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A NOTE ON GEODESICS AND CURVATURES OF CERTAIN 4-SPACES

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Abstract. This work is a continuation of the papers [3] and [4], in which we studied the metrics (1.1) and (1.2) on R^4_+. The metric (1.1) with $a = 0$:

$$ds^2 = \frac{dx_1dx_1 + dx_2dx_2 + dx_3dx_3 - dx_4dx_4}{x_4x_4}$$

is analogous to the metric of the hyperbolic 4-space. We considered fundamentally metrics on R^4_+ based on this hyperbolic type metric, not Euclidean or Minkowsky types.

1. Geodesics

We studied the following metrics on ${R^4}_+ = R^3 \times R_+$

$$ds^2 = \frac{1}{x_4x_4}\Big\{\sum_{b,c=1}^{3}\Big(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\Big)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\} \tag{1.1}$$

and

$$ds^2 = \frac{1}{x_4x_4}\Big\{\sum_{b,c=1}^{3}\Big(\frac{8}{(x_3+3r)^2}(r^2\delta_{bc} - x_bx_c) + \frac{x_bx_c}{r^2(1+ar^2)}\Big)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\}, \tag{1.2}$$

where $r^2 = \sum_{b=1}^{3} x_bx_b$, $a =$ constant, in [1],[2] and [3],[4], respectively. They are derived as special ones from the metric on ${R^4}_+$

$$ds^2 = \frac{1}{u_4u_4}\sum_{i,j=1}^{4} F_{ij}du_idu_j, \quad F_{ij} = F_{ji},$$

where $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, $u_4 = x_4$ and

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta,$$

and (r,θ,ϕ) are the polar coordinates of R^3, which satisfies the Einstein condition and

$$F_{ij} = F_{ij}(u_1,u_2) \quad \text{except for} \quad F_{44} = F_{44}(u_1,u_2,u_4)$$

and

$$F_{12} = F_{\alpha\lambda} = 0 \quad (\alpha = 1,2;\ \lambda = 3,4).$$

The metric (1.1) is the one such that

$$\frac{\partial F_{11}}{\partial u_2} = \frac{\partial F_{22}}{\partial u_2} = 0 \quad \text{and} \quad F_{33} = \psi(u_1)\sin^2 u_2$$

and the metric (1.2) is the one essentially depending on the longitude ϕ. In this paper we shall show that their geodesics have special features quite different for the two metrics.

In general for the metric

$$(1.3) \qquad ds^2 = \sum_{i,j=1}^{4} g_{ij} dx_i dx_j = \frac{1}{x_4 x_4} \sum_{i,j=1}^{4} F_{ij}(x) dx_i dx_j, \quad g_{ij} = \frac{F_{ij}(x)}{x_4 x_4},$$

its Christoffel symbols become

$$(1.4) \qquad \{_j{}^i{}_h\} = \frac{1}{2}\sum_{k=1}^{4} g^{ik}\Big(\frac{\partial g_{jk}}{\partial x_h} + \frac{\partial g_{kh}}{\partial x_j} - \frac{\partial g_{jh}}{\partial x_k}\Big)$$
$$= \frac{1}{2}\sum_{k} F^{ik}\Big(\frac{\partial F_{jk}}{\partial x_h} + \frac{\partial F_{kh}}{\partial x_j} - \frac{\partial F_{jh}}{\partial x_k}\Big)$$
$$- \frac{1}{x_4}(\delta^i_j \delta_{4k} + \delta^i_h \delta_{4j} - F^{i4} F_{jh}),$$

where $(g^{ij}) = (g_{ij})^{-1}$, $(F^{ij}) = (F_{ij})^{-1}$.

The equations of a geodesic of the metric (1.3) are

$$(1.5) \qquad \frac{d^2 x_i}{dt^2} + \sum_{j,h} \{_j{}^i{}_h\} \frac{dx_j}{dt}\frac{dx_h}{dt} = 0, \quad i = 1,\ 2,\ 3,\ 4.$$

Proposition 1. *For any geodesic* $(x_i(t)), 1 \leq i \leq 4$, *of the metric* (1.1), *the curve* $(x_b(t)), 1 \leq b \leq 3$, *in* R^3 *is a plane curve.*

Proof. Since we have for the metric (1.1)

$$(1.6) \qquad F_{bc} = \delta_{bc} - \frac{a x_b x_c}{1 + ar^2}, \quad F_{b4} = 0, \quad F_{44} = -\frac{1}{1 + a x_4 x_4}$$

where $b,\ c,\ e = 1, 2, 3$, we obtain by (1.4)

$$\{_b{}^e{}_c\} = -a x_e\Big(\delta_{bc} - \frac{a x_b x_c}{1 + ar^2}\Big), \quad \{_b{}^4{}_c\} = -\frac{1 + a x_4 x_4}{x_4}\Big(\delta_{bc} - \frac{a x_b x_c}{1 + ar^2}\Big),$$
$$\{_b{}^e{}_4\} = -\frac{1}{x_4}\delta^e_b, \quad \{_b{}^4{}_4\} = 0, \quad \{_4{}^e{}_4\} = 0,$$
$$\{_4{}^4{}_4\} = -\frac{1}{x_4}\,\frac{1 + 2a x_4 x_4}{1 + a x_4 x_4} = -\frac{2}{x_4} + \frac{1}{x_4(1 + a x_4 x_4)}.$$

For the geodesic $x_i(t)$ we have

$$\begin{aligned}
&\frac{d^2x_e}{dt^2}+\sum_{j,h}\{{}_j{}^e{}_h\}\frac{dx_j}{dt}\frac{dx_h}{dt}\\
&\quad=\frac{d^2x_e}{dt^2}-ax_e\sum_{b,c}\Big(\delta_{bc}-\frac{ax_bx_c}{1+ar^2}\Big)\frac{dx_b}{dt}\frac{dx_c}{dt}-\frac{2}{x_4}\sum_b\delta^e_b\frac{dx_b}{dt}\frac{dx_4}{dt}\\
&\quad=\frac{d^2x_e}{dt^2}-ax_e\Big(\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}-\frac{a}{1+ar^2}\Big(\sum_b x_b\frac{dx_b}{dt}\Big)^2\Big)-\frac{2}{x_4}\frac{dx_e}{dt}\frac{dx_4}{dt}\\
&\quad=0,
\end{aligned}$$

that is

$$\frac{d^2x_e}{dt^2}=ax_e\Big(\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}-\frac{ar^2}{1+ar^2}\frac{dr}{dt}\frac{dr}{dt}\Big)+\frac{2}{x_4}\frac{dx_e}{dt}\frac{dx_4}{dt},\tag{1.7}$$

and

$$\begin{aligned}
&\frac{d^2x_4}{dt^2}+\sum_{j,h}\{{}_j{}^4{}_h\}\frac{dx_j}{dt}\frac{dx_h}{dt}\\
&=\frac{d^2x_4}{dt^2}-\frac{1+ax_4x_4}{x_4}\sum_{b,c}\Big(\delta_{bc}-\frac{ax_bx_c}{1+ar^2}\Big)\frac{dx_b}{dt}\frac{dx_c}{dt}-\frac{1}{x_4}\frac{1+2ax_4x_4}{1+ax_4x_4}\frac{dx_4}{dt}\frac{dx_4}{dt}\\
&=\frac{d^2x_4}{dt^2}-\frac{1+ax_4x_4}{x_4}\Big(\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}-\frac{ar^2}{1+ar^2}\frac{dr}{dt}\frac{dr}{dt}\Big)-\frac{1+2ax_4x_4}{x_4(1+ax_4x_4)}\frac{dx_4}{dt}\frac{dx_4}{dt}\\
&=0,
\end{aligned}$$

that is

$$\begin{aligned}\frac{d^2x_4}{dt^2}=\frac{1+ax_4x_4}{x_4}\Big(\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}-\frac{ar^2}{1+ar^2}\frac{dr}{dt}\frac{dr}{dt}\Big)\\
+\frac{1+2ax_4x_4}{x_4(1+ax_4x_4)}\frac{dx_4}{dt}\frac{dx_4}{dt}.\end{aligned}\tag{1.8}$$

