frac{\partial F_{\sigma 4}}{\partial u_\beta} - \{{}_\rho{}^\lambda{}_4\}_\Lambda\frac{\partial F_{\sigma 3}}{\partial u_\beta}\Big)$$

$$- \frac{1}{2u_4}\sum_{\sigma}F^{4\sigma}\Big(\delta^\lambda_3\frac{\partial F_{\sigma 4}}{\partial u_\beta} - \delta^\lambda_4\frac{\partial F_{\sigma 3}}{\partial u_\beta}\Big)$$

$$= -\frac{1}{2}\frac{\partial F^{\lambda 3}}{\partial u_4}\frac{\partial F_{33}}{\partial u_\beta} - \frac{1}{2}F^{\lambda 3}\frac{\partial^2 F_{33}}{\partial u_4 \partial u_\beta} - \frac{1}{2}F^{43}\{{}_4{}^\lambda{}_4\}_\Lambda\frac{\partial F_{33}}{\partial u_\beta}$$

$$- \frac{1}{2u_4}\Big(F^{44}\delta^\lambda_3\frac{\partial F_{44}}{\partial u_\beta} - F^{44}\delta^\lambda_4\frac{\partial F_{43}}{\partial u_\beta}\Big) = 0,$$

that is

(2.23) $$R_\beta{}^\lambda{}_{34} = 0.$$

Next, we obtain by (1.15) in [4]

$$R_\lambda{}^\alpha{}_{34} = \frac{1}{2}F^{\alpha\alpha}\frac{\partial^2 F_{\lambda 3}}{\partial u_\alpha \partial u_4} - \frac{1}{2}F^{\alpha\alpha}\sum_{\rho}\Big(\frac{\partial F_{\rho 3}}{\partial u_\alpha}\{{}_\lambda{}^\rho{}_4\}_\Lambda - \frac{\partial F_{\rho 4}}{\partial u_\alpha}\{{}_\lambda{}^\rho{}_3\}_\Lambda\Big)$$

$$- \frac{1}{2u_4}F^{\alpha\alpha}\sum_{\rho}F^{4\rho}\Big(\frac{\partial F_{\rho 3}}{\partial u_\alpha}F_{\lambda 4} - \frac{\partial F_{\rho 4}}{\partial u_\alpha}F_{\lambda 3}\Big)$$

$$= -\frac{1}{2}F^{\alpha\alpha}\frac{\partial F_{43}}{\partial u_4}\{{}_\lambda{}^4{}_4\}_\Lambda - \frac{1}{2u_4}F^{\alpha\alpha}F^{43}\frac{\partial F_{33}}{\partial u_\alpha}F_{\lambda 4} = 0,$$

that is

$$R_\lambda{}^\alpha{}_{34} = 0. \tag{2.24}$$

Finally, we obtain by (1.16) in [4]

