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GEOMETRY ON GRASSMANN MANIFOLDS $G(2,8)$ AND $G(3,8)$

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Abstract. In this paper, we use the Clifford algebra $C\ell_8$ to construct fibre bundles $\tau_1\colon G(2,8)\to S^6$, $\tau_1'\colon G(2,7)\to S^6$ and $\tau_2\colon G(3,8)\to S^7$, the fibres are CP^3, CP^2 and $ASSOC = G_2/SO(4)$ respectively. We show that $G(2,5)$, CP^3 and S^6 are the homologically volume minimizing submanifolds of $G(2,8)$ by calibrations and they generate the homology group $H_6(G(2,8))$. The submanifolds S^7 and $ASSOC$ of $G(3,8)$ generate $H_7(G(3,8))$ and $H_8(G(3,8))$ respectively.

§1. Introduction

As is well-known, the Grassmann manifold $G(2,4)$ is a fibre bundle over S^2. In this paper, we use the Clifford algebra $C\ell_8$ to define maps $\tau_1\colon G(2,8)\to S^6$, $\tau_1'\colon G(2,7)\to S^6$ and $\tau_2\colon G(3,8)\to S^7$, which make the Grassmann manifolds $G(2,8)$, $G(2,7)$ and $G(3,8)$ fibre bundles. The fibres are complex projective spaces CP^3, CP^2 and $ASSOC = G_2/SO(4)$ respectively. The fibres of these bundles are also the totally geodesic submanifolds of $G(2,8)$, $G(2,7)$ and $G(3,8)$ respectively.

By calibrations, we show that the submanifolds $G(2,5)$, CP^3 and S^6 of $G(2,8)$ are the volume-minimizing cycles in $G(2,8)$ and they generate the homology group $H_6(G(2,8))$. These gives an answer to the problem (5) in [3]. The submanifolds S^7 and $ASSOC$ are also the generators of $H_7(G(3,8))$, $H_8(G(3,8))$ respectively.

In this paper, we also show that the Stiefel manifold $V_{8,2}$ is homeomorphic to the product of two spheres $S^7\times S^6$.

§2. Grassmann manifolds $G(2,8)$ and $G(3,8)$

Let $C\ell_8$ be the Clifford algebra associated to the Euclidean space R^8. Let $\bar e_1,\bar e_2,\dots,\bar e_8$ be a fixed orthonormal basis of R^8, the Clifford product be determined by the relations:

$$\bar e_B\bar e_C+\bar e_C\bar e_B=-2\delta_{BC}\ (B,C=1,2,\dots,8).$$

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Define the subspace $V = V^+ \oplus V^-$ of $C\ell_8$ by $V^+ = C\ell_8^{\mathrm{even}} A$ and $V^- = C\ell_8^{\mathrm{odd}} A$, where

$$A = \mathrm{Re}[(\bar{e}_1 + \sqrt{-1}\bar{e}_2)\cdots(\bar{e}_7 + \sqrt{-1}\bar{e}_8)(1 + \bar{e}_1\bar{e}_3\bar{e}_5\bar{e}_7)].$$

Lemma 1. *The space $V = V^+ \oplus V^-$ is an irreducible module over $C\ell_8$. The spaces V^+ and V^- are generated by $\bar{e}_1\bar{e}_B A$ and $\bar{e}_B A$ $(B = 1, \ldots, 8)$ respectively.*

Proof. For proof see [7], [9]. In the following we give another proof. By a computation, we have

$$\begin{aligned} A = {} & \bar{e}_1\bar{e}_3\bar{e}_5\bar{e}_7 + \bar{e}_2\bar{e}_4\bar{e}_6\bar{e}_8 - \bar{e}_1\bar{e}_3\bar{e}_6\bar{e}_8 - \bar{e}_2\bar{e}_4\bar{e}_5\bar{e}_7 \\ & - \bar{e}_1\bar{e}_4\bar{e}_5\bar{e}_8 - \bar{e}_1\bar{e}_4\bar{e}_6\bar{e}_7 - \bar{e}_2\bar{e}_3\bar{e}_5\bar{e}_8 - \bar{e}_2\bar{e}_3\bar{e}_6\bar{e}_7 \\ & + 1 + \bar{e}_1\bar{e}_2\bar{e}_3\bar{e}_4\bar{e}_5\bar{e}_6\bar{e}_7\bar{e}_8 - \bar{e}_5\bar{e}_6\bar{e}_7\bar{e}_8 - \bar{e}_1\bar{e}_2\bar{e}_3\bar{e}_4 \\ & - \bar{e}_3\bar{e}_4\bar{e}_7\bar{e}_8 - \bar{e}_1\bar{e}_2\bar{e}_5\bar{e}_6 - \bar{e}_1\bar{e}_2\bar{e}_7\bar{e}_8 - \bar{e}_3\bar{e}_4\bar{e}_5\bar{e}_6. \end{aligned}$$