Now, for the curve $(x_1(t),x_2(t),x_3(t))$ in R^3 we set

$$V_1:=\frac{dx_2}{dt}\frac{d^2x_3}{dt^2}-\frac{dx_3}{dt}\frac{d^2x_2}{dt^2},\quad V_2:=\frac{dx_3}{dt}\frac{d^2x_1}{dt^2}-\frac{dx_1}{dt}\frac{d^2x_3}{dt^2},$$

$$V_3:=\frac{dx_1}{dt}\frac{d^2x_2}{dt^2}-\frac{dx_2}{dt}\frac{d^2x_1}{dt^2},\quad\text{and}\quad\Phi:=\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}-\frac{ar^2}{1+ar^2}\frac{dr}{dt}\frac{dr}{dt},$$

then we have

$$V_1=\frac{dx_2}{dt}\Big(ax_3\Phi+\frac{2}{x_4}\frac{dx_3}{dt}\frac{dx_4}{dt}\Big)-\frac{dx_3}{dt}\Big(ax_2\Phi+\frac{2}{x_4}\frac{dx_2}{dt}\frac{dx_4}{dt}\Big)$$

$$= -a\Phi\Big(x_2\frac{dx_3}{dt} - x_3\frac{dx_2}{dt}\Big)$$

and analogously

$$V_2 = -a\Phi\Big(x_3\frac{dx_1}{dt} - x_1\frac{dx_3}{dt}\Big), \quad V_3 = -a\Phi\Big(x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}\Big).$$

As vectors in R^3, we have the relation

$$V = (V_1, V_2, V_3) = \frac{dx}{dt} \times \frac{d^2x}{dt^2} = -a\Phi(x \times \frac{dx}{dt}),$$

which implies

$$\begin{aligned}\frac{dV}{dt} &= -a\frac{d\Phi}{dt}(x \times \frac{dx}{dt}) - a\Phi(x \times \frac{d^2x}{dt^2}) \\ &= -a\frac{d\Phi}{dt}(x \times \frac{dx}{dt}) - a\Phi\Big(x \times \Big(ax\Phi + \frac{2}{x_4}\frac{dx}{dt}\frac{dx_4}{dt}\Big)\Big) \\ &= -a\frac{d\Phi}{dt}\Big(x \times \frac{dx}{dt}\Big) - \frac{2a}{x_4}\Phi\frac{dx_4}{dt}\Big(x \times \frac{dx}{dt}\Big),\end{aligned}$$

that is

$$\frac{dV}{dt} = \Big(\frac{1}{\Phi}\frac{d\Phi}{dt} + \frac{2}{x_4}\frac{dx_4}{dt}\Big)V.$$

Hence we see that the normal direction of the osculating plane of the curve $(x_1(t), x_2(t), x_3(t))$ in R^3 is constant. Therefore this curve must be a plane curve in R^3. □

Proposition 2. *For a geodesic $(x_i(t))$ of the metric* (1.2), *the curve $(x_1(t), x_2(t), x_3(t))$ in R^3 is a plane curve, if and only if it satisfies the condition*:

$$\frac{d}{dt}\Big(\frac{C}{A}\Big)\Big(x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}\Big) = 0, \tag{1.9}$$

where

$$\begin{aligned}A := &\Big\{-\frac{3}{r^2} + \frac{1}{r^2(1+ar^2)} + \frac{8(1+ar^2)}{(x_3+3r)^2} + \frac{3}{r(x_3+3r)}\Big\}\frac{dr}{dt}\frac{dr}{dt} \\ &+ \Big\{\frac{3}{r(x_3+3r)} - \frac{2(1+ar^2)}{(x_3+3r)^2}\Big\}\sum_b \frac{dx_b}{dt}\frac{dx_b}{dt} + \frac{2}{r(x_3+3r)}\frac{dr}{dt}\frac{dx_3}{dt},\end{aligned} \tag{1.10}$$

$$C := \frac{1}{x_3+3r}\Big(\sum_b \frac{dx_b}{dt}\frac{dx_b}{dt} - \frac{dr}{dt}\frac{dr}{dt}\Big). \tag{1.11}$$

Proof. From the metric (1.2) we have

$$F_{bc} = \Big(\frac{1}{r^2(1+ar^2)} - \frac{8}{(x_3+3r)^2}\Big)x_bx_c + \frac{8r^2}{(x_3+3r)^2}\delta_{bc}, \tag{1.12}$$

$$F_{b4} = 0, \quad F_{44} = -\frac{1}{1+ax_4x_4} \tag{1.13}$$

and

$$F^{bc} = \Big(-\frac{(x_3+3r)^2}{8r^4} + \frac{1+ar^2}{r^2}\Big)x_bx_c + \frac{(x_3+3r)^2}{8r^2}\delta_{bc}, \quad (F^{bc}) = (F_{bc})^{-1},$$

from which we obtain

$$\begin{aligned}\frac{\partial F_{bc}}{\partial x_e} = {} & \Big\{\frac{2}{r^4}\Big(-\frac{2}{1+ar^2} + \frac{1}{(1+ar^2)^2}\Big) + \frac{48}{r(x_3+3r)^3}\Big\}x_bx_cx_e \\ & + \frac{16}{(x_3+3r)^3}\{\delta_{3e}(x_bx_c - r^2\delta_{bc}) + \delta_{bc}x_3x_e\} \\ & + \Big(\frac{1}{r^2(1+ar^2)} - \frac{8}{(x_3+3r)^2}\Big)(\delta_{eb}x_c + \delta_{ec}x_b).\end{aligned}$$

The Christoffel symbols (1.4) are computed as follows:

(1.14)

$$\begin{aligned}\{{}_b{}^e{}_c\} = {} & \frac{1+ar^2}{2}\Big[\Big\{\frac{2}{r^4}\Big(-\frac{2}{1+ar^2} + \frac{1}{(1+ar^2)^2}\Big) + \frac{48}{r(x_3+3r)^3}\Big\}x_bx_cx_e \\ & + \frac{16}{(x_3+3r)^3}x_e(\delta_{3b}x_c + \delta_{3c}x_b - \delta_{bc}x_3) \\ & + 2\Big(\frac{1}{r^2(1+ar^2)} - \frac{8}{(x_3+3r)^2}\Big)x_e\delta_{bc}\Big] \\ & + \frac{8}{(x_3+3r)^3}\frac{1+ar^2}{r^2}x_e(x_b\phi_{3c} + x_c\phi_{3b} - x_3\phi_{bc}) \\ & - \frac{1}{r^4(x_3+3r)}(\phi_{eb}\phi_{3c} + \phi_{ec}\phi_{3b} - \phi_{e3}\phi_{bc}),\end{aligned}$$

where we set

$$\phi_{bc} := x_bx_c - r^2\delta_{bc}, \tag{1.15}$$

and

$$\{{}_b{}^4{}_c\} = -\frac{1+ax_4x_4}{x_4}F_{bc}, \quad \{{}_b{}^e{}_4\} = -\frac{1}{x_4}\delta_b^e, \quad \{{}_b{}^4{}_4\} = 0, \tag{1.16}$$

$$\{{}_4{}^e{}_4\} = 0, \quad \{{}_4{}^4{}_4\} = -\frac{1+2ax_4x_4}{x_4(1+ax_4x_4)} = -\frac{2}{x_4} + \frac{1}{x_4(1+ax_4x_4)}.$$

On the above auxiliary functions ϕ_{bc}, we see easily that

$$\sum_c \phi_{bc}x_c = 0, \qquad \sum_c \phi_{bc}\phi_{ce} = -r^2\phi_{be}, \tag{1.17}$$

and
(1.18)
$$F_{bc} = \frac{x_b x_c}{r^2(1+ar^2)} - \frac{8}{(x_3+3r)^2}\phi_{bc}, \quad F^{bc} = \frac{1+ar^2}{r^2}x_b x_c - \frac{(x_3+3r)^2}{8r^4}\phi_{bc}.$$

By means of (1.14), (1.16), the geodesic $(x_i(t))$ satisfies the following differential equations:

