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ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA. XI

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Let $t(G)$ be the nilpotency index of the radical $J(KG)$ of a group algebra KG of a finite p-solvable group G over a field K of characteristic $p > 0$. Then it is well known by D. A. R. Wallace [7] that

$$p^e \geq t(G) \geq e(p-1)+1,$$

where p^e is the order of a Sylow p-subgroup of G.

H. Fukushima [1] characterized a group G of p-length 2 satisfying $t(G) = e(p-1)+1$, see also [4]. Unfortunately, his characterization holds under a condition such that the p'-part $V = O_{p',p}(G)/O_p(G)$ of G is abelian.

In this paper, using Dickson near fields, we shall give an explicit example (see Example 1) such that a group G of p-length 2 has the non abelian p'-part V and satisfies $t(G) = e(p-1)+1$. This example will be new and have a contributions in our research. Example 2 is also very interesting because quite different objects (see [3] and [5]) are unified on the ground of Dickson near fields.

Let H be a sharply 2-fold transitive group on $\Delta = \{0, 1, \alpha, \beta, \ldots, \gamma\}$ (see [8, p. 22]). Let $V = H_0$ be a stabilizer of 0, and let U be the set consisting of the identity ε and fixed point-free permutations in H. Then U is an elementary abelian p-subgroup of H with the order p^s (see Lemma 1). Let σ be a permutation of order p on Δ satisfying conditions

$$\sigma H \sigma^{-1} \subseteq H, \quad \sigma^p = 1, \quad \sigma(0) = 0 \quad \text{and} \quad \sigma(1) = 1.$$

Then it is easy to see $\sigma U \sigma^{-1} \subseteq U$ and $\sigma V \sigma^{-1} \subseteq V$. We set $W = \langle \sigma \rangle$ and $C_V(\sigma) = \{v \in V \mid \sigma v = v\sigma\}$. Assume that there exists a normal subgroup T of WV contained in V such that V is a semi-direct product of T by $C_V(\sigma)$. We set $G = \langle W, T, U \rangle$.

Now, we present the well known results Lemmas 1 and 2 for completeness of this paper.

Lemma 1. *U is a normal and elementary abelian p-subgroup of H.*

Proof. First we shall prove, for $k \in \Delta^* = \Delta \setminus \{0\}$, there exists only one $u_k \in U$ with $u_k(0) = k$, equivalently, the following map ν from U to Δ is bijective:

$$\nu \colon u \to u(0).$$

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For $\tau \in U \setminus \{\varepsilon\}$, there exists $\rho \in H_0$ with $\rho(\tau(0)) = k$ since $\tau(0) \neq 0$ and H_0 is transitive on Δ^*. We set $u_k = \rho\tau\rho^{-1}$. Then $u_k \in U$ and $u_k(0) = k$. Thus ν is surjective. It follows from definition of H and U that

$$U = H \setminus \bigcup_{a\in\Delta} (H_a \setminus \{\varepsilon\}), \quad (H_a \setminus \{\varepsilon\}) \cap (H_b \setminus \{\varepsilon\}) = \emptyset \text{ for } a \neq b.$$

Using $|H| = |H_a||a^H| = |H_a||\Delta|$, where a^H is an orbit of a, we can see $|U| = |\Delta|$. Hence ν is injective.

Assume $\eta\tau$ has a fixed point ℓ for $\eta, \tau \in U$. Then we may assume $\ell = 0$ since H is transitive on Δ and $\rho U \rho^{-1} = U$ for $\rho \in H$. Thus $\tau = \eta^{-1}$ follows from $\eta^{-1} \in U$, $\tau(0) = \eta^{-1}(0)$ and the above observation. This means $\eta\tau \in U$. Hence U is a normal subgroup of H because $\rho U \rho^{-1} = U$ for all $\rho \in H$.

Now, we shall show U is elementary abelian. Let p be a prime factor of $|U|$ and let τ be an element of order p in the center of a Sylow p-subgroup of U. We set $\eta \in U \setminus \{\varepsilon\}$. Then there exists $\rho \in H_0$ with $\rho(\tau(0)) = \eta(0)$. Thus $\rho\tau\rho^{-1} = \eta$ follows from $\rho\tau\rho^{-1} \in U$ and $\rho\tau\rho^{-1}(0) = \eta(0)$. Thus the order of every element in U is p or 1 and so η is in the center of a p-group U. Thus U is elementary abelian. □

The next shows Δ is a near field of characteristic p.

Lemma 2. *Δ is a near field of characteristic p and σ is an automorphism of Δ.*

Proof. First, we shall prove that Δ is a near field. We can set a structure of a near field in a set Δ by the following method. It follows from Lemma 1 that there exists only one $u_a \in U$ with $u_a(0) = a$ for $a \in \Delta$. It is easy to see that for $a \in \Delta^* = \Delta \setminus \{0\}$, there exists only one $v_a \in V = H_0$ with $v_a(1) = a$. It is clear from definition that $u_0 = v_1 = \varepsilon$.

