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ON COMMUTATIVITY OF RINGS WITH GENERALIZED DERIVATIONS

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Abstract. The concept of derivations as well as of generalized inner derivations have been generalized as an additive function $F\colon R \to R$ satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation on R, such a function F is said to be a generalized derivation. In the present paper we have discussed the commutativity of prime rings admitting a generalized derivation F satisfying (i) $[F(x), x] = 0$, (ii) $F([x, y]) = [x, y]$, and (iii) $F(x \circ y) = x \circ y$ for all x, y in some appropriate subset of R.

1. Introduction

Let R denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes for anti-commutator $xy + yx$. Recall that a ring R is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$. An additive mapping $d\colon R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a\colon R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be inner derivation.

Many analysts have studied generalized derivation in the context of algebras on certain normed spaces (see [10] for reference). By a generalized derivation on an algebra A one usually means a map of the form $x \mapsto ax+xb$, where a and b are fixed elements in A. We prefer to call such maps generalized inner derivations for the reason they present a generalization of the concept of inner derivations (i.e. the maps of the form $x \mapsto ax - xa$). Now in a ring R, let F be a generalized inner derivation of R given by $F(x) = ax + xb$. Notice that $F(xy) = F(x)y + xI_b(y)$, where $I_b(y) = yb - by$ is an inner derivation.

Motivated by these observation Hvala [10] introduced the notions of generalized derivations in rings. An additive mapping $F\colon R \to R$ is called a generalized derivation if there exists a derivation $d\colon R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Obviously, every derivation generalized inner derivation and left multiplier (i.e. an additive mapping $F\colon R \to R$ such that $F(xy) = F(x)y$ for all $x, y \in R$) are generalized derivations.

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In the present paper we shall attempt to generalize some known results for derivations to generalized derivations.

2. Preliminary results

Throughout the present paper, we shall make use of the following two basic identities without any specific mention:

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z,$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

We begin with the following known results which will be used extensively to prove our theorems.

Lemma 2.1 ([4, Lemma 4]). *If $U \not\subseteq Z(R)$ is Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([4, Lemma 5]). *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $d(U) = 0$, then $U \subseteq Z(R)$.*

Lemma 2.3 ([2, Theorem 7]). *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $[u, d(u)] \in Z(R)$ for all $u \in U$, then $U \subseteq Z(R)$.*

Lemma 2.4 ([3, Theorem 4]). *Let R be a prime ring and I a nonzero left ideal of R. If R admits a nonzero derivation d such that $[x, d(x)]$ is central for all $x \in I$, then R is commutative.*

Lemma 2.5 ([11, Lemma 3]). *If a prime ring R contains a nonzero commutative right ideal, then R is commutative.*

Now, we prove the following.

Lemma 2.6. *Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R. If U is a commutative Lie ideal of R, i.e. $[u, v] = 0$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

Proof. Since U is a commutative Lie ideal of R, i.e.

$$[u, v] = 0, \quad \text{for all } u, v \in U. \tag{2.1}$$

Replacing v by $[u, r]$ in (2.1), we get $[u, [u, r]] = 0$ for all $u \in U$, $r \in R$. Again replace r by rs, to get $[u, [u, rs]] = 0$ for all $u \in U$, $r, s \in R$, that is

$$[u, [u, r]]s + r[u, [u, s]] + 2[u, r][u, s] = 0, \quad \text{for all } u \in U,\ r, s \in R.$$

This implies that $2[u, r][u, s] = 0$ for all $u \in U$, $r, s \in R$. Since $\mathrm{char}(R) \neq 2$, we get $[u, r][u, s] = 0$. Replacing s by sr, we get $[u, r]R[u, r] = (0)$ for all $u \in U$, $r \in R$. Thus primeness of R forces that $[u, r] = 0$ for all $u \in U$, $r \in R$, and hence $U \subseteq Z(R)$. □

3. Lie ideals and generalized derivations of prime rings

Theorem 3.1. *Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a nonzero generalized derivation F with d such that $[F(u), u] = 0$ for all $u \in U$, and if $d \neq 0$, then $U \subseteq Z(R)$.*