$$\begin{aligned}
R_\lambda{}^\mu{}_{34} = &-\frac{\partial}{\partial u_4}\{_\lambda{}^\mu{}_3\}_\Lambda + \sum_\sigma\Big(\{_\sigma{}^\mu{}_3\}_\Lambda\{_\lambda{}^\sigma{}_4\}_\Lambda - \{_\sigma{}^\rho{}_4\}_\Lambda\{_\lambda{}^\sigma{}_3\}_\Lambda\Big)\\
&-\frac{1}{u_4}\delta^\mu_3\Big(\{_\lambda{}^4{}_4\}_\Lambda + \frac{1}{u_4}F^{44}F_{\lambda 4}\Big) + \frac{1}{u_4}\delta^\mu_4\Big(\{_\Lambda{}^4{}_3\}_\Lambda + \frac{1}{u_4}F^{44}F_{\lambda 3}\Big)\\
&+\frac{1}{u_4}\sum_\sigma F^{\mu\sigma}\Big(F_{\lambda 3}\{_\sigma{}^4{}_4\}_\Lambda - F_{\lambda 4}\{_\sigma{}^4{}_3\}_\Lambda\Big)\\
&+\frac{1}{4}\sum_{\rho,\sigma}F^{\alpha\alpha}F^{\mu\sigma}\Big(\frac{\partial F_{\lambda 3}}{\partial u_\alpha}\frac{\partial F_{\sigma 4}}{\partial u_\alpha} - \frac{\partial F_{\lambda 4}}{\partial u_\alpha}\frac{\partial F_{\sigma 3}}{\partial u_\alpha}\Big)\\
=&-\frac{1}{u_4}\delta^\mu_3\Big(\delta^4_\lambda\{_4{}^4{}_4\}_\Lambda + \frac{1}{u_4}F^{44}F_{\lambda 4}\Big) + \frac{1}{u_4u_4}\delta^\mu_4F^{44}F_{\lambda 3}\\
&+\frac{1}{u_4}F^{\mu 4}F_{\lambda 3}\{_4{}^4{}_4\}_\Lambda + \frac{1}{4}F^{\mu\mu}\sum_\alpha F^{\alpha\alpha}\Big(\frac{\partial F_{\lambda 3}}{\partial u_\alpha}\frac{\partial F_{\mu 4}}{\partial u_\alpha} - \frac{\partial F_{\lambda 4}}{\partial u_\alpha}\frac{\partial F_{\mu 3}}{\partial u_\alpha}\Big)\\
=&\,\frac{1}{u_4}\{_4{}^4{}_4\}_\Lambda(F^{\mu 4}F_{\lambda 3} - \delta^\mu_3\delta^4_\lambda) + \frac{1}{u_4u_4}(-\delta^\mu_3\delta^4_\lambda + \delta^\mu_4\delta^3_\lambda F^{44}F_{33})\\
=&-\frac{a}{1+au_4u_4}(\delta^\mu_4\delta^3_\lambda F^{44}F_{33} - \delta^\mu_3\delta^4_\lambda) + \frac{1}{u_4u_4}(-\delta^\mu_3\delta^4_\lambda + \delta^\mu_4\delta^3_\lambda F^{44}F_{33})\\
=&\,\frac{1}{u_4u_4(1+au_4u_4)}(\delta^\mu_4\delta^3_\lambda F^{44}F_{33} - \delta^\mu_3\delta^4_\lambda)\\
=&-\frac{1}{u_4u_4}(\delta^\mu_4F_{\lambda 3} - \delta^\mu_3F_{\lambda 4}),
\end{aligned}$$

that is

$$R_\lambda{}^\mu{}_{34} = \frac{1}{u_4u_4}(\delta^\mu_3F_{\lambda 4} - \delta^\mu_4F_{\lambda 3}). \tag{2.25}$$

Thus we have proved the following theorem,

Main theorem. *For the metric on $R^4{}_+$*

$$ds^2 = \frac{1}{u_4u_4}\sum_{i,j}F_{ij}du_idu_j,$$

given by

$$F_{11} = \frac{1}{1+au_1u_1}, \quad F_{22} = \frac{b_0u_1u_1}{H^2}, \quad F_{33} = \frac{b_0u_1u_1\sin^2 u_2}{H^2},$$
$$F_{34} = 0, \quad F_{44} = -\frac{1}{1+au_4u_4}, \quad F_{12} = F_{\alpha\lambda} = 0,$$

$$\alpha = 1,\, 2, \;\; \textit{and}\; \lambda = 3,\, 4, \quad b_0 = \text{constant},$$
$$H = (b_0 - \frac{1}{4}) \cos u_2 + b_0 + \frac{1}{4},$$

its curvature tensor $R_j{}^i{}_{hk}$ *satisfies the equality*:

$$R_j{}^i{}_{hk} = \frac{1}{u_4 u_4}(\delta^i_h F_{jk} - \delta^i_k F_{jh}), \quad i,\, j,\, h,\, k = 1,\, 2,\, 3,\, 4.$$

Furthermore, we have easily the following (see [7]).

Corollary. *The space with the metric in the above theorem has constant sectional curvature* -1.

Proof. Let Π be any two-dimensional nondegenerate tangent subspace at a point and v^i, w^i be tangent non zero vectors belonging to Π and orthgonal each other to. Then, using the Einstein convention for summation , we have

$$\begin{aligned} R_{jihk} v^j w^i v^h w^k &= \frac{1}{u_4 u_4}(g_{ih} F_{jk} - g_{ik} F_{jh}) v^j w^i v^h w^k \\ &= (g_{ih} g_{jk} - g_{ik} g_{jh}) v^j w^i v^h w^k \\ &= g_{ih} w^i v^h g_{jk} v^j w^k - g_{ik} w^i w^k g_{jh} v^j v^h \\ &= -(g_{ik} w^i w^k)(g_{jh} v^j v^h), \end{aligned}$$

which shows the sectional curvature of Π is -1. □

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