It is easy to see that for any $1 \le i_1 < i_2 < i_3 \le 8$, there is a term in $A - 1 - \bar{e}_1 \cdots \bar{e}_8$ which contains $\bar{e}_{i_1}\bar{e}_{i_2}\bar{e}_{i_3}$. Furthermore, A is invariant by acting every summand of A on itself, then $A \cdot A = 16A$. These shows V^+ and V^- are generated by $\bar{e}_1\bar{e}_B A$ and $\bar{e}_B A$ $(B = 1, \ldots, 8)$ respectively. For the dimensional reason, V is an irreducible module over $C\ell_8$. These generators of V can be used to construct the isomorphism between the Clifford algebra $C\ell_8$ and the matrix algebra $R(16)$. □

Let $G(k, 8)$ be the Grassmann manifold formed by all oriented k-dimensional subspaces of R^8. For any $x \in G(k, 8)$, there are orthonormal vectors $e_1, \ldots, e_k$ such that x can be represented by $e_1 \wedge \cdots \wedge e_k$. Thus $G(k, 8)$ becomes a submanifold of the space $\bigwedge^k(R^8)$. The spaces $\bigwedge(R^8)$ and $C\ell_8$ are isomorphic as a vector space. Identify the elements $e_1 \wedge \cdots \wedge e_k$ with $e_1 \cdots e_k$, $G(k, 8)$ can also be viewed as a subset of the Clifford algebra $C\ell_8$. Then for any $x \in G(k, 8)$, there is $v \in R^8$ such that $xA = \bar{e}_1 vA$ or $xA = vA$ according to the number k being even or odd. With the inner product defined on $C\ell_8$ naturally, we can show $|v| = 1$. Thus we have a map $G(k, 8) \to S^7$, $x \mapsto v$. It is not difficult to see that if $k = 4$, the map $G(4, 8) \to S^7$ can not be a fibre bundle: the dimensions of the fibres over $\pm\bar{e}_1 \in S^7$ are different from that of the other fibres. Since the Grassmann manifolds $G(2, 8)$ and $G(6, 8)$, $G(3, 8)$ and $G(5, 8)$ are isometric respectively, we need only to study $G(2, 8)$ and $G(3, 8)$.

Let $e_1, e_2, \ldots, e_8$ be an orthonormal frame fields on R^8 such that $e_1 \wedge \cdots \wedge e_k$ generate a neighborhood of x in $G(k, 8)$. By

$$d(e_1 \wedge \cdots \wedge e_k) = \sum_{i=1}^{k} \sum_{\alpha=k+1}^{8} \omega_i^\alpha E_{i\alpha}, \quad \omega_i^\alpha = \langle de_i, e_\alpha \rangle,$$

we know that the elements $E_{i\alpha} = e_1 \dots e_{i-1} e_\alpha e_{i+1} \dots e_k$ ($i = 1, \dots, k$, $\alpha = k+1, \dots, 8$) can be looked as a basis of $T_{e_1 \dots e_k} G(k,8)$, and ω_i^α are its dual. The metric on $G(k,8)$ is $ds^2 = \sum_{i=1}^{k} \sum_{\alpha=k+1}^{8} (\omega_i^\alpha)^2$. Differentiate $E_{i\alpha}$ we get the Riemannian connection ∇ on $G(k,8)$,

$$\nabla E_{i\alpha} = \sum_{j=1}^{k} \omega_i^j E_{j\alpha} + \sum_{\beta=k+1}^{8} \omega_\alpha^\beta E_{i\beta}.$$

Bryant [1] has shown that the Lie group $Spin_7$ is the isotropy group of $SO(8)$ acting on A, that is, $Spin_7 = \{G \in SO(8) \mid G(A) = A\}$. He also shows that $Spin_7$ acts on $G(2,8)$, $G(3,8)$ and S^7 transitively. The subgroup $G_2 = \{G \in Spin_7 \mid G(\bar{e}_1) = \bar{e}_1\}$ acts transitively on $S^6 = \{v \in S^7 \mid v \perp \bar{e}_1\}$.

