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WHEN IS $RP^n \times Spin(n)$ DIFFEOMORPHIC TO $S^n \times SO(n)$ AND HOW

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Abstract. We show that the spaces in the title, whose corresponding homotopy groups are isomorphic, are homotopy equivalent only when $n = 3$ or $n = 7$. We produce an explicit diffeomorphism in the only non trivial case, $n = 7$.

Introduction

For $n \geq 3$, $Spin(n)$ is the universal covering group of the rotation group $SO(n)$, whose fundamental group is Z_2 (see [5]). This implies that $S^n \times Spin(n)$ is the fundamental cover of both $S^n \times SO(n)$ and of $RP^n \times Spin(n)$, for all $n \geq 3$ and that the corresponding homotopy groups of these two spaces are isomorphic. For $n = 3$ the algebra of quaternions implies that $Spin(3)$ is isomorphic with S^3 and $SO(3)$ is isomorphic with RP^3 (see [5]). A simple switching of the factors provides the diffeomorphism $RP^3 \times Spin(3) \cong S^3 \times SO(3)$. Two obvious question arise: The one in the title and "To what extend does the existence of an algebra structure on R^{n+1} describe adequately the solution to the first question?".

In section 1 we show that if $RP^n \times Spin(n)$ is homotopy equivalent to $S^n \times SO(n)$ then S^n is an H-space and therefore $n = 3$ or 7 (see [1]) (remember, here $n \geq 3$).

In section 2 we use the Cayley algebra and the principle of triality (see [2]) to produce an explicit formula for a diffeomorphism in the case $n = 7$.

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1. Topological obstructions

If n is even RP^n is not orientable and there is no homotopy equivalence between $RP^n \times Spin(n)$ and $S^n \times SO(n)$. So, let n be odd and let $h : S^n \times SO(n) \to RP^n \times Spin(n)$ be a homotopy equivalence. Composing with the obvious inclusions and projections we have:

$$RP^n \to RP^n \times Spin(n) \to S^n \times SO(n) \to SO(n) \to$$
$$\to S^n \times SO(n) \to RP^n \times Spin(n) \to RP^n$$

where we have employed first h^{-1} and then h. The induced maps in rational cohomology compose to

$$F : H^*(RP^n; Q) \to H^*(RP^n; Q).$$

From [5], p. 177, Th. 2.19 (2) and Cor. 3.15 (2), p. 122 we see that the projection induces an isomorphism $H^*(SO(n); Q) \cong H^*(Spin(n); Q)$ which is, for odd n, isomorphic to the exterior algebra in the generators e_3, e_7, …, e_{2n-3}. The projection $S^n \to RP^n$ also induces an isomorphism in cohomology with rational coefficients and $H^*(RP^n; Q)$ is the exterior algebra in one generator, s, of degree n. If we follow the composition F around we easily conclude that $F(s) = \lambda s$.
Claim: λ is an odd integer.
Proof: It is easy to see that the maps $f : RP^n \to SO(n)$ and $g : SO(n) \to RP^n$, composed as is obvious from some of the maps that make up F, induce isomorphisms on the fundamental groups that are isomorphic to Z_2. Consequently, $(g \circ f)^* : H^*(RP^n; Z_2) \to H^*(RP^n; Z_2)$ is an isomorphism since $H^*(RP^n; Z_2)$ is generated by an element of degree 1, the dual of the generator of the fundamental group. In particular, $(g \circ f)^*$ is an isomorphism.
Corollary: $(g \circ f)^* : H^*(RP^n; Z) \to H^*(RP^n; Z)$ is multiplication by an odd integer.
Corollary: $(g \circ f)^* : H^*(RP^n; Q) \to H^*(RP^n; Q)$ is multiplication by an odd integer.

As a consequence we have that the map $g \circ f$ is a homotopy equivalence on the 2-primary localizations of RP^n and $SO(n)$, which implies that $RP^n_{(2)}$ is an H-space (see [4]). Localization is a functor that preserves coverings, so $S^n_{(2)}$ is an H-space. Now apply the 2-primary localization to the Hopf construction (see [6]) to obtain a map $S^{2n+1}_{(2)} \to S^{n+1}_{(2)}$, whose Hopf invariant is unit in $Z_{(2)}$, the integers localized at 2. Corollary 5.13, p. 89 of [4] implies now that some odd integer multiple of this must arise from localizing an actual map $S^{2n+1} \to S^{n+1}$, Corollary 15.14, p. 409 of [5] implies now, using [1], that $n = 3$ or 7 (recall that $n \geq 3$).

