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SYMBOLS OF PSEUDODIFFERENTIAL OPERATORS ASSOCIATED TO GEVREY KERNEL'S TYPE

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Abstract. In this article, we aim at proving the truthfulness of the inverse Theorem (1) of [5]. More precisely, we associated symbols of Gevrey type to pseudodifferential operators when the latter are given by their kernels.

1. Introduction

In [4], we gave a description of pseudodifferential operators when defined by their Kohn-Nirenberg's symbols or their kernels (cf. [2], [3]); and this in the C^∞-case. It appeared that the two approches are equivalent. In [1], it is shown that this equivalence remains also true in the analytic case. This paper is a continuation of [5], where we obtained that the definition by symbols implies the one by kernels. Here, we deal with the converse. And this, in the more fine context of Gevrey classes.

2. Definitions and notations

We recall three definitions as they were given in [5].

Definition 1. Let n be a non zero positive integer, Ω an open subset of $\mathbb{R}^n$ and s any real number larger or equal to 1. A real fonction φ in $C^\infty(\Omega)$ is said of Gevrey class with order s if, for any compact subset $K \subset \Omega$, there exists constant $C_K > 0$ such that

$$\forall\, \alpha \in \mathbb{N}^n \;\; \|D^\alpha f\| \le C_K^{|\alpha|+1} \left(|\alpha|!\right)^s . \tag{1}$$

Definition 2. Given $n \in \mathbb{N}^\star$, $m \in \mathbb{R}$, $s \ge 1$ and Ω an open subset of $\mathbb{R}^n$. We say that a real function $a = a(z,x)$ in $C^\infty(\Omega \times \mathbb{R}^n)$, is a symbol (or amplitude) of Gevrey type with class s on Ω if, and only if, it satisfies

For any compact subset $K \subset \Omega$, there exist three positive constants C_0, C_1, R such that

$$\left| D_\xi^\alpha D_z^\beta a(z,\xi) \right| \le C_0 C_1^{|\alpha+\beta|} \left(|\alpha|!\right)^s \left(|\beta|!\right)^s \left(1 + |\xi|^2\right)^{m-|\alpha|}, \tag{2}$$

for any $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ with, $|\xi| \ge R$, α and $\beta \in \mathbb{N}^n$.
We denote by ${}_{1,0}\mathcal{S}^m_{G^s}(\Omega \times \mathbb{R}^n)$ the set of such symbols.

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Notice that, in general, a is supposed analytic in z. (This is done by taking $s = 1$ in the factor $(|\beta|!)^s$ corresponding to the variable z.)

Definition 3. We keep the notations of Definition 2. Let U be an open neighborhood of 0 in $\mathbb{R}^n$ and m a positive real number. We say that a distribution $T = T(z,x)$, on $\Omega \times U$, is a Gevrey kernel of order m if, and only if, the following assertions are satisfied:

a) The restriction f of T to $(U \setminus \{0\})$ is Gevrey of order s such that: For every compact K of Ω, there is an open neighborhood V of $U \setminus \{0\}$ and a scalar $C > 0$ such that

$$\left| D_x^\alpha D_z^\beta f\ (z,x) \right| \leq C^{|\alpha+\beta|+1} (|\alpha|!)^s (|\beta|!)^s \left(1 + |x|^2\right)^{-m-n-|\alpha|}, \quad \forall (z,x) \in K \times V. \tag{3}$$

b) The distribution T is of the form

$$T(z,\cdot) = P_{f_\theta}(z,\cdot) + \sum_{|\alpha| \leq m} C_\alpha(z) \delta^{(\alpha)}. \tag{4}$$

where $(C_\alpha)_\alpha$ is a family of Gevrey functions of order s in $\Omega$and θ is a map of $C_0^\infty(U)$, verifying $\theta \equiv 1$ in a neighborhood of zero, while P_{f_θ} is a distribution given for any ψ in $C_0^\infty(U)$ by

$$\langle P_{f_\theta} f, \psi \rangle = \int_\Omega f(x) \left(\psi(x) - \sum_{|\alpha| \leq m} \frac{D^\alpha \psi(0)}{\alpha!} x^\alpha \theta(x) \right) dx \tag{5}$$

c) If m is a positive integer, then, for any compact K in Ω, there is a scalar $C_K > 0$ such that: For every α in $\mathbb{N}^n$ with $|\alpha| = m$ and any $\varepsilon > 0$

$$\left| \int_{|x| \geq \varepsilon, x \in U} x^\alpha f(z,x) dx \right| \leq C_K, \forall z \in K. \tag{6}$$

The set of Gevrey kernels, so defined, is designated by ${}_{1,0}\mathcal{K}_{G^s}^m(\Omega \times \mathbb{R}^n)$.