Theorem 2. *There is a map $\tau_1 \colon G(2,8) \to S^6$ which makes $G(2,8)$ a fibre bundle. The fibres are diffeomorphic to the complex projective space CP^3.*

Proof. First we show that if $xA = \bar{e}_1 v A$, $x \in G(2,8)$, then $v \perp \bar{e}_1$. Denote $\langle \quad , \quad \rangle$ the inner product on $C\ell_8$. Let $v = a\bar{e}_1 + bv'$, $v' \perp \bar{e}_1$. By $A \cdot A = 16A$, $x \cdot x = -1$, $\langle xA, A \rangle = 16 \langle x, A \rangle = 0$ and

$$16 = \langle xA, \bar{e}_1 v A \rangle = \langle xA, b\bar{e}_1 v' A \rangle \leq 16|b|,$$

we have $v \perp \bar{e}_1$. Hence we have a map $\tau_1 \colon G(2,8) \to S^6$, $\tau_1(x) = v$.

For any $G \in Spin_7$, we have the following commutative diagram

$$\begin{array}{ccc} G(2,8) & \xrightarrow{G} & G(2,8) \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_1} \\ S^6 & \xrightarrow{\bar{G}} & S^6, \end{array}$$

where $\bar{G}$ is defined by $G(x)A = \bar{e}_1 \bar{G}(v) A$, $v = \tau_1(x)$. Thus the fibres of τ_1 are all diffeomorphic. Let J_0 be the complex structure on R^8: $J_0 \bar{e}_{2s-1} = \bar{e}_{2s}$ ($s = 1, \dots, 4$). As $A \cdot A = 16A$, we can show that $xA = \bar{e}_1 \bar{e}_2 A$ if and only if $\langle x, \bar{e}_1 \bar{e}_2 A \rangle = \frac{1}{16} \langle xA, \bar{e}_1 \bar{e}_2 A \rangle = 1$. For any $x \in G(2,8)$, we can write $x = v \wedge (aJ_0 v + w)$, $w \perp v, J_0 v$. Since the two form part of $\bar{e}_1 \bar{e}_2 A$ is invariant under the action of the unitary group $U(n)$, we can show $vJ_0 vA = \bar{e}_1 \bar{e}_2 A$ and $\langle vwA, \bar{e}_1 \bar{e}_2 A \rangle = 0$. Then we have

$$\tau_1^{-1}(\bar{e}_2) = \{w \wedge J_0(w) \mid w \in S^7\} \approx CP^3. \qquad \square$$

Lemma 3. *Let $e_1, e_2, \dots, e_8$ be $Spin_7$ frame fields on R^8. Then the 1-forms $\omega_B^C = \langle de_B, e_C \rangle$ satisfy*

$$\omega_1^2 + \omega_3^4 + \omega_5^6 + \omega_7^8 = 0, \qquad \omega_1^3 - \omega_2^4 + \omega_6^8 - \omega_5^7 = 0,$$
$$\omega_1^4 + \omega_2^3 + \omega_5^8 + \omega_6^7 = 0, \qquad \omega_1^5 - \omega_2^6 + \omega_3^7 - \omega_4^8 = 0,$$
$$\omega_1^6 + \omega_2^5 - \omega_3^8 - \omega_4^7 = 0, \qquad \omega_1^7 - \omega_2^8 - \omega_3^5 + \omega_4^6 = 0,$$
$$\omega_1^8 + \omega_2^7 + \omega_3^6 + \omega_4^5 = 0.$$

Proof. As the Lie group $Spin_7$ is the isotropy group of A, the element A can also be represented by

$$A = \mathrm{Re}[(e_1 + \sqrt{-1}e_2) \cdots (e_7 + \sqrt{-1}e_8)(1 + e_1e_3e_5e_7)].$$

Since $-e_1e_2e_3e_4A = A$, differentiate it we can get the last four equations of the lemma. The other equations can be proved similarly. $\square$

In the following, we study the fibres of τ_1. Let $x \in \tau_1^{-1}(v)$, $v \in S^6$, and $e_1, e_2, \dots, e_8$ be $Spin_7$ frame fields on R^8 such that the elements $e_1 \wedge e_2$ generate a neighborhood $U \subset \tau_1^{-1}(v)$ of x.