Proof. We have

$$[F(u), u] = 0, \quad \text{for all } u \in U. \tag{3.1}$$

Linearizing (3.1) and using (3.1), we obtain

$$[F(u), v] + [F(v), u] = 0, \quad \text{for all } u, v \in U. \tag{3.2}$$

Notice that $vw + wv = (v + w)^2 - v^2 - w^2$ for all $v, w \in U$. Since $u^2 \in U$ for all $u \in U$, $vw + wv \in U$. Also $vw - wv \in U$ for all $v, w \in U$. Hence we find that $2vw \in U$ for all $v, w \in U$. Replacing v by $2vu$ in (3.2) and use (3.1) and (3.2), to get

$$v[d(u), u] + [v, u]d(u) = 0, \quad \text{for all } u, v \in U. \tag{3.3}$$

Again replacing v by $2wv$ in (3.3) and using (3.3), we get $[w, u]vd(u) = 0$ for all $u, v, w \in U$, and hence $[w, u]Ud(u) = (0)$ for all $u, w \in U$. Thus for each $u \in U$, by Lemma 2.1 we find that either $[w, u] = 0$ or $d(u) = 0$. Now, let $A = \{u \in U \mid d(u) = 0\}$ and $B = \{u \in U \mid [w, u] = 0 \text{ for all } w \in U\}$. Then A and B are additive subgroups of U and $U = A \cup B$. But a group can not be a union of two its proper subgroups, and hence $U = A$ or $U = B$. If $U = A$, then $d(u) = 0$ for all $u \in U$. Thus by Lemma 2.2, we get the required result. On the other hand if $[w, u] = 0$ for all $w, u \in U$, then by Lemma 2.6, we get $U \subseteq Z(R)$. This completes the proof of the theorem. □

Using the same techniques with necessary variations, we can prove the following corollary even without the characteristic assumption on the ring.

Corollary 3.2. *Let R be a prime ring. If R admits a nonzero generalized derivation F with d such that $[F(x), x] = 0$ for all $x \in R$, and if $d \neq 0$, then R is commutative.*

In a recent paper, Daif and Bell [7] established that a semiprime ring R must be commutative if it admits a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in R$. Further, Ashraf and Rehman [1] extended the mentioned result for Lie ideals of R. In the present section we generalize this result for generalized derivations and Lie ideals of R.

Theorem 3.3. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a generalized derivation F with d such that $F([u, v]) = [u, v]$ for all $u, v \in U$, and if $F = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Proof. Given that F is a generalized derivation of R such that $F([u,v]) = [u,v]$ for all $u,v \in U$. If $F = 0$, then $[u,v] = 0$ for all $u,v \in U$. Thus by Lemma 2.6, we get the required result.

Now, onward we assume that $F \neq 0$. Suppose on contrary that $U \not\subseteq Z(R)$. For any $u,v \in U$, we have $F([u,v]) = [u,v]$. This can be rewritten as

$$F(u)v + ud(v) - F(v)u - vd(u) = [u,v], \quad \text{for all } u,v \in U. \tag{3.4}$$

Replacing v by $2vu$ in (3.4) and using the fact that $\mathrm{char}(R) \neq 2$, we find that

$$F(u)vu + ud(v)u + [u,v]d(u) - F(v)u^2 - vd(u)u = [u,v]u, \quad \text{for all } u,v \in U,$$

and hence application of (3.4) gives that $[u,v]d(u) = 0$ for all $u,v \in U$. Again replace v by $2wv$, to get $[u,w]vd(u) = 0$ for all $u,v,w \in U$, and hence $[u,w]Ud(u) = (0)$ for all $u,w \in U$. Thus for each $u \in U$, by Lemma 2.1, either $[u,w] = 0$ or $d(u) = 0$. Now, let $U_1 = \{u \in U \mid [u,w] = 0 \text{ for all } w \in U\}$ and $U_2 = \{u \in U \mid d(u) = 0\}$. Then U_1 and U_2 both are additive subgroups of U and $U_1 \cup U_2 = U$. Thus either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $[u,w] = 0$ for all $u,w \in U$. Hence by Lemma 2.6, we get $U \subseteq Z(R)$, contradiction. On the other hand if $U_2 = U$, then $d(u) = 0$ for all $u \in U$. Thus by Lemma 2.2, we get $U \subseteq Z(R)$, again a contradiction. This completes the proof of the theorem. □

Using the same techniques with necessary variations we get the following.

Theorem 3.4. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a generalized derivation F with d such that $F([u,v]) + [u,v] = 0$ for all $u,v \in U$, and if $F = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Corollary 3.5. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a generalized derivation F with d such that $F(uv) = uv$ for all $u,v \in U$, and if $F = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Proof. For any $u,v \in U$, $F(uv - vu) = F(uv) - F(vu) = uv - vu$, and hence by Theorem 3.3, we get the required result. □

Similarly, in view of the Theorem 3.4, we get the following.