(1.19)
$$\begin{aligned}
&\frac{d^2x_e}{dt^2} + \sum_{b,c}\{{}_b{}^e{}_c\}\frac{dx_b}{dt}\frac{dx_c}{dt} + 2\sum_b\{{}_b{}^e{}_4\}\frac{dx_b}{dt}\frac{dx_4}{dt} \\
&= \frac{d^2x_e}{dt^2} + \frac{1+ar^2}{2}\Big[\Big\{\frac{2}{r^4}\Big(-\frac{2}{1+ar^2} + \frac{1}{(1+ar^2)^2} + \frac{48}{r(x_3+3r)^3}\Big\} \\
&\times x_e\Big(\sum_b x_b\frac{dx_b}{dt}\Big)^2 + \frac{16}{(x_3+3r)^3}x_e\Big(2\frac{dx_3}{dt}\sum_b x_b\frac{dx_b}{dt} - x_3\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}\Big) \\
&\quad + 2\Big(\frac{1}{r^2(1+ar^2)} - \frac{2}{(x_3+3r)^2}\Big)x_e\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}\Big] \\
&+ \frac{8(1+ar^2)}{r^2(x_3+3r)^3}x_e\Big(2\sum_b x_b\frac{dx_b}{dt}\sum_c\phi_{3c}\frac{dx_c}{dt} - x_3\sum_{b,c}\phi_{bc}\frac{dx_b}{dt}\frac{dx_c}{dt}\Big) \\
&- \frac{1}{r^4(x_3+3r)}\Big(2\sum_b\phi_{eb}\frac{dx_b}{dt}\sum_c\phi_{3c}\frac{dx_c}{dt} - \phi_{e3}\sum_{b,c}\phi_{bc}\frac{dx_b}{dt}\frac{dx_c}{dt}\Big) \\
&\qquad - 2\frac{1}{x_4}\frac{dx_e}{dt}\frac{dx_4}{dt} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d^2x_4}{dt^2} + \sum_{b,c}\{{}_b{}^4{}_c\}\frac{dx_b}{dt}\frac{dx_c}{dt} + \{{}_4{}^4{}_4\}\frac{dx_4}{dt}\frac{dx_4}{dt} \\
&= \frac{d^2x_4}{dt^2} - \frac{1+ax_4x_4}{x_4}\sum_{b,c}F_{bc}\frac{dx_b}{dt}\frac{dx_c}{dt} + \Big(-\frac{2}{x_4} + \frac{1}{x_4(1+ax_4x_4)}\Big)\frac{dx_4}{dt}\frac{dx_4}{dt} \\
&= \frac{d^2x_4}{dt^2} - \frac{1+ax_4x_4}{x_4}\sum_{b,c}\Big(\frac{x_bx_c}{r^2(1+ar^2)} - \frac{8}{(x_3+3r)^2}\phi_{bc}\Big)\frac{dx_b}{dt}\frac{dx_c}{dt} \\
&\qquad + \Big(-\frac{2}{x_4} + \frac{1}{x_4(1+ax_4x_4)}\Big)\frac{dx_4}{dt}\frac{dx_4}{dt} = 0
\end{aligned}$$

that is

(1.20)

$$\frac{d^2x_4}{dt^2}-\frac{1+ax_4x_4}{x_4}\Big\{\frac{1}{1+ar^2}\Big(\frac{dr}{dt}\Big)^2-\frac{8r^2}{(x_3+3r)^2}\Big(\Big(\frac{dr}{dt}\Big)^2-\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}\Big)\Big\}$$
$$+\Big(-\frac{2}{x_4}+\frac{1}{x_4(1+ax_4x_4)}\Big)\Big(\frac{dx_4}{dt}\Big)^2=0.$$

Since we have

$$\sum_b x_b\frac{dx_b}{dt}=r\frac{dr}{dt},\qquad \sum_c\phi_{bc}\frac{dx_c}{dt}=x_br\frac{dr}{dt}-r^2\frac{dx_b}{dt},$$
$$\sum_{b,c}\phi_{bc}\frac{dx_b}{dt}\frac{dx_c}{dt}=r^2\Big(\frac{dr}{dt}\frac{dr}{dt}-\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}\Big),$$

the coefficients of x_e of the above expression (1.19) are arranged as

$$\frac{1+ar^2}{2}\Big[\Big\{\frac{2}{r^4}\Big(-\frac{2}{1+ar^2}+\frac{1}{(1+ar^2)^2}\Big)+\frac{48}{r(x_3+3r)^3}\Big\}r^2\frac{dr}{dt}\frac{dr}{dt}$$
$$+\frac{16}{(x_3+3r)^3}\Big(2\frac{dx_3}{dt}r\frac{dr}{dt}-x_3\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}\Big)$$
$$+2\Big(\frac{1}{r^2(1+ar^2)}-\frac{2}{(x_3+3r)^2}\Big)\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}\Big]$$
$$+\frac{8(1+ar^2)}{(x_3+3r)^3r^2}\Big\{2r\frac{dr}{dt}\Big(x_3r\frac{dr}{dt}-r^2\frac{dx_3}{dt}\Big)-x_3r^2\Big(\frac{dr}{dt}\frac{dr}{dt}-\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}\Big)\Big\}$$
$$-\frac{1}{r^4(x_3+3r)}\Big\{2r\frac{dr}{dt}\Big(x_3r\frac{dr}{dt}-r^2\frac{dx_3}{dt}\Big)-x_3r^2\Big(\frac{dr}{dt}\frac{dr}{dt}-\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}\Big)\Big\}$$
$$=\Big\{-\frac{3}{r^2}+\frac{1}{r^2(1+ar^2)}+\frac{8(1+ar^2)}{(x_3+3r)^2}+\frac{3}{r(x_3+3r)}\Big\}\frac{dr}{dt}\frac{dr}{dt}$$
$$+\Big\{\frac{3}{r(x_3+3r)}-\frac{2(1+ar^2)}{(x_3+3r)^2}\Big\}\sum_b\frac{dx_b}{dt}\frac{dx_b}{dt}+\frac{2}{r(x_3+3r)}\frac{dr}{dt}\frac{dx_3}{dt},$$

which is the expression A in (1.10). Therefore (1.19) can be written as

$$\frac{d^2x_e}{dt^2}+Ax_e+B\frac{dx_e}{dt}+C\delta_{3e}=0, \tag{1.21}$$

where

$$B=\frac{2}{r(x_3+3r)}\Big(x_3\frac{dr}{dt}-r\frac{dx_3}{dt}\Big)-\frac{2}{x_4}\frac{dx_4}{dt}, \tag{1.22}$$

and C is the expression given by (1.11).

Now, for the curve $(x_1(t), x_2(t), x_3(t))$ in R^3 we compute the vector (V_1, V_2, V_3) given by

$$V_1 := \frac{dx_2}{dt}\frac{d^2x_3}{dt^2} - \frac{dx_3}{dt}\frac{d^2x_2}{dt^2}, \quad V_2 := \frac{dx_3}{dt}\frac{d^2x_1}{dt^2} - \frac{dx_1}{dt}\frac{d^2x_3}{dt^2},$$
$$V_3 := \frac{dx_1}{dt}\frac{d^2x_2}{dt^2} - \frac{dx_2}{dt}\frac{d^2x_1}{dt^2}.$$

Since we have

$$\begin{aligned}
&\frac{dx_b}{dt}\frac{d^2x_c}{dt^2} - \frac{dx_c}{dt}\frac{d^2x_b}{dt^2}\\
&\qquad = -\frac{dx_b}{dt}\Big(Ax_c + B\frac{dx_c}{dt} + C\delta_{3c}\Big) + \frac{dx_c}{dt}\Big(Ax_b + B\frac{dx_b}{dt} + C\delta_{3b}\Big)\\
&\qquad = A\Big(x_b\frac{dx_c}{dt} - x_c\frac{dx_b}{dt}\Big) + C\Big(\delta_{3b}\frac{dx_c}{dt} - \delta_{3c}\frac{dx_b}{dt}\Big),
\end{aligned}$$

$$\begin{aligned}
V_1 : V_2 : V_3 &= A\Big(x_2\frac{dx_3}{dt} - x_3\frac{dx_2}{dt}\Big) - C\frac{dx_2}{dt}\\
&\qquad : A\Big(x_3\frac{dx_1}{dt} - x_1\frac{dx_3}{dt}\Big) + C\frac{dx_1}{dt} : A\Big(x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}\Big)\\
&= \frac{x_2\frac{dx_3}{dt} - x_3\frac{dx_2}{dt}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}} - \frac{C}{A}\frac{\frac{dx_2}{dt}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}}\\
&\qquad : \frac{x_3\frac{dx_1}{dt} - x_1\frac{dx_3}{dt}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}} + \frac{C}{A}\frac{\frac{dx_1}{dt}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}} : 1.
\end{aligned}$$