From $d(e_1e_2A) = d(e_1e_2)A = d(\bar{e}_1vA) = 0$, we have

$$\omega_2^{2s+1} + \omega_1^{2s+2} = 0,\ \omega_2^{2s+2} - \omega_1^{2s+1} = 0\ (s = 1, 2, 3).$$

Then on U,

$$d(e_1 \wedge e_2) = \sum_{s=1}^{3} [\omega_1^{2s+1}(E_{1\,2s+1} + E_{2\,2s+2}) + \omega_1^{2s+2}(E_{1\,2s+2} - E_{2\,2s+1})].$$

Hence the vectors $\widetilde{E}_{2s+1} = E_{1\,2s+1} + E_{2\,2s+2}$ and $\widetilde{E}_{2s+2} = E_{1\,2s+2} - E_{2\,2s+1}$ form a basis of $T_{e_1 \wedge e_2}\tau_1^{-1}(v)$, and ω_1^α $(\alpha = 3, \dots, 8)$ are its dual.

Lemma 4. *The Riemannian connection D on the fibre $\tau_1^{-1}(v)$ is given by*

$$D\widetilde{E}_{2s+1} = -\omega_1^2 \widetilde{E}_{2s+2} + \sum_{\beta=3}^{8} \omega_{2s+1}^\beta \widetilde{E}_\beta,\ D\widetilde{E}_{2s+2} = \omega_1^2 \widetilde{E}_{2s+1} + \sum_{\beta=3}^{8} \omega_{2s+2}^\beta \widetilde{E}_\beta.$$

The fibre $\tau_1^{-1}(v)$ is a totally geodesic submanifold of $G(2, 8)$.

Proof. Restricting the Riemannian connection of $G(2, 8)$ on $\tau_1^{-1}(v)$, by Lemma 3, we have $\nabla \widetilde{E}_\alpha = D\widetilde{E}_\alpha$. Then $\tau_1^{-1}(v)$ is a totally geodesic submanifold of $G(2, 8)$. $\square$

The action of $Spin_7$ on $G(2, 8)$ is isometry and preserves the fibres of τ_1. To prove $\tau_1^{-1}(v)$ is a totally geodesic submanifold we need only to show this is true for $v = \bar{e}_2$ which can be proved directly (without Lemma 3).

Let $G(2,7)$ be the Grassmann manifold on $R^7 = \{v \in R^8 \mid v \perp \bar{e}_1\}$. Restricting the map τ_1 on $G(2,7)$, we have map $\tau_1' \colon G(2,7) \to S^6$. For any $G \in G_2$, we have the following commutative diagram

$$\begin{array}{ccc} G(2,7) & \xrightarrow{G} & G(2,7) \\ {\scriptstyle \tau_1'}\downarrow & & \downarrow{\scriptstyle \tau_1'} \\ S^6 & \xrightarrow{G} & S^6. \end{array}$$

These shows

Theorem 5. *There is a map $\tau_1' \colon G(2,7) \to S^6$ which makes $G(2,7)$ a fibre bundle. The fibres are homeomorphic to the complex projective space CP^2 and are totally geodesic submanifolds of $G(2,7)$.*

The fibre of τ_1' over $\bar{e}_2$ is ${\tau_1'}^{-1}(\bar{e}_2) = \{u \wedge J_0 u \mid u \in S^5\}$, where $S^5 = \{u \in S^7 \mid u \perp \bar{e}_1, \bar{e}_2\}$. Restricting the map τ_1 on $G(2,6)$ is not a fibre bundle. We can also show that the map $\tau_1 \colon G(2,8) \to S^6(\frac{\sqrt{2}}{2})$ is a Riemannian submersion, but $\tau_1' \colon G(2,7) \to S^6(r)$ can not be a Riemannian submersion for any $r > 0$.

Similar to the case of $G(2,8)$, there is a commutative diagram for each $G \in Spin_7$:

$$\begin{array}{ccc} G(3,8) & \xrightarrow{G} & G(3,8) \\ \downarrow{\scriptstyle \tau_2'} & & \downarrow{\scriptstyle \tau_2'} \\ S^7 & \xrightarrow{G} & S^7. \end{array}$$

Denote $\tau_2^{-1}(\bar{e}_1)$ by $ASSOC$ which is homeomorphic to $G_2/SO(4)$, see [3], [4].