3. The main result

We prove the following result.

Theorem 4. *Let m be a positive real number. If T is in ${}_{1,0}\mathcal{K}_{G^s}^m(\Omega \times \mathbb{R}^n)$, then there is a symbol f in ${}_{1,0}\mathcal{S}_{G^s}^m(\Omega \times \mathbb{R}^n)$ such that*

$$T = \mathcal{F}^{-1} f$$

is of Gevrey type of order s on $\Omega \times U'$, where U' is an open 0 neighborhood in $\mathbb{R}^n$.

WE recall here that $\mathcal{F}^{-1}f$ denotes the image of f by the inverse of Fourier transform.

To prove this theorem we need the following lemma which is an analogous of the one in [1].

Lemma 5. *Let m be a positive real number, U and U' open neighborhoods of zero, in $\mathbb{R}^n$, such that $U' \subset\subset U$.*

i) *Let T be a distribution on U, the restriction g of which, to $U\setminus\{0\}$, is C^∞ and satisfies*

$$\left|D_x^\beta g(x)\right| \leq C\,|x|^{m-n-|\beta|}, \quad |\beta| \leq m. \tag{7}$$

ii) *We suppose $x^\alpha T$ that is an integrable on U, for $|\alpha| > m$.*

iii) *If m is a positive integer, we suppose the existence of a map φ in $C_0^\infty(U)$, identically equal to 1 on $U\prime$ and such that, for $|\alpha| = m$,*

$$\underset{0<\varepsilon<1}{sup}\,|\langle x^\alpha T_\varepsilon(x), \varphi_\varepsilon(x)\rangle| \leq C, \tag{8}$$

with

$$\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right).$$

Then, for every ψ in $C_0^\infty(U)$, there exists a scalar $M_\psi > 0$ such that

$$\left|D_\xi^\alpha\widehat{(\psi T)}(\xi)\right| \leq CM_\psi(1+|\xi|)^{m-|\alpha|},$$

and this for every α in $\mathbb{N}^n$, $m \leq |\alpha| < m+1$, and every ξ in $\mathbb{R}^n$.

Proof. Since α is in $\mathbb{N}^n$, we have, for $|\xi| \geq 1$,

$$\begin{aligned} D_\xi^\alpha(\widehat{\psi T})(\xi) &= (-i)^{|\alpha|}(\widehat{x^\alpha\psi T})(\xi) = (-i)^{|\alpha|}\langle x^\alpha\psi T, e^{ix\xi}\rangle \\ &= (-i)^{|\alpha|}\left(I_1 + I_2 + I_3\right) \end{aligned}$$

where

$$\begin{aligned} I_1 &= \langle x^\alpha T, \varphi_\varepsilon\psi\rangle, \\ I_2 &= \langle x^\alpha T, \varphi_\varepsilon\psi\left(e^{-ix\xi}-1\right)\rangle, \\ I_3 &= \langle x^\alpha T, (1-\varphi_\varepsilon)\,\psi e^{-ix\xi}\rangle. \end{aligned}$$

Let us examine these expressions one by one. First, fix

$$\varepsilon = \frac{1}{|\xi|}. \tag{9}$$

Using (7), (8) and (9), we get

$$|I_1| \leq C\ J\ |\xi|^{m-|\alpha|}$$

with a constant $J > 0$.

Concerning the estimation of I_2, the definition of φ assures that

$$\left|e^{-ix\xi} - 1\right| \leq |\xi| \ \ |x| \, .$$

This inequality and (7) yield to

$$|I_2| \leq C \ S \ \ |\xi|^{m-|\alpha|} \, ,$$

with a constant $S > 0$.

Concerning I_3, we can write

$$I_3 = x^{\alpha} T \, \widehat{(1 - \varphi_{\varepsilon})} \, \psi.$$

Then

$$\xi_j I_3 = -i \ \langle D_{x_j} \left(x^{\alpha} \left(1 - \varphi_{\varepsilon}\right) \psi T, e^{-ix\xi} \right) \rangle, \quad \forall j = 1, ..., n.$$

Since the support of $1 - \varphi_{\varepsilon}$ is contained in $U \backslash U'$, we can find a scalar μ such that

$$0 < \mu \leq |\xi| \ \ |x| \, ,$$

which permits to have

$$|\xi_j I_3| \leq C \ K \ \ |\xi|^{m-|\alpha|+1} \, ,$$

where K is a strictly positive scalar.