2. The diffeomorphism

Recall (see e.g. [3]) that $Spin(8)$ is identified with the subgroup of all triples $(A, B, C) \in SO(8) \times SO(8) \times SO(8)$ with the property

$$\text{(T)} \qquad A(xy) = B(x)C(y), \text{ for all } x, y \in Ca, \text{ the Cayley field.}$$

One really needs just two copies of $SO(8)$ as C is determined from A and the sign of B, but it seems to be more convenient to use all three to express the triality automorphisms.

The subgroup $Spin(7) \subset Spin(8)$ can be identified with all $(N, M, \widetilde{M})$, where $\widetilde{M}(x) = \overline{M(\overline{x})}$, for all $x \in Ca$, the bar denoting the usual conjugation of a Cayley number. This is equivalent to $N(1) = 1$.

If γ is the usual triality automorphism of order 3, then

$$\gamma(Spin(7)) = \{(M, N, M) \text{ in (T), with } N(1) = 1\}$$

Lemma 1. *The map $\gamma(Spin(7)) \to SO(8)$ with $(M, N, M) \mapsto M$ is an injective group morphism.*

Proof. It is a group morphism by its definition and the kernel is (I, I, I), because if $(I, N, I) \mapsto I$, then $I(y) = I(y1) = N(y)I(1) = N(y)$ for all $y \in Ca$, which implies $N = I$. □

From now on $Spin(7)$ is the subgroup of $SO(8)$ with $(M, N, M) \in \gamma(Spin(7))$, equivalently, $N(1) = 1$.

Lemma 2. *The map $\pi : SO(8) \to RP^7$ with $\pi(X) = \pm Y(1)$ is well defined.*

Proof. Note that $(X, \pm(Y, Z))$ is a well defined pair of points in $Spin(8)$, namely the fiber of the projection onto the first $SO(8)$ factor. □

Claim 3. *The fiber $\pi^{-1}(1)$ consists of all $X \in SO(8)$ with $(X, \pm(Y, Z)) \in Spin(8)$ and $\pm Y(1) = 1$, i.e., $Y \in O(7) = SO(7) \cup -SO(7)$.*

Proof. $Y(1) = 1$. The element $(X, Y, X) \in \gamma(Spin(7))$is represented by $X \in SO(8)$. The element $-Y(1) = 1$ is $(X, -Y, -X) \in Spin(8)$, for it is (X, Y, Z) for some Z, so $X(y) = Y(1)Z(y) = -1Z(y)$ and $Z = -X$. □

Note that the image in $SO(8)$ is the same: X. Also that $\pi^{-1}(\pm 1)$ is a subgroup of $SO(8)$ as the first factor projection of

$$Pin(7) = \{(X, Y, X)\} \cup \{(X, -Y, -X)\}$$

into $SO(8)$. This projection coincides with the inclusion of $Spin(7) \subset SO(8)$ of Lemma 1.

Proposition 4. *The map π of Lemma 2 is the projection of the fibration*

$$Spin(7) \cdots SO(8) \to SO(8)/Spin(7).$$

Proof. Consider the right action by a subgroup multiplication $SO(8) \times Spin(7) \to SO(8)$ with $X(M, N, M) \mapsto XM$. Then $(X, \pm(Y, Z))(M, N, M) = (XM, \pm(YN, ZM))$ and the whole orbit XM is mapped through π to $\pm YN(1) = \pm Y(1)$: the point $\pi(X) \in RP^7$. □

Consider now the following map $\chi : RP^7 \to SO(8)$ defined by $\chi(\pm\alpha) = L_{\pm\alpha} \circ R_{\pm\alpha} = L_\alpha \circ R_\alpha$, where $L_\alpha(x) = \alpha x$ and $R_\alpha(x) = x\alpha$, Cayley products.

Proposition 5. *χ is a well defined section of the principal bundle π.*

Proof. It is clearly well defined. From the Moufang identity $\alpha(xy)\alpha = (\alpha x)(y\alpha)$ (see e.g. [3]) we see that $(L_\alpha \circ R_\alpha, \pm(L_\alpha, R_\alpha)) \in Spin(8)$ and $\pi(\chi(\pm\alpha)) = \pm L_\alpha(1) = \pm\alpha$. □

Corollary 6. *$SO(8)$ is diffeomorphic to $RP^7 \times Spin(7)$ as follows:*
$RP^7 \times Spin(7) \ni (\pm\alpha, M) \mapsto (L_\alpha \circ R_\alpha)M \in SO(8)$ whose inverse is
$SO(8) \ni X \mapsto (\pm Y(1), (L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}})X) \in RP^7 \times Spin(7)$.