In order to show that the curve is a plane curve in R^3 it is necessary and sufficient that the normal direction of its osculating plane is constant along it, therefore

$$\frac{x_2\frac{dx_3}{dt} - x_3\frac{dx_2}{dt}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}} - \frac{C}{A}\frac{\frac{dx_2}{dt}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}} = \text{constant},$$

and

$$\frac{x_3\frac{dx_1}{dt} - x_1\frac{dx_3}{dt}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}} + \frac{C}{A}\frac{\frac{dx_1}{dt}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}} = \text{constant}$$

must hold good. From the first equation we obtain the equivalent equation by differentiation as follows

$$\frac{x_2\frac{d^2x_3}{dt^2} - (x_3 + \frac{C}{A})\frac{d^2x_2}{dt^2} - \frac{d}{dt}(\frac{C}{A})\frac{dx_2}{dt}}{x_2\frac{dx_3}{dt} - (x_3 + \frac{C}{A})\frac{dx_2}{dt}} - \frac{x_1\frac{d^2x_2}{dt^2} - x_2\frac{d^2x_1}{dt^2}}{x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}} = 0,$$

which becomes

$$\Big(x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}\Big)\Big\{x_2\frac{d^2x_3}{dt^2} - (x_3 + \frac{C}{A})\frac{d^2x_2}{dt^2} - \frac{d}{dt}(\frac{C}{A})\frac{dx_2}{dt}\Big\}$$

$$- \Big\{ x_2 \frac{dx_3}{dt} - (x_3 + \frac{C}{A}) \frac{dx_2}{dt} \Big\} \Big(x_1 \frac{d^2x_2}{dt^2} - x_2 \frac{d^2x_1}{dt^2} \Big)$$

$$\begin{aligned}
&= \Big(x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \Big) \Big\{ -x_2 (Ax_3 + B\frac{dx_3}{dt} + C) \\
&\quad + (x_3 + \frac{C}{A})(Ax_2 + B\frac{dx_2}{dt}) - \frac{d}{dt}(\frac{C}{A}) \frac{dx_2}{dt} \Big\} \\
&\quad - \Big\{ x_2 \frac{dx_3}{dt} - (x_3 + \frac{C}{A}) \frac{dx_2}{dt} \Big\} \Big\{ -x_1 (Ax_2 + B\frac{dx_2}{dt}) + x_2 (Ax_1 + B\frac{dx_1}{dt}) \Big\} \\
&= (x_1 \frac{dx_2}{ddt} - x_2 \frac{dx_1}{dt}) \Big\{ -Bx_2 \frac{dx_3}{dt} + (Bx_3 + \frac{BC}{A} - \frac{d}{dt}(\frac{C}{A})) \frac{dx_2}{dt} \Big\} \\
&\qquad - \Big\{ x_2 \frac{dx_3}{dt} - (x_3 + \frac{C}{A}) \frac{dx_2}{dt} \Big\} B \Big(-x_1 \frac{dx_2}{dt} + x_2 \frac{dx_1}{dt} \Big) \\
&= -\frac{d}{dt}(\frac{C}{A}) \frac{dx_2}{dt} (x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt}) = 0,
\end{aligned}$$

that is

$$\frac{d}{dt}(\frac{C}{A}) \frac{dx_2}{dt} (x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt}) = 0. \tag{1.23}$$

Analogously, from the second equation we obtain

$$\frac{x_1 \frac{d^2x_3}{dt^2} - (x_3 + \frac{C}{A}) \frac{d^2x_1}{dt^2} - \frac{d}{dt}(\frac{C}{A}) \frac{dx_1}{dt}}{x_1 \frac{dx_3}{dt} - (x_3 + \frac{C}{A}) \frac{dx_1}{dt}} - \frac{x_1 \frac{d^2x_2}{dt^2} - x_2 \frac{d^2x_1}{dt^2}}{x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt}} = 0,$$

which becomes

$$\begin{aligned}
&\Big(x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \Big) \Big\{ (x_3 + \frac{C}{A}) \frac{d^2x_1}{dt^2} - x_1 \frac{d^2x_3}{dt^2} + \frac{d}{dt}(\frac{C}{A}) \frac{dx_1}{dt} \Big\} \\
&\qquad - \Big\{ (x_3 + \frac{C}{A}) \frac{dx_1}{dt} - x_1 \frac{dx_3}{dt} \Big\} \Big(x_1 \frac{d^2x_2}{dt^2} - x_2 \frac{d^2x_1}{dt^2} \Big) \\
&= \Big(x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \Big) \Big\{ -(x_3 + \frac{C}{A})(Ax_1 + B\frac{dx_1}{dt}) \\
&\quad + x_1 (Ax_3 + B\frac{dx_3}{dt} + C) + \frac{d}{dt}(\frac{C}{A}) \frac{dx_1}{dt} \Big\} \\
&\quad - \Big\{ (x_3 + \frac{C}{A}) \frac{dx_1}{dt} - x_1 \frac{dx_3}{dt} \Big\} \Big\{ -x_1 (Ax_2 + B\frac{dx_2}{dt}) + x_2 (Ax_1 + B\frac{dx_1}{dt}) \Big\} \\
&= \Big(x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \Big) \Big\{ Bx_1 \frac{dx_3}{dt} - (Bx_3 + \frac{BC}{A} - \frac{d}{dt}(\frac{C}{A})) \frac{dx_1}{dt} \Big\} \\
&\qquad - \Big\{ (x_3 + \frac{C}{A}) \frac{dx_1}{dt} - x_1 \frac{dx_3}{dt} \Big\} \Big\{ -B (x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt}) \Big\} \\
&= \frac{d}{dt}(\frac{C}{A}) \frac{dx_1}{dt} \Big(x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \Big) = 0,
\end{aligned}$$

that is

$$\frac{d}{dt}(\frac{C}{A})\frac{dx_1}{dt}\Big(x_1\frac{dx_2}{dt}-x_2\frac{dx_1}{dt}\Big)=0. \tag{1.24}$$

From the conditions (1.23) and (1.24) we obtain the equation:

$$\frac{d}{dt}(\frac{C}{A})\Big(x_1\frac{dx_2}{dt}-x_2\frac{dx_1}{dt}\Big)^2=0,$$

which is equivalent to

$$\frac{d}{dt}(\frac{C}{A})\Big(x_1\frac{dx_2}{dt}-x_2\frac{dx_1}{dt}\Big)=0,$$

and conversely this equation implies (1.23) and (1.24). Hence, we obtain the claim of this Proposition. □

2. Curvatures

For the metrics (1.1) and (1.2) in §1, we compute their curvature tensors, by using the Einstein convention for summation, given by

$$R_j{}^i{}_{hk}=\frac{\partial}{\partial x_h}\{_j{}^i{}_k\}-\frac{\partial}{\partial x_k}\{_j{}^i{}_h\}+\{_\ell{}^i{}_h\}\{_j{}^\ell{}_k\}-\{_\ell{}^i{}_k\}\{_j{}^\ell{}_h\} \tag{2.1}$$

where $i,j,h,k=1,2,3,4$ and we shall show the following

Proposition 3. *We have the equalities*

$$R_j{}^i{}_{hk}=\delta^i_h g_{jk}-\delta^i_k g_{jh}. \tag{2.2}$$

Proof. For the metric (1.1), using (1.6) we obtain easily

$$R_\alpha{}^e{}_{bc}=\delta^e_b g_{\alpha c}-\delta^e_c g_{\alpha b},\quad R_\alpha{}^4{}_{bc}=R_4{}^e{}_{bc}=R_4{}^4{}_{bc}=0,$$
$$R_4{}^e{}_{4c}=-\delta^e_c g_{44},\quad R_4{}^4{}_{4c}=R_b{}^e{}_{4c}=0,\quad R_b{}^4{}_{4c}=g_{bc},$$

where $\alpha,b,c,e=1,2,3$, and they are explained as (2.2).