We define the sum and the product of elements a, b in Δ by using the above v_a and u_b:

$$a + b := u_b(a), \quad ab := v_a(b) \text{ for } a \neq 0 \quad \text{and} \quad 0b := 0.$$

First we shall prove the next equations:

$$u_a u_b = u_{b+a}, \quad v_a v_b = v_{ab} \quad \text{and} \quad v_a u_b v_a^{-1} = u_{ab}.$$

These follow from

$$\begin{aligned}
&u_a u_b(0) = u_a(b) = b + a = u_{b+a}(0),\\
&v_a v_b(1) = v_a(b) = ab = v_{ab}(1),\\
&v_a u_b v_a^{-1}(0) = v_a u_b(0) = v_a(b) = ab = u_{ab}(0).
\end{aligned}$$

Next we shall prove the next equations from the first equation and the commutativity of U:

$$\begin{aligned}
&a+(b+c)=u_{b+c}(a)=u_c u_b(a)=u_c(a+b)=(a+b)+c,\\
&a+b=u_{a+b}(0)=u_b u_a(0)=u_a u_b(0)=u_a(b)=b+a,\\
&a+0=0+a=u_a(0)=a,\\
&a+u_a^{-1}(0)=u_a^{-1}(0)+a=u_a(u_a^{-1}(0))=\varepsilon(0)=0.
\end{aligned}$$

We shall prove the next equations from the second equation for $a,b\in\Delta^*$. For $a=0$ or $b=0$, it is easy to prove our equations:

$$\begin{aligned}
&a(bc)=v_a(bc)=v_a(v_b(c))=v_a v_b(c)=v_{ab}(c)=(ab)c,\\
&a1=v_a(1)=a=\varepsilon(a)=v_1(a)=1a,\\
&av_a^{-1}(1)=v_a(v_a^{-1}(1))=\varepsilon(1)=1.
\end{aligned}$$

For $a\in\Delta^*$, $v_a^{-1}(1)\neq 0$ follows from $v_a(0)=0\neq 1$ and we can see $v_{v_a^{-1}(1)}=v_a^{-1}$ by $v_{v_a^{-1}(1)}(1)=v_a^{-1}(1)$. Thus we have

$$v_a^{-1}(1)a=v_{v_a^{-1}(1)}(a)=v_a^{-1}(a)=v_a^{-1}(v_a(1))=1.$$

The next equation follows from the third equation:

$$a(b+c)=v_a(b+c)=v_a(u_c(b))=v_a u_c v_a^{-1}(v_a(b))=u_{ac}(ab)=ab+ac.$$

Thus Δ is a near field by our definition of the sum and the product. Moreover Δ is of characteristic p because $u_{p\cdot 1}=u_1^p=\varepsilon=u_0$.

Next we shall show σ is an automorphism of Δ. It is easy to see from the definitions of U and V that

$$\sigma U\sigma^{-1}\subseteq U \quad\text{and}\quad \sigma V\sigma^{-1}\subseteq V.$$

It follows from the definitions of u_a and v_a that

$$\sigma u_a\sigma^{-1}=u_{\sigma(a)} \quad\text{and}\quad \sigma v_b\sigma^{-1}=v_{\sigma(b)}$$

by equations

$$\sigma u_a\sigma^{-1}(0)=\sigma u_a(0)=\sigma(a)=u_{\sigma(a)}(0)$$

and

$$\sigma v_b\sigma^{-1}(1)=\sigma v_b(1)=\sigma(b)=v_{\sigma(b)}(1).$$

Since σ is a permutation on Δ, it follows from the next equations that σ is an automorphism of Δ:

$$u_{\sigma(a+b)}=\sigma u_{a+b}\sigma^{-1}=\sigma u_a\sigma^{-1}\sigma u_b\sigma^{-1}=u_{\sigma(a)}u_{\sigma(b)}=u_{\sigma(a)+\sigma(b)}$$

and

$$v_{\sigma(ab)}=\sigma v_{ab}\sigma^{-1}=\sigma v_a\sigma^{-1}\sigma v_b\sigma^{-1}=v_{\sigma(a)}v_{\sigma(b)}=v_{\sigma(a)\sigma(b)}. \qquad \square$$

We can see from Lemma 2 and the classification of finite near fields (see [9]) that Δ is a Dickson near field because Δ has an automorphism of order p where p is the characteristic of Δ.