Theorem 6. *The map $\tau_2 \colon G(3,8) \to S^7$ is a fibre bundle, the fibre type is $ASSOC$ and the fibres are totally geodesic submanifolds.*

The proof of Theorem 6 is similar to that of Theorem 2 and Lemma 4. By Theorem 6 we have the following corollaries:

(1) If $v = \tau_2(x)$, we can show $v \perp x$. Then we have another fibre bundle $G(3,8) \to CAYLEY$, $x \mapsto x \wedge v$, the fibre type is $G(3,4) \approx S^3$, where $CAYLEY = \{y \in G(4,8) \mid yA = A\}$ is a totally geodesic submanifolds of $G(4,8)$;

(2) The space $E = \{(x,w) \in G(3,8) \times R^8 \mid w \perp x,\ w \perp v = \tau_2(x)\}$ is a vector bundle over $G(3,8)$ with fibre R^4;

(3) Combing Hopf bundles $S^7 \to CP^3$ and $S^7 \to HP^1$, we get two fibre bundles $G(3,8) \to CP^3$ and $G(3,8) \to S^4$.

In the following we give more applications of the Clifford algebra $C\ell_8$. Let $V_{8,2}$ be the Stiefel manifold. We have maps

$$V_{8,2} \xrightarrow{\pi} G(2,8) \xrightarrow{\tau_1} S^6.$$

For any $v \in S^6$, define a linear map $J_v\colon R^8 \to R^8$ by $J_v(\bar{e}_1) = v$, $J_v(v) = -\bar{e}_1$, and for any $w \perp \bar{e}_1, v$, $J_v(w)$ is defined by $J_v(w)A = -\bar{e}_1 vwA$. It is not difficult to see that J_v is a complex structure on R^8 and $J_{\bar{e}_2} = J_0$ as defined in the proof of Theorem 2. It is easy to show that $J_v = G^{-1}J_0G$ for any $G \in G_2$ such that $G(v) = \bar{e}_2$. Then the fibre of τ_1 over $v \in S^6$ can be represented by $\tau_1^{-1}(v) = \{u \wedge J_v u \mid u \in S^7\}$. This shows (see also [6], p. 37)

$$V_{8,2} = \{(u, J_v u) \mid u \in S^7,\ v \in S^6\} \approx S^7 \times S^6.$$

Similarly, for the Stiefel manifold $V_{7,2}$, there are maps

$$V_{7,2} \xrightarrow{\pi'} G(2,7) \xrightarrow{\tau_1'} S^6.$$

We can show that

$$V_{7,2} = \{(u, J_v u) \mid u, v \in S^6,\ u \perp v\} \subset S^6 \times S^6.$$

Now we give some representation of Hopf maps. Let $w_1 = \sum_{i=1}^{4} v_i \bar{e}_i$, $w_2 = \sum_{j=5}^{8} v_j \bar{e}_j$ with $\sum_{i=1}^{8} v_i^2 = 1$. By $w_s J_0 w_s A = |w_s|^2 \bar{e}_1 \bar{e}_2 A$ $(s = 1, 2)$ and $w_1 J_0 w_2 A = -w_2 J_0 w_1 A$, we have

$$(w_1 + w_2)(J_0 w_1 - J_0 w_2)A = (|w_1|^2 - |w_2|^2)\bar{e}_1\bar{e}_2 A + 2w_2 J_0 w_1 A.$$

Computing the right hand side of this equation, we find a map $\eta\colon S^7 \to S^4$,

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_8 \end{pmatrix} \mapsto \begin{pmatrix} |w_1|^2 - |w_2|^2 \\ 2(v_1v_6 + v_2v_5 + v_3v_8 + v_4v_7) \\ 2(v_1v_5 - v_2v_6 - v_3v_7 + v_4v_8) \\ 2(v_1v_8 + v_2v_7 - v_3v_6 - v_4v_5) \\ 2(v_1v_7 - v_2v_8 + v_3v_5 - v_4v_6) \end{pmatrix}.$$

Let $z_1 = v_1 + iv_2 + jv_3 + kv_4$, $z_2 = v_6 + iv_5 + jv_8 + kv_7$, where i, j, k are the quaternion numbers with $k = ij$. The map η can also be obtained from the map

$$(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2\bar{z}_1 \cdot z_2).$$

It is easy to see that η is the Hopf map $S^7 \to HP^1$.