Next, for the metric (1.2) we can explain (1.14) as

$$\{_b{}^e{}_c\}=x_e(Ax_bx_c+B\delta_{bc})-\frac{1}{r^4(x_3+3r)}(\phi_{eb}\phi_{3c}+\phi_{ec}\phi_{3b}-\phi_{e3}\phi_{bc}), \tag{2.3}$$

where we set

$$A=-\frac{2}{r^4}+\frac{1}{r^4(1+ar^2)}+\frac{8(1+ar^2)}{r^2(x_3+3r)^2},\quad B=\frac{1}{r^2}-\frac{8(1+ar^2)}{(x_3+3r)^2}. \tag{2.4}$$

We set

$$\frac{\partial A}{\partial x_b}=A_1x_b+A_2\delta_{3b},\quad \frac{\partial B}{\partial x_b}=B_1x_b+B_2\delta_{3b},$$

where

$$A_1=\frac{8}{r^6}-\frac{4}{r^6(1+ar^2)}-\frac{2a}{r^4(1+ar^2)^2}-\frac{16}{r^4(x_3+3r)^2}-\frac{48(1+ar^2)}{r^3(x_3+3r)^3},$$

$$A_2 = -\frac{16}{r^2(x_3+3r)^3},$$
$$B_1 = -\frac{2}{r^4} - \frac{16a}{(x_3+3r)^2} + \frac{48(1+ar^2)}{r(x_3+3r)^3}, \quad B_2 = \frac{16(1+ar^2)}{(x_3+3r)^3}.$$

Regarding ϕ_{bc}, in addition to (1.17) we have

$$\frac{\partial \phi_{bc}}{\partial x_e} = \delta_{be}x_c + \delta_{ce}x_b - 2\delta_{bc}x_e. \tag{2.5}$$

First, we compute

$$R_\alpha{}^e{}_{bc} = \Big(\frac{\partial}{\partial x_b}\{{}_\alpha{}^e{}_c\} - \frac{\partial}{\partial x_c}\{{}_\alpha{}^e{}_b\} + \{{}_\varepsilon{}^e{}_b\}\{{}_\alpha{}^\varepsilon{}_c\} - \{{}_\varepsilon{}^e{}_c\}\{{}_\alpha{}^\varepsilon{}_b\}\Big)$$
$$+ \{{}_4{}^e{}_b\}\{{}_\alpha{}^4{}_c\} - \{{}_4{}^e{}_c\}\{{}_\alpha{}^4{}_b\},$$

where $\alpha, \beta, \varepsilon, b, c, e = 1, 2, 3$. We obtain easily

$$\{{}_4{}^e{}_b\}\{{}_\alpha{}^4{}_c\} - \{{}_4{}^e{}_c\}\{{}_\alpha{}^4{}_b\} = (1 + ax_4x_4)(\delta_b^e g_{\alpha c} - \delta_c^e g_{\alpha b}). \tag{2.6}$$

by (1.16). Then, we have by (2.3)

$$\frac{\partial}{\partial x_b}\{{}_\alpha{}^e{}_c\} - \frac{\partial}{\partial x_c}\{{}_\alpha{}^e{}_b\} + \{{}_\varepsilon{}^e{}_b\}\{{}_\alpha{}^\varepsilon{}_c\} - \{{}_\varepsilon{}^e{}_c\}\{{}_\alpha{}^\varepsilon{}_b\}$$
$$= \frac{\partial}{\partial x_b}\{x_e(Ax_\alpha x_c + B\delta_{\alpha c}) - \frac{1}{r^4(x_3+3r)}(\phi_{e\alpha}\phi_{3c} + \phi_{ec}\phi_{3\alpha} - \phi_{e3}\phi_{\alpha c})\}$$
$$- \frac{\partial}{\partial x_c}\{x_e(Ax_\alpha x_b + B\delta_{\alpha b}) - \frac{1}{r^4(x_3+3r)}(\phi_{e\alpha}\phi_{3b} + \phi_{eb}\phi_{3\alpha} - \phi_{e3}\phi_{\alpha b})\}$$
$$+ \{x_e(Ax_\varepsilon x_b + B\delta_{\varepsilon b}) - \frac{1}{r^4(x_3+3r)}(\phi_{e\varepsilon}\phi_{3b} + \phi_{eb}\phi_{3\varepsilon} - \phi_{e3}\phi_{\varepsilon b})\}$$
$$\times \{x_\varepsilon(Ax_\alpha x_c + B\delta_{\alpha c}) - \frac{1}{r^4(x_3+3r)}(\phi_{\varepsilon\alpha}\phi_{3c} + \phi_{\varepsilon c}\phi_{3\alpha} - \phi_{\varepsilon 3}\phi_{\alpha c})\}$$
$$- \{x_e(Ax_\varepsilon x_c + B\delta_{\varepsilon c}) - \frac{1}{r^4(x_3+3r)}(\phi_{e\varepsilon}\phi_{3c} + \phi_{ec}\phi_{3\varepsilon} - \phi_{e3}\phi_{\varepsilon c})\}$$
$$\times \{x_\varepsilon(Ax_\alpha x_b + B\delta_{\alpha b}) - \frac{1}{r^4(x_3+3r)}(\phi_{\varepsilon\alpha}\phi_{3b} + \phi_{\varepsilon b}\phi_{3\alpha} - \phi_{\varepsilon 3}\phi_{\alpha b})\}$$

which is arranged by means of (1.7), (2.4) and (2.5) as

$$= \delta_b^e\Pi_1 - \delta_c^e\Pi_2 + (\delta_{3b}x_c - \delta_{3c}x_b)\Pi_3 + (\delta_{\alpha b}x_c - \delta_{\alpha c}x_b)\Pi_4$$
$$+ (\delta_{3b}\delta_{\alpha c} - \delta_{3c}\delta_{\alpha b})\Pi_5,$$

where we set

$$\Pi_1 = Ax_\alpha x_c + B\delta_{\alpha c} + \Big(\frac{4}{r^4(x_3+3r)} + \frac{3}{r^3(x_3+3r)^2}\Big)\phi_{3\alpha}x_c$$

$$+\frac{1}{r^2(x_3+3r)^2}\phi_{3\alpha}\delta_{3c}-\frac{1}{r^4(x_3+3r)}\{(3\phi_{3\alpha}-2r^2\delta_{3\alpha})x_c+2r^2x_3\delta_{\alpha c}\} + \frac{1}{r^4(x_3+3r)^2}(\phi_{3\alpha}\phi_{3c}-\phi_{33}\phi_{\alpha c}),$$

and

$$\Pi_2 = Ax_\alpha x_b + B\delta_{\alpha b} + \Big(\frac{4}{r^4(x_3+3r)}+\frac{3}{r^3(x_3+3r)^2}\Big)\phi_{3\alpha}x_b + \frac{1}{r^2(x_3+3r)^2}\phi_{3\alpha}\delta_{3b}-\frac{1}{r^4(x_3+3r)}\{(3\phi_{3\alpha}-2r^2\delta_{3\alpha})x_b+2r^2x_3\delta_{\alpha b}\} + \frac{1}{r^4(x_3+3r)^2}(\phi_{3\alpha}\phi_{3b}-\phi_{33}\phi_{\alpha b}),$$

$$\Pi_3 = x_e x_\alpha A_2 + \Big(\frac{4}{r^4(x_3+3r)}+\frac{3}{r^3(x_3+3r)^2}\Big)\phi_{e\alpha} + \frac{1}{r^4(x_3+3r)^2}(\phi_{e\alpha}x_3+\phi_{3\alpha}x_e-\phi_{e3}x_\alpha)+\frac{1}{r^4(x_3+3r)}(2r^2\delta_{e\alpha}-3\phi_{e\alpha}) - \frac{2B}{r^2(x_3+3r)}x_e x_\alpha + \frac{1}{r^4(x_3+3r)^2}(\phi_{e3}x_\alpha-\phi_{3\alpha}x_e),$$