Lemma 3. *WT is a Frobenius group with kernel T and complement W.*

Proof. We note $W \cap V = \{\varepsilon\}$ since $\sigma(1) = 1$. Let $x = \sigma^k v$ be an element of $WT \setminus W$, where $v \in T$, and let $x^{-1}\sigma^s x = \sigma^t \neq \varepsilon$ be an element of $x^{-1}Wx \cap W$. Then we may assume $s = 1$ because the order of σ is p. Thus $x^{-1}Wx \cap W$ contains $v^{-1}\sigma v = \sigma^t$. The element $\sigma^{t-1} = v^{-1} \cdot \sigma v \sigma^{-1}$ is contained in $W \cap V = \{\varepsilon\}$. Hence $\sigma v = v\sigma$. Thus $v \in C_V(\sigma) \cap T = \{\varepsilon\}$ and $x = \sigma^k v = \sigma^k$ is contained in W. Therefore we have

$$x^{-1}Wx \cap W = \{\varepsilon\} \text{ for } x \in WT \setminus W. \qquad \square$$

Lemma 4. *$G = TC_G(\sigma)T$.*

Proof. Clearly $TC_G(\sigma)T$ contains T and W. Let u_δ be an arbitrary element of $U \setminus \{\varepsilon\}$, where δ is an arbitrary element in $\Delta^* = \Delta \setminus \{0\}$. Then $v_\delta = v_\gamma v_\lambda = v_{\gamma\lambda}$ where $v_\gamma \in T$ and $v_\lambda \in C_V(\sigma)$, namely, $\sigma(\lambda) = \lambda$. Thus $\delta = \gamma\lambda$ and so $u_\delta = v_\gamma u_\lambda v_\gamma^{-1} \in TC_G(\sigma)T$. It follows from $U \subset TC_G(\sigma)T$ that $G = TC_G(\sigma)T$. $\square$

Lemma 5. *$(J(KW)\hat{T}KG)^n \subseteq J(KW)^n\hat{T}KG$, where $\hat{T} = \sum_{t\in T} t$.*

Proof. Since T is normal in WV and $G = TC_G(\sigma)T$ by Lemma 4, we can see $s\sigma = \sigma s$ for every $s \in \hat{T}KG\hat{T} = \hat{T}KC_G(\sigma)\hat{T}$. Clearly the result holds for $n = 1$. Assume that the result holds for n. Then using the last assertion, we conclude that

$$\begin{aligned}(J(KW)\hat{T}KG)^{n+1} &\subseteq J(KW)^n\hat{T}KGJ(KW)\hat{T}KG \\ &= J(KW)^n\hat{T}KG\hat{T}J(KW)KG \\ &\subseteq J(KW)^{n+1}\hat{T}KG.\end{aligned} \qquad \square$$

Theorem. *Let S be a subgroup of V containing T and let p^{s+1} be the order of a Sylow p-subgroup WU of $M = \langle S, W, U\rangle$. Then $t(M) = (s+1)(p-1)+1$.*

Proof. Let v be an arbitrary element of S. Then $v = tc$ where $t \in T$ and $c \in C_V(\sigma)$. Hence we have

$$v\sigma v^{-1} = tc\sigma c^{-1}t^{-1} = t\sigma t^{-1} \in G = \langle T, W, U\rangle.$$

Noting T is normal in V, we have that G is a normal in M and the index $|M : G|$ is relatively prime to p. Therefore we obtain $t(M) = t(G)$ and it is enough to prove in case $M = G$. Since the radical $J(KG)$ contains the kernel $J(KU)KG$ of the natural homomorphism ν of the group algebra KG onto $K(G/U)$, it follows that $\nu(J(KG)) = \nu(J(KW)\hat{T})$ by Lemma 3 and

[2, Theorem 4] and so $J(KG) = J(KW)\hat{T}KG + J(KU)KG$. Since U is a normal and elementary abelian subgroup of order p^s, it is clear that the nilpotency index of $J(KU)KG$ is $s(p-1)+1$. On the other hand, Lemma 5 shows that $(J(KW)\hat{T}KG)^p = 0$. Since $J(KW)\hat{T}KG$ and $J(KU)KG$ are right ideals of KG, it follows that

$$J(KG)^{(s+1)(p-1)+1} = (J(KW)\hat{T}KG + J(KU)KG)^{p+s(p-1)} = 0,$$

and so $t(G) \leq (s+1)(p-1)+1$. On the other hand $(s+1)(p-1)+1 \leq t(G)$ by [7, Theorem 3.3]. This completes the proof. □

Example 1. Let (q, n) be a Dickson pair where p is a prime and $q = p^r$ for a positive integer r. Then (q^p, n) is also a Dickson pair because $q^p \equiv -1 \bmod 4$ if and only if $q \equiv -1 \bmod 4$. Let $\boldsymbol{F} = \boldsymbol{F}_{q^{pn}}$ be a finite field of order q^{pn} and let $\boldsymbol{D} = \boldsymbol{D}_{q^{pn}}$ be a finite Dickson near field defined by the automorphism $\tau\colon x \to x^{q^p}$ of $\boldsymbol{F}$. Then an automorphism $\sigma\colon x \to x^{q^n}$ of $\boldsymbol{F}$ is also of $\boldsymbol{D}$ by [9, Satz 18] or [6, Theorem 5] because $p^{rn} = q^n \equiv 1 \bmod n$ (see also [6, Theorem 1]).