With these three estimations we obtain (12) in the case $|\xi| \geq 1$ considered.

As (12) remains obviously true when $|\xi| < 1$ the proof of Lemma 5 is finished. □

The proof of Theorem 4 need also the following proposition.

Proposition 6. *Let U be an open 0-neighborhood in $\mathbb{R}^n$, a given α in $\mathbb{N}^n$ and T an element of ${}_{1,0}\mathcal{K}^m_{G^s}(\Omega \times \mathbb{R}^n)$ such that its restriction f to $U \backslash \{0\}$, satisfies*

$$|D^{\alpha}_x f(x)| \leq C^{|\alpha|+1} (|\alpha|!) \, |x|^{-m-n-|\alpha|} \, . \tag{10}$$

For any map φ in $C^{\infty}_0(U)$ which is identically equal to 1 on an open 0-neighborhood $U' \subset\subset U$, we get

$$\left| D^{\alpha}_{\xi} (\widehat{\varphi T})(\xi) \right| \leq C^{|\alpha|+1} (|\alpha|!)^s \left(1 + |\xi|\right)^{m-|\alpha|} . \tag{11}$$

Proof. We define a function ψ on $C^{\infty}_0(U)$ as follows

$$\psi(x) = \begin{cases} 1 & \text{for} \quad |x| \leq 1, \\ 0 & \text{for} \quad |x| \geq 3. \end{cases} \tag{12}$$

For any β in $\mathbb{N}^n$ such that $|\beta| = |\alpha|$ we get

$$\left| \xi^{\alpha} D^{\alpha} (\widehat{\varphi T})(\xi) \right| \leq |I_1| + |I_2| \, ,$$

where

$$\begin{aligned} |I_1| &= \left| \int_\Omega D_x^\alpha \left(x^\alpha (\varphi T)(x) \right) \left[1 - \psi \left(|\xi|\, x \right) \right] e^{-ix\xi} dx \right|, \\ |I_2| &= \left| \int_\Omega D_x^\alpha \left(x^\alpha (\varphi T)(x) \right) \psi \left(|\xi|\, x \right) e^{-ix\xi} dx \right|. \end{aligned}$$

Let us estimate these expressions. We have

$$|I_2| \le \int_{|x| < \frac{3}{|\xi|}} \left| D_x^\alpha \left(x^\alpha (\varphi T)(x) \right) \right| dx.$$

The Leibniz formula and (10) permits to get

$$\begin{aligned} |I_2| &\le \int_{|x| < \frac{3}{|\xi|}} \left| \sum_{\upsilon \le \beta} \tbinom{\beta}{\upsilon} \partial^{\beta - \upsilon} T \partial^\upsilon \left(x^\alpha \varphi(x) \right) \right| dx \\ &\le \int_{|x| < \frac{3}{|\xi|}} \sum_{\upsilon \le \beta} \tbinom{\beta}{\upsilon} \left| \partial^\upsilon \left(x^\alpha \varphi(x) \right) \right| C^{|\beta - \upsilon| + 1} (|\beta - \upsilon|!)^s \ |x|^{-n-m-|\beta-\nu|} \, dx. \end{aligned}$$

Whence

$$\begin{aligned} |I_2| &\le \int_{|x| < \frac{3}{|\xi|}} A_2 C^{|\beta|+1} (|\beta|!)^s \ |x|^{-n-m} \, dx \\ &\le B_2 C^{|\beta|+1} (|\beta|!)^s \ (1 + |\xi|)^m \, . \end{aligned} \tag{13}$$

We proceed in the same way for $|I_1|$. We obtain

$$\begin{aligned} |I_1| &\le \int_{\frac{1}{|\xi|} < |x| < \frac{3}{|\xi|}} A_1 C^{|\beta|+1} (|\beta|!)^s \ |x|^{-n-m} \, dx \\ &\le B_1 C^{|\beta|+1} (|\beta|!)^s \ (1 + |\xi|)^m \, . \end{aligned} \tag{14}$$

Notice that the scalars A_1 and A_2 depend on β while B_1 and B_2 depend on β, m and n. The constant C does not depend on β, nor n, nor m.

On the other hand, using (13) and (14), we easily obtain

$$\left| D_\xi^\alpha \widehat{(\varphi T)}(\xi) \right| \le A C^{|\alpha|+1} (|\alpha|!)^s \, (1 + |\xi|)^{m - |\alpha|} \, .$$

This inequality completes the proof of the Relation (11). □

The proof of Theorem 4 follows directly from the Lemma 5 and the Proposition 6.

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