From

$$\left(\sum_{i=1}^{4} v_i\bar{e}_i \sum_{i=1}^{4} v_i\bar{e}_{i+4}\right) A$$
$$= \bar{e}_1[(v_1^2 - v_2^2 + v_3^2 - v_4^2)\bar{e}_5 + 2(v_1v_2 - v_3v_4)\bar{e}_6 + 2(v_1v_4 + v_2v_3)\bar{e}_8]A,$$

we have the Hopf map $S^3 \to S^2 = CP^1$,

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \mapsto \begin{pmatrix} v_1^2 - v_2^2 + v_3^2 - v_4^2 \\ 2v_1v_2 - 2v_3v_4 \\ 2v_1v_4 + 2v_2v_3 \end{pmatrix}.$$

§3. The calibrations on $G(2,8)$ and $G(3,8)$

Let ξ be a closed k-form on a Riemannian manifold M. If $\xi(X_1, \ldots, X_k) \leq 1$ for any $p \in M$ and for any orthonormal vectors $X_1, \ldots, X_k \in T_pM$, we call ξ a calibration on M, see [3], [4]. If H is a k-dimensional oriented submanifold of M so that the restriction of ξ on H is the volume elements on H, we call H a ξ-submanifold or an integral submanifold of ξ. The ξ-submanifolds have the minimizing volume in its homology class.

In the following we determine the volume-minimizing cycles of dimension 6 in Grassmann manifold $G(2,8)$. As is well known that $G(2,8)$ is a Kaehler manifold and the Euler class $\omega = -\sum \omega_1^\alpha \wedge \omega_2^\alpha$ of vector bundle

$$E = \{(x, v) \in G(2,8) \times R^8 \mid v \in x\} \to G(2,8)$$

is also the Kaehler form on $G(2,8)$. As is well known, $\frac{1}{k!}\omega^k$ is a calibration on $G(2,8)$ for each $k = 1, 2, 3$. Together with the Euler class of the vector bundle F defined below, they generate the cohomology groups $H^*(G(2,8))$ (see [3]).

Theorem 7. *The submanifolds $G(2,5)$ and the fibres of τ_1 are the integral submanifold of the calibration $\xi = \frac{1}{3!}\omega^3$.*

Proof. It is easy to see that $G(2,5) \subset G(2,8)$ is a ξ-submanifold. By

$$\omega = \frac{1}{2}\sum_{l=1}^{3}[(\omega_1^{2l+1} + \omega_2^{2l+2}) \wedge (\omega_1^{2l+2} - \omega_2^{2l+1}) - (\omega_1^{2l+1} - \omega_2^{2l+2}) \wedge (\omega_1^{2l+2} + \omega_2^{2l+1})],$$

we have $\frac{1}{2^3}\xi(\widetilde{E}_3, \ldots, \widetilde{E}_8) = 1$, where $\widetilde{E}_\alpha$ is a basis of $T_{e_1\wedge e_2}\tau_1^{-1}(v)$ defined in §2. These shows $\tau_1^{-1}(v)$ is a ξ-submanifold. $\square$

Let J be a complex structure on $R^{2m+2} \subset R^N$. Then $\{v \wedge Jv \mid v \in S^{2m+1}\} \subset G(2, N)$ is homeomorphic to CP^m. Theorem 7 can be generalized as follows:

Grassmann manifold $G(2, m+2)$ and complex projective space CP^m are two integral submanifolds of the calibration $\xi = \frac{1}{m!}\omega^m$, where ω is the Kaehler form on $G(2, N)$ $(m+2 < N,\ 2m+2 \leq N)$.