Proof. To X corresponds $(X, \pm(Y, Z))$, we have also

$$(L_{Y(1)} \circ R_{Y(1)}, \pm(L_{Y(1)}, R_{Y(1)}))$$

and their product in $Spin(8)$ is

$$\begin{aligned}(L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}}, \pm(L_{\overline{Y(1)}}, R_{\overline{Y(1)}}))(X, \pm(Y, Z))\\ = ((L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}})X, \pm(L_{\overline{Y(1)}}Y, R_{\overline{Y(1)}}Z)).\end{aligned}$$

But $\pm(L_{\overline{Y(1)}}Y)(1) = \pm 1$, so $(L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}})X$ is in $Spin(7) \subset SO(8)$. □

On the other hand, $SO(8)$ is diffeomorphic to $S^7 \times SO(7)$ as follows:

$$SO(8) \ni W \mapsto (W(1), L_{\overline{W(1)}} \circ W) \in S^7 \times SO(7)$$

whose inverse is $S^7 \times SO(7) \ni (\beta, A) \mapsto L_\beta \circ A \in SO(8)$. Now we can compose these two diffeomorphisms, i.e., given (β, A) in $S^7 \times SO(7)$ we look for its image in $RP^7 \times Spin(7)$. Note that $A(1) = 1$, $(A, \pm(B, \widetilde{B})) \in Spin(7) \subset Spin(8)$ and $L_\beta \circ A = X$ will go to $(\pm Y(1), L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}} \circ X)$. From the Moufang identity $\beta(xy) = (\beta x \beta)(\overline{\beta} y)$ (see e.g. [3]) we obtain $(L_\beta, \pm(L_\beta \circ R_\beta, L_{\overline{\beta}})) \in Spin(8)$. So the triality triple $(X, \pm(Y, Z))$ will be the product

$$(L_\beta, \pm(L_\beta \circ R_\beta, L_{\overline{\beta}}))(A, \pm(B, \widetilde{B})) = (L_\beta \circ A, \pm(L_\beta \circ R_\beta \circ B, L_{\overline{\beta}} \circ \widetilde{B})).$$

Through the identification of $SO(8)$ with $RP^7 \times Spin(7)$ this will go to

$$\begin{aligned}(\pm L_\beta \circ R_\beta \circ B(1), L_{\overline{\beta B(1)\beta}} \circ R_{\overline{\beta B(1)\beta}} \circ L_\beta \circ A)\\ = (\pm\beta B(1)\beta, L_{\overline{\beta B(1)\beta}} \circ R_{\overline{\beta B(1)\beta}} \circ L_\beta \circ A),\end{aligned}$$

which we denote by λ.

The following little calculation now

$$\begin{aligned}\xi \ \mapsto\ (\overline{\beta B(1)\beta})(\beta A(\xi))(\overline{\beta B(1)\beta})\\ = (\overline{\beta B(1)})[A(\xi)(\overline{\beta B(1)\beta})] = (L_{\overline{\beta B(1)}})(R_{\overline{\beta B(1)\beta}}(A(\xi)))\end{aligned}$$

and the associativity of the subalgebra generated by the two elements β and $B(1)$ imply that the operators $L_{\overline{\beta B(1)}}$ and $R_{\overline{\beta B(1)\beta}}$ commute and therefore

$$\lambda = (\pm\beta B(1)\beta, R_{\overline{\beta B(1)\beta}} \circ L_{\overline{\beta B(1)}} \circ A) \in RP^7 \times Spin(7)$$

is the image of $(\beta, A) \in S^7 \times SO(7)$.

The inverse of this map is $RP^7 \times Spin(7) \ni (\pm\alpha, M) \mapsto W \in S^7 \times SO(7)$, where

$$W = (L_\alpha \circ R_\alpha) \circ M \mapsto ((L_\alpha \circ R_\alpha)(M(1)), (L_{\overline{\overline{\alpha}M(1)\overline{\alpha}}} \circ L_\alpha \circ R_\alpha) \circ M)$$

To verify that the matrix coordinate is really in $SO(7)$:

$$((L_{\overline{\overline{\alpha}M(1)\overline{\alpha}}} \circ L_\alpha \circ R_\alpha) \circ M)(1) = (\overline{\overline{\alpha}M(1)\overline{\alpha}})(\alpha M(1)\alpha) = 1.$$

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