$$\Pi_4 = Ax_e - B_1x_e - \Big(\frac{4}{r^4(x_3+3r)}+\frac{3}{r^3(x_3+3r)^2}\Big)\phi_{e3} - \frac{1}{r^4(x_3+3r)}(2x_3x_e+2r^2\delta_{e3}-3\phi_{e3})-(Ar^2+B)Bx_e + \frac{2Bx_ex_3}{r^2(x_3+3r)}+\frac{1}{r^4(x_3+3r)^2}(\phi_{33}x_e-\phi_{e3}x_3),$$

$$\Pi_5 = x_eB_2 - \frac{2}{r^2(x_3+3r)}x_e + \frac{2}{x_3+3r}Bx_e.$$

We compute these expressions in detail. We have first

$$\Pi_1 = \Big\{A + \Big(\frac{4}{r^4(x_3+3r)}+\frac{3}{r^3(x_3+3r)^2}\Big)x_3 - \frac{3x_3}{r^4(x_3+3r)} + \frac{1}{r^4(x_3+3r)^2}(x_3x_3-\phi_{33})\Big\}x_\alpha x_c + \Big\{B-\frac{2x_3}{r^2(x_3+3r)} + \frac{\phi_{33}}{r^2(x_3+3r)^2}\Big\}\delta_{\alpha c} - \Big(\frac{4}{r^2(x_3+3r)}+\frac{3}{r(x_3+3r)^2}\Big)\delta_{3\alpha}x_c + \frac{1}{r^2(x_3+3r)^2}(x_3x_\alpha\delta_{3c}-r^2\delta_{3\alpha}\delta_{3c})+\frac{5}{r^2(x_3+3r)}\delta_{3\alpha}x_c - \frac{1}{r^2(x_3+3r)^2}(\delta_{3\alpha}x_3x_c+x_3x_\alpha\delta_{3c}-r^2\delta_{3\alpha}\delta_{3c})$$

$$
\begin{aligned}
&= \Big\{-\frac{2}{r^4} + \frac{1}{r^4(1+ar^2)} + \frac{8(1+ar^2)}{r^2(x_3+3r)^2} + \frac{x_3}{r^4(x_3+3r)} \\
&\qquad + \frac{3x_3}{r^3(x_3+3r)^2} + \frac{1}{r^2(x_3+3r)^2}\Big\}x_\alpha x_c + \Big\{\frac{1}{r^2} - \frac{8(1+ar^2)}{(x_3+3r)^2} \\
&\qquad - \frac{2x_3}{r^4(x_3+3r)} + \frac{x_3x_3}{r^2(x_3+3r)^2} - \frac{1}{(x_3+3r)^2}\Big\}\delta_{\alpha c} \\
&= \Big\{-\frac{2}{r^4} + \frac{1}{r^4(1+ar^2)} + \frac{8(1+ar^2)}{r^2(x_3+3r)^2} + \frac{(x_3+3r)^2-8r^2}{r^4(x_3+3r)^2}\Big\}x_\alpha x_c \\
&\qquad + \Big\{\frac{1}{r^2} - \frac{8(1+ar^2)}{(x_3+3r)^2} + \frac{-2x_3(x_3+3r)+x_3x_3-r^2}{r^2(x_3+3r)^2}\Big\}\delta_{\alpha c} \\
&= \Big\{-\frac{1}{r^4} + \frac{1}{r^4(1+ar^2)} + \frac{8ar^2}{r^2(x_3+3r)^2}\Big\}x_\alpha x_c \\
&\qquad + \Big\{\frac{8}{(x_3+3r)^2} - \frac{8(1+ar^2)}{(x_3+3r)^2}\Big\}\delta_{\alpha c} \\
&= -a\Big\{\Big(\frac{1}{r^2(1+ar^2)} - \frac{8}{(x_3+3r)^2}\Big)x_\alpha x_c + \frac{8r^2}{(x_3+3r)^2}\delta_{\alpha c}\Big\} \\
&= -aF_{\alpha c}
\end{aligned}
$$

by (1.12). We obtain analogously

$$\Pi_2 = -aF_{\alpha b}.$$

Next, we see that

$$
\begin{aligned}
\Pi_3 &= -\frac{16(1+ar^2)}{r^2(x_3+3r)^3}x_e x_\alpha + \Big(\frac{4}{r^4(x_3+3r)} + \frac{3}{r^3(x_3+3r)^2}\Big)(x_e x_\alpha - r^2\delta_{e\alpha}) \\
&\qquad + \frac{1}{r^4(x_3+3r)^2}\{x_e x_\alpha x_3 - r^2(\delta_{e\alpha}x_3 + \delta_{3\alpha}x_e - \delta_{e3}x_\alpha)\} \\
&\qquad + \frac{1}{r^4(x_3+3r)}(-3x_e x_\alpha + 5r^2\delta_{e\alpha}) - \frac{2B}{r^2(x_3+3r)}x_e x_\alpha \\
&\qquad + \frac{1}{r^2(x_3+3r)^2}(\delta_{3\alpha}x_e - \delta_{e3}x_\alpha) \\
&= \Big\{-\frac{16(1+ar^2)}{r^2(x_3+3r)^3} + \frac{4}{r^4(x_3+3r)} + \frac{3}{r^3(x_3+3r)^2} + \frac{x_3}{r^4(x_3+3r)^2} \\
&\qquad - \frac{3}{r^4(x_3+3r)} - \frac{2}{r^2(x_3+3r)}\Big(\frac{1}{r^2} - \frac{8(1+ar^2)}{(x_3+3r)^2}\Big)\Big\}x_e x_\alpha \\
&\qquad - \Big(\frac{4}{r^2(x_3+3r)} + \frac{3}{r(x_3+3r)^2} + \frac{x_3}{r^2(x_3+3r)^2} - \frac{5}{r^2(x_3+3r)}\Big)\delta_{e\alpha} \\
&= 0,
\end{aligned}
$$

and

$$\Pi_4 = (A - B_1 - (Ar^2 + B)B + \frac{2Bx_3}{r^2(x_3+3r)})x_e - \frac{5}{r^4(x_3+3r)}\phi_{3e} + \frac{1}{r^4(x_3+3r)}(x_3x_e - 5r^2\delta_{3e}) + \frac{1}{r^4(x_3+3r)^2}\phi_{33}x_e$$

$$= \Big\{A - B_1 + \Big(\frac{2x_3}{r^2(x_3+3r)} - Ar^2 - B\Big)B - \frac{4x_3}{r^4(x_3+3r)} + \frac{\phi_{33}}{r^4(x_3+3r)^2}\Big\}x_e$$

$$= \Big\{-\frac{2}{r^4} + \frac{1}{r^4(1+ar^2)} + \frac{8(1+ar^2)}{r^2(x_3+3r)^2} + \frac{2}{r^4} + \frac{16a}{(x_3+3r)^2} - \frac{48(1+ar^2)}{r(x_3+3r)^3} + \Big(\frac{2x_3}{r^2(x_3+3r)} + \frac{1}{r^2} - \frac{1}{r^2(1+ar^2)}\Big)\Big(\frac{1}{r^2} - \frac{8(1+ar^2)}{(x_3+3r)^2}\Big) - \frac{4x_3}{r^4(x_3+3r)} + \frac{x_3x_3}{r^4(x_3+3r)^2} - \frac{1}{r^2(x_3+3r)^2}\Big\}x_e$$

$$= \Big\{\frac{1}{r^4(1+ar^2)} + \frac{8(1+ar^2)}{r^2(x_3+3r)^2} + \frac{16a}{(x_3+3r)^2} - \frac{48(1+ar^2)}{r(x_3+3r)^3} + \frac{2x_3}{r^4(x_3+3r)} + \frac{1}{r^4} - \frac{1}{r^4(1+ar^2)} - \frac{16(1+ar^2)x_3}{r^2(x_3+3r)^3} - \frac{8(1+ar^2)}{r^2(x_3+3r)^2} + \frac{8}{r^2(x_3+3r)^2} - \frac{4x_3}{r^4(x_3+3r)} + \frac{x_3x_3}{r^4(x_3+3r)^2} - \frac{1}{r^2(x_3+3r)^2}\Big\}x_e$$