Define a vector bundle on $G(2, 8)$ by

$$p\colon F = \{(x, w) \in G(2, 8) \times R^8 \mid w \perp x\} \to G(2, 8), \quad p(x, w) = x.$$

Let $e_1, e_2, \ldots, e_8$ be an orthonormal frame fields on R^8. Then $e_3, e_4, \ldots, e_8$ can be viewed as local sections of the vector bundle F. The connection $\overline{\nabla}$ on F is defined by $\overline{\nabla} e_\alpha = \sum_{\beta=3}^{8} \omega_\alpha^\beta e_\beta$ $(\alpha = 3, \ldots, 8)$, and curvature forms are $\Omega_\alpha^\beta = \omega_\alpha^1 \wedge \omega_1^\beta + \omega_\alpha^2 \wedge \omega_2^\beta$. From the Euler class of F, we have a closed form

$$\zeta = -\frac{1}{6!} \sum \delta^{3\ldots8}_{\alpha_1\ldots\alpha_6} \Omega_{\alpha_1\alpha_2} \Omega_{\alpha_3\alpha_4} \Omega_{\alpha_5\alpha_6}.$$

Theorem 8. *The 6-form ζ is a calibration on $G(2, 8)$ and $G(1, 7) = \{\bar{e}_1 \wedge v \mid v \in S^6\}$ is a ζ-submanifold.*

Proof. It is easy to see that $-\delta^{3\ldots8}_{\alpha_1\ldots\alpha_6} \Omega_{\alpha_1\alpha_2} \Omega_{\alpha_3\alpha_4} \Omega_{\alpha_5\alpha_6}(X_1, \ldots, X_6) \leq 1$ holds for any orthonormal vectors $X_1, \ldots, X_6 \in TG(2, 8)$ and for any fixed $\alpha_1, \alpha_2, \ldots, \alpha_6$. Hence

$$-\frac{1}{6!} \sum \delta^{3\ldots8}_{\alpha_1\ldots\alpha_6} \Omega_{\alpha_1\alpha_2} \Omega_{\alpha_3\alpha_4} \Omega_{\alpha_5\alpha_6}(X_1, \ldots, X_6) \leq 1.$$

This shows ζ is a calibration on $G(2, 8)$.

Write ζ as

$$\zeta = \omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^8 + \omega_2^3 \wedge \omega_2^4 \wedge \cdots \wedge \omega_2^8 + \cdots.$$

It is easy to see that the $G(1, 7) = \{\bar{e}_1 \wedge v \mid v \in S^6\}$ is a totally geodesic submanifold of $G(2, 8)$ and is a ζ-submanifold. $\square$

It is easy to see that $\int_{G(2,5)} \zeta = \int_{\tau_1^{-1}(v)} \zeta = 0$. Then $G(2, 5)$ and $\tau_1^{-1}(v)$ belong to the same homology class of $H_6(G(2, 8))$. The complex projective space CP^3 and $G(1, 7) \approx S^6$ are two generators of the homology group $H_6(G(2, 8))$. These gives an answer to the problem (5) in [3]. As Theorem 7, 8 can be generalized to the Grassmann manifold $G(2, 2n)$. Then the homology classes of the Grassmann manifold $G(2, N)$ can be all represented by the integral submanifolds of $G(2, N)$ for some calibrations. For the homology groups of $G(2, N)$, see the table 2.1 in [3].

Let dV_{S^6} be the volume element of S^6. Then $\frac{1}{8}\tau_1^*(dV_{S^6})$ is a calibration on $G(2, 8)$, but there is no integral submanifold for this calibration, even locally. We can show that

$$\frac{1}{8}\tau_1^*(dV_{S^6}) = \frac{1}{8} \prod_{l=1}^{3} (\omega_1^{2l+1} - \omega_2^{2l+2}) \wedge (\omega_1^{2l+2} + \omega_2^{2l+1})$$

is a summand of $\frac{1}{3!}\omega^3$.

Let dV_{S^7} be the volume element of S^7. Then $\tau_2^*(dV_{S^7})$ also determines a calibration on $G(3,8)$ and there is also no integral submanifold for this calibration. Let I, J, K be the quaternion structures on $R^8 \cong H^2$. The map

$$f\colon S^7 \to G(3,8),\ v \mapsto IvJvKv$$

gives a section of the fibre bundle $\tau_2\colon G(3,8) \to S^7$. We can show that $\int_{S^7} f^*(\tau_2^*(dV_{S^7})) \neq 0$. Then $f(S^7)$ is a generator of $H_7(G(3,8))$. Similarly we can show that $ASSOC$ is a generator of $H_8(G(3,8))$.

From the various fibre bundles defined on the Grassmann manifold $G(3,8)$ in §2, we can get many calibrations on $G(3,8)$, but we can not find any integral submanifolds for them.

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