Let ω be a generator of the multiplicative group $\boldsymbol{F}^*$ and we set $a = \omega^n$, $b = \omega$ in $\boldsymbol{F}^*$. Then the multiplicative group $\boldsymbol{D}^*$ of $\boldsymbol{D}$ has the structure

$$\boldsymbol{D}^* = \langle a, b \mid a^m = 1,\ b^n = a^t,\ bab^{-1} = a^{q^p}\rangle,$$

where $m = \frac{q^{pn}-1}{n}$, $t = \frac{m}{q^p-1}$. Here we use the usual symbol as the product in $\boldsymbol{D}$ for simplicity. Do not confuse with the product in $\boldsymbol{F}$. We consider some permutations on $\boldsymbol{D}$:

$$u_c\colon x \to x + c \text{ for } c \in \boldsymbol{D}, \quad v_c\colon x \to cx \text{ for } c \in \boldsymbol{D}^*.$$

Then we have some relations

$$u_c u_d = u_{d+c},\ v_c v_d = v_{cd},\ v_c u_d v_c^{-1} = u_{cd},\ \sigma u_c \sigma^{-1} = u_{\sigma(c)},\ \sigma v_c \sigma^{-1} = v_{\sigma(c)}$$

on u_c, v_c, σ. We set

$$U = \{u_c \mid c \in \boldsymbol{D}\},\ V = \{v_c \mid c \in \boldsymbol{D}^*\},\ W = \langle\sigma\rangle$$

and

$$T = \{v_c \in V \mid c \in \langle a^{\frac{q^n-1}{n}}\rangle\}.$$

It is easy to see that UV is sharply 2-fold transitive on $\boldsymbol{D}$, T is normal in WV and the order of T is $\frac{q^{pn}-1}{q^n-1}$ because products of a and x in $\boldsymbol{D}$ are the same in $\boldsymbol{F}$. On the other hand, the set $C_V(\sigma)$ is equal to $\boldsymbol{F}^*_{q^n}$ as a set and the order of $C_V(\sigma)$ is $q^n - 1$. Since $\frac{q^{pn}-1}{q^n-1}$ and $q^n - 1$ are relatively prime, we have $V = C_V(\sigma)T$, $C_V(\sigma) \cap T = \{\varepsilon\}$. Let S be a subgroup of V containing T and $M = \langle S, W, U\rangle$. Then $t(M) = (rpn+1)(p-1)+1$ by Theorem, where p^{rpn+1} is the order of a Sylow p-subgroup WU of M.

If we put $\boldsymbol{D} = \boldsymbol{F}$ for the extreme case $n = 1$, we have the same example as in [3]. $\square$

Example 2. If $(q, n) \neq (3, 2)$ and p is not a divisor of r, then $\boldsymbol{D}_{q^n}$ has no automorphisms of order p, and so we consider $\boldsymbol{D}_{q^{pn}}$. But $\boldsymbol{D}_{3^2}$ has an automorphism σ of order 3 and we can consider the affine group $G = \langle \sigma, V, U \rangle$ over $\boldsymbol{D}_{3^2}$ where $\boldsymbol{D}_{3^2}$ is a Dickson near fields defined by an automorphism $x \to x^3$ of $\boldsymbol{F}_{3^2} = \boldsymbol{F}_3[x]/(x^2+1) = \{s + ti \mid i^2 = -1,\ s, t \in \boldsymbol{F}_3\}$, σ is defined by $\sigma(s + ti) = s + t + ti$, and the permutation group U, V are defined as in Example 1. This group G is isomorphic to $Qd(3)$, namely, a group defined by semi-direct product of $\boldsymbol{F}_3^{(2)}$ by $SL(2, 3)$ using the natural action, where $\boldsymbol{F}_3^{(2)}$ is 2-dimensional vector space over $\boldsymbol{F}_3$ and $SL(2, 3)$ is the special linear group over $\boldsymbol{F}_3^{(2)}$. In this case 3^3 is the order of a Sylow 3-subgroup of G and it is known form [5] that $t(G) = 9 > 7 = 3(3 - 1) + 1$.

This observation is very interesting because quite different objects (see [3] and [5]) are unified on the ground of Dickson near fields. $\square$

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