$$= \Big\{\frac{16a}{(x_3+3r)^2} - \frac{16(1+ar^2)}{r^2(x_3+3r)^2} - \frac{2x_3}{r^4(x_3+3r)} + \frac{1}{r^4} + \frac{8}{r^2(x_3+3r)^2} + \frac{x_3x_3}{r^4(x_3+3r)^2} - \frac{1}{r^2(x_3+3r)^2}\Big\}x_e$$

$$= \Big\{-\frac{16}{r^2(x_3+3r)^2} - \frac{2x_3}{r^4(x_3+3r)} + \frac{1}{r^4} + \frac{7}{r^2(x_3+3r)^2} + \frac{x_3x_3}{r^4(x_3+3r)^2}\Big\}x_e$$

$$= 0,$$

and

$$\Pi_5 = x_eB_2 - \frac{2}{r^2(x_3+3r)}x_e + \frac{2B}{x_3+3r}x_e$$

$$= \Big\{\frac{16(1+ar^2)}{(x_3+3r)^3} - \frac{2}{r^2(x_3+3r)} + \frac{2}{x_3+3r}\Big(\frac{1}{r^2} - \frac{8(1+ar^2)}{(x_3+3r)^2}\Big)\Big\}x_e$$

$$= 0.$$

From these results, we obtain the equalities

$$(2.7)\quad \frac{\partial}{\partial x_b}\{{}_{\alpha}{}^{e}{}_{c}\}-\frac{\partial}{\partial x_c}\{{}_{\alpha}{}^{e}{}_{b}\}+\{{}_{\varepsilon}{}^{e}{}_{b}\}\{{}_{\alpha}{}^{\varepsilon}{}_{c}\}-\{{}_{\varepsilon}{}^{e}{}_{c}\}\{{}_{\alpha}{}^{\varepsilon}{}_{b}\}=-a(\delta^e_b F_{\alpha c}-\delta^e_c F_{\alpha b}).$$

By means of (2.6) and (2.7) we obtain

$$R_{\alpha}{}^{e}{}_{bc}=\Big(\frac{1+ax_4x_4}{x_4x_4}-a\Big)(\delta^e_b F_{\alpha c}-\delta^e_c F_{\alpha b}),$$

i.e.,

$$(2.8)\qquad R_{\alpha}{}^{e}{}_{bc}=\frac{1}{x_4x_4}(\delta^e_b F_{\alpha c}-\delta^e_c F_{\alpha b})=\delta^e_b g_{\alpha c}-\delta^e_c g_{\alpha b}.$$

Next, by (1.15), (1.16) and (2.3), we have

$$\begin{aligned}
R_{\alpha}{}^{4}{}_{bc}&=\frac{\partial}{\partial x_b}\{{}_{\alpha}{}^{4}{}_{c}\}-\frac{\partial}{\partial x_c}\{{}_{\alpha}{}^{4}{}_{b}\}+\{{}_{\varepsilon}{}^{4}{}_{b}\}\{{}_{\alpha}{}^{\varepsilon}{}_{c}\}-\{{}_{\varepsilon}{}^{4}{}_{c}\}\{{}_{\alpha}{}^{\varepsilon}{}_{b}\}\\
&=-\frac{1+ax_4x_4}{x_4}\Big(\frac{\partial}{\partial x_b}F_{\alpha c}-\frac{\partial}{\partial x_c}F_{\alpha b}\Big)-\frac{1+ax_4x_4}{x_4}F_{\varepsilon b}\Big(x_\varepsilon(Ax_\alpha x_c+B\delta_{\alpha c})\\
&\quad-\frac{1}{r^4(x_3+3r)}(\phi_{\varepsilon\alpha}\phi_{3c}+\phi_{\varepsilon c}\phi_{3\alpha}-\phi_{\varepsilon 3}\phi_{\alpha c})\Big)+\frac{1+ax_4x_4}{x_4}F_{\varepsilon c}\\
&\quad\times\Big(x_\varepsilon(Ax_\alpha x_b+B\delta_{\alpha b})-\frac{1}{r^4(x_3+3r)}(\phi_{\varepsilon\alpha}\phi_{3b}+\phi_{\varepsilon b}\phi_{3\alpha}-\phi_{\varepsilon 3}\phi_{\alpha b})\Big).
\end{aligned}$$

Since we have

$$F_{\alpha c}=\frac{B}{1+ar^2}x_\alpha x_c+\frac{8r^2}{(x_3+3r)^2}\delta_{\alpha c},$$

we obtain

$$\begin{aligned}
\frac{\partial F_{\alpha c}}{\partial x_b}&=\Big(\frac{2}{r^4(1+ar^2)^2}-\frac{4}{r^4(1+ar^2)}+\frac{48}{r(x_3+3r)^3}\Big)x_\alpha x_b x_c\\
&\quad+\frac{16}{(x_3+3r)^3}\delta_{3b}x_\alpha x_c+\Big(\frac{1}{r^2(1+ar^2)}-\frac{8}{(x_3+3r)^2}\Big)(\delta_{\alpha b}x_c+\delta_{bc}x_\alpha)\\
&\quad+\frac{16}{(x_3+3r)^3}\phi_{3b}\delta_{\alpha c}
\end{aligned}$$

and we have also

$$(2.9)\qquad F_{bc}x_c=\frac{x_b}{1+ar^2},\quad F_{bc}\phi_{ce}=\frac{8r^2}{(x_3+3r)^2}\phi_{be}.$$

Therefore, the above expression becomes

$$\begin{aligned}
R_{\alpha}{}^{4}{}_{bc}=-\frac{1+ax_4x_4}{x_4}\Big\{&\frac{16}{(x_3+3r)^3}x_\alpha(\delta_{3b}x_c-\delta_{3c}x_b)\\
&+\Big(\frac{1}{r^2(1+ar^2)}-\frac{8}{(x_3+3r)^2}\Big)(\delta_{\alpha b}x_c-\delta_{\alpha c}x_b)
\end{aligned}$$

$$+\frac{16}{(x_3+3r)^3}(\phi_{3b}\delta_{\alpha c}-\phi_{3c}\delta_{\alpha b})-\frac{1}{1+ar^2}B(\delta_{\alpha b}x_c-\delta_{\alpha c}x_b)$$
$$+\frac{16}{r^2(x_3+3r)^3}(\phi_{b3}\phi_{\alpha c}-\phi_{c3}\phi_{\alpha b})\Big\}=0.$$

Next, we have

$$\begin{aligned}R_4{}^e{}_{bc}&=\frac{\partial}{\partial x_b}\{{}_4{}^e{}_c\}-\frac{\partial}{\partial x_c}\{{}_4{}^e{}_b\}+\{{}_\varepsilon{}^e{}_b\}\{{}_4{}^\varepsilon{}_c\}-\{{}_\varepsilon{}^e{}_c\}\{{}_4{}^\varepsilon{}_b\}\\&=-\frac{1}{x_4}\{{}_c{}^e{}_b\}+\frac{1}{x_4}\{{}_b{}^e{}_c\}=0,\end{aligned}$$

and

$$\begin{aligned}R_4{}^4{}_{bc}&=\frac{\partial}{\partial x_b}\{{}_4{}^4{}_c\}-\frac{\partial}{\partial x_c}\{{}_4{}^4{}_b\}+\{{}_\varepsilon{}^4{}_b\}\{{}_4{}^\varepsilon{}_c\}-\{{}_\varepsilon{}^4{}_c\}\{{}_4{}^\varepsilon{}_b\}\\&=\frac{1+ax_4x_4}{x_4}F_{\varepsilon b}\frac{1}{x_4}\delta^\varepsilon_c-\frac{1+ax_4x_4}{x_4}F_{\varepsilon c}\frac{1}{x_4}\delta^\varepsilon_b=0.\end{aligned}$$

Analogously, we obtain easily the equalities

$$R_b{}^e{}_{4c}=0.$$

We have also

$$\begin{aligned}R_b{}^4{}_{4c}&=\frac{\partial}{\partial x_4}\{{}_b{}^4{}_c\}-\frac{\partial}{\partial x_c}\{{}_b{}^4{}_4\}+\{{}_\varepsilon{}^4{}_4\}\{{}_b{}^\varepsilon{}_c\}-\{{}_\varepsilon{}^4{}_c\}\{{}_b{}^\varepsilon{}_4\}+\{{}_4{}^4{}_4\}\{{}_b{}^4{}_c\}\\&=-\frac{\partial}{\partial x_4}\Big(\frac{1+ax_4x_4}{x_4}F_{bc}\Big)-\frac{1+ax_4x_4}{x_4}F_{\varepsilon c}\frac{1}{x_4}\delta^\varepsilon_b\\&\qquad+\frac{1+2ax_4x_4}{x_4(1+ax_4x_4)}\frac{1+ax_4x_4}{x_4}F_{bc}\\&=\frac{1-ax_4x_4}{x_4x_4}F_{bc}-\frac{1+ax_4x_4}{x_4x_4}F_{bc}+\frac{1+2ax_4x_4}{x_4x_4}F_{bc}\\&=\frac{1}{x_4x_4}F_{bc},\end{aligned}$$

i.e.,

$$R_b{}^4{}_{4c}=\frac{1}{x_4x_4}F_{bc}=g_{bc}.$$

Last, we have

$$\begin{aligned}R_4{}^e{}_{4c}&=\frac{\partial}{\partial x_4}\{{}_4{}^e{}_c\}-\frac{\partial}{\partial x_c}\{{}_4{}^e{}_4\}+\{{}_\varepsilon{}^e{}_4\}\{{}_4{}^\varepsilon{}_c\}-\{{}_\varepsilon{}^e{}_c\}\{{}_4{}^\varepsilon{}_4\}+\{{}_4{}^e{}_4\}\{{}_4{}^4{}_c\}\\&\qquad-\{{}_4{}^e{}_c\}\{{}_4{}^4{}_4\}\\&=\frac{1}{x_4x_4}\delta^e_c+\frac{1}{x_4}\delta^e_\varepsilon\frac{1}{x_4}\delta^\varepsilon_c-\frac{1}{x_4}\delta^e_c\frac{1+2ax_4x_4}{x_4(1+ax_4x_4)}\end{aligned}$$

$$= \Big(\frac{2}{x_4x_4} - \frac{1+2ax_4x_4}{x_4x_4(1+ax_4x_4)}\Big)\delta^e_c$$
$$= \frac{1}{x_4x_4(1+ax_4x_4)}\delta^e_c,$$

i.e.,

$$R_4{}^e{}_{4c} = -\frac{1}{x_4x_4}\delta^e_c F_{44} = -\delta^e_c g_{44}.$$

We obtain easily

$$R_4{}^4{}_{4c} = 0.$$

These results can be explained simply as

$$R_j{}^i{}_{hk} = \frac{1}{x_4x_4}(\delta^i_h F_{jk} - \delta^i_k F_{jh}) = \delta^i_h g_{jk} - \delta^i_k g_{jh}.$$

□

3. Related Otsuki Connections

Let $\{{}_j{}^i{}_h\}$ be the Levi-Civita connection by a pseudo-Riemannian metric $ds^2 = g_{ij}du^i du^j$ and $P = (P^i_j)$ a tenser field of type (1,1). We consider the general (Otsuki) connection $\Gamma = (P^i_j, \Gamma_j{}^i{}_h) = P(\delta^i_j, \{{}_j{}^i{}_h\})$, where we set

$$\Gamma_j{}^i{}_h = P^i_k\{{}_j{}^k{}_h\}.$$

The curvature tensor of Γ is defined by

$$\text{(3.1)}\quad \bar{R}_j{}^i{}_{hk} = \Big\{P^i_\ell\Big(\frac{\partial}{\partial u^h}\Gamma_m{}^\ell{}_k - \frac{\partial}{\partial u^k}\Gamma_m{}^\ell{}_h\Big) + \Gamma_\ell{}^i{}_h\Gamma_m{}^\ell{}_k - \Gamma_\ell{}^i{}_k\Gamma_m{}^\ell{}_h\Big\}P^m_j - \delta^i_{m;h}\Lambda_j{}^m{}_k + \delta^i_{m;k}\Lambda_j{}^m{}_h$$

where ";" denotes the covariant derivatives by Γ and

$$\Lambda_j{}^i{}_h = \Gamma_j{}^i{}_h - \frac{\partial}{\partial u^h}P^i_j$$

are the covariant components of Γ. For the tensor field Q^i_j, $Q^i_{j;h}$ are defined as

$$\text{(3.2)}\qquad Q^i_{j;h} = P^i_\ell\frac{\partial Q^\ell_m}{\partial u^h}P^m_j + \Gamma_k{}^i{}_h Q^k_m P^m_j - P^i_\ell Q^\ell_m \Lambda_j{}^m{}_h$$

and hence we have

$$\delta^i_{j;h} = \Gamma_k{}^i{}_h P^k_j - P^i_k\Lambda_j{}^k{}_h = P^i_k P^k_{j}{}_{,h}$$

where "," denotes the covariant derivatives by $\{{}_j{}^i{}_h\}$. Therefore $\bar{R}_j{}^i{}_{hk}$ can be written as

$$\text{(3.3)}\qquad \bar{R}_j{}^i{}_{hk} = P^i_\ell(P^\ell_k R_m{}^p{}_{hk}P^m_j + P^\ell_{m,h}P^m_{j}{}_{,k} - P^\ell_{m,k}P^m_{j}{}_{,h})$$

for this case. The Ricci curvature of Γ defined by

$$\bar{R}_{jk} := \bar{R}_j{}^i{}_{ik}$$

becomes as

$$\bar{R}_{jk} = P^i_\ell (P^\ell_p R_m{}^p{}_{ik} P^m_j + P^\ell_{m,i} P^m_j{}_{,k} - P^\ell_{m,k} P^m_j{}_{,h}) \tag{3.4}$$

and we have

$$\begin{aligned}\bar{R}_{jk} - \bar{R}_{kj} = P^i_\ell \{ & P^\ell_p (R_m{}^p{}_{ik} P^m_j - R_m{}^p{}_{ij} P^m_k) \\ & + P^\ell_{m,i}(P^m_j{}_{,k} - P^m_k{}_{,j}) - P^\ell_{m,k} P^m_j{}_{,i} + P^\ell_{m,j} P^m_k{}_{,i} \}\end{aligned} \tag{3.5}$$

which does not vanish in general.

Now, suppose the metric $g_{ij} du^i du^j$ be the metric (1.1) or (1.2) on R^4_+. Then by means of Proposition 3, we obtain their curvature tensor for Γ as follows.

$$\bar{R}_j{}^i{}_{hk} = P^i_\ell \{ P^\ell_p (\delta^p_h g_{mk} - \delta^p_k g_{mh}) P^m_j + P^\ell_{m,h} P^m_j{}_{,k} - P^\ell_{m,k} P^m_j{}_{,h} \},$$

i.e.

$$\bar{R}_j{}^i{}_{hk} = P^i_j (P^\ell_h P_{jk} - P^\ell_k P_{jh} + P^\ell_{m,h} P^m_j{}_{,k} - P^\ell_{m,k} P^m_j{}_{,h}), \tag{3.6}$$

where we set

$$P_{jk} = g_{mk} P^m_j. \tag{3.7}$$

And Ricci tensor $\bar{R}_{jk} = \bar{R}_j{}^\ell{}_{\ell k}$ can be written as

$$\bar{R}_{jk} = (trP^2) P_{jk} - (P^2)^i_k P_{ji} + P^i_\ell (P^\ell_{m,i} P^m_j{}_{,k} - P^\ell_{m,k} P^m_j{}_{,i}), \tag{3.8}$$

where we set

$$(P^2)^i_j = P^i_\ell P^\ell_j.$$

Remark. We see that geodesics of $\Gamma = P(\delta^i_j, \{{}_j{}^i{}_h\})$ are the same ones of the metrics (1.1) or (1.2) including their affine parameters and we can take P so that it vanishes the singularities of these metrics and it gives a kind of rotation of the spaces.

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