Corollary 3.6. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a generalized derivation F with d such that $F(uv) = vu$ for all $u,v \in U$, and if $F = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Theorem 3.7. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a generalized derivation F with d such that $F(u \circ v) = u \circ v$ for all $u, v \in U$, and if $F = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Proof. If $F = 0$, then we have

$$u \circ v = 0, \quad \text{for all } u, v \in U. \tag{3.5}$$

Replacing v by $2vw$ in (3.5) and using (3.5), we have $2v[u, w] = 0$ for all $u, v, w \in U$. This implies that $v[u, w] = 0$ for all $u, v, w \in U$. Again replace v by $[u, r]$, to get $[u, r][u, w] = 0$ for all $u, w \in U$, $r \in R$. For any $s \in R$, replacing r by rs, we get $[u, r]R[u, w] = (0)$ for all $u, w \in U$, $r \in R$. Thus, in particular we have $[u, w]R[u, w] = (0)$ for all $u, w \in U$. Thus primeness of R yields that $[u, w] = 0$ for all $u, w \in U$, and hence by Lemma 2.6, we get the required result.

Therefore now onward we shall assume that $F \neq 0$. Suppose on contrary that $U \nsubseteq Z(R)$. For any $u, v \in U$, we have $F(u \circ v) = u \circ v$. This can be rewritten as

$$F(u)v + ud(v) + F(v)u + vd(u) = u \circ v, \quad \text{for all } u, v \in U. \tag{3.6}$$

Replacing v by $2vu$ in (3.6), we find that

$$(F(u)v + ud(v) + F(v)u + vd(u) - u \circ v)u + (u \circ v)d(u) = 0, \quad \text{for all } u, v \in U.$$

Thus an application of (3.6) gives that $(u \circ v)d(u) = 0$ for all $u, v \in U$. Again replace v by $2wv$, to get $[u, w]vd(u) = 0$ for all $u, v, w \in U$. Note that the arguments given in the proof of Theorem 3.3 are still valid in the present situation and hence repeating the same process we get the required result. $\square$

Using similar arguments one can also prove the following.

Theorem 3.8. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a generalized derivation F with d such that $F(u \circ v) + u \circ v = 0$ for all $u, v \in U$, and if $F = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

4. Ideals and generalized derivations of prime rings

In the hypothesis of Theorems 3.7 and 3.8, if we choose the underlying subset as an ideal instead of a Lie ideal, then we can prove the following result even without the characteristic assumption on the ring.

Theorem 4.1. *Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation F with d such that $F(x \circ y) = x \circ y$ holds for all $x, y \in I$, and if $F = 0$ or $d \neq 0$, then R is commutative.*

Proof. For any $x, y \in I$, we have $F(x \circ y) = x \circ y$. If $F = 0$, then $x \circ y = 0$ for all $x, y \in I$. Replacing y by yz and using the fact that $xy = -yx$, we find that $y[x, z] = 0$ for all $x, y, z \in I$, and hence $IR[x, z] = (0)$ for all $x, z \in I$. Since $I \neq (0)$ and R is prime, we get $[x, z] = 0$ for all $x, z \in I$, and hence by Lemma 2.5, R is commutative. Hence onward we assume that $F \neq 0$. For any $x, y \in I$, we have $F(x \circ y) = x \circ y$. This can be rewritten as

$$F(x)y + xd(y) + F(y)x + yd(x) = x \circ y, \quad \text{for all } x, y \in I. \tag{4.1}$$

Replacing y by yx in (4.1), we get

$$(F(x)y + xd(y) + F(y)x + yd(x) - (x \circ y))x + (x \circ y)d(x) = 0, \quad \text{for all } x, y \in I.$$

In view of (4.1) the above relation yields that $(x \circ y)d(x) = 0$ for all $x, y \in I$. Again replace y by zy, to get $z(x \circ y)d(x) + [x, z]yd(x) = 0$ for all $x, y, z \in I$, and hence $[x, z]IRd(x) = (0)$ for all $x, z \in I$. Thus primeness of R forces that for each $x \in I$ either $d(x) = 0$ or $[x, z]I = (0)$ for all $z \in I$. The set of $x \in I$ for which these two properties hold are additive subgroups of I whose union is I, and therefore $d(x) = 0$ for all $x \in I$ or $[x, z]I = (0)$ for all $x, z \in I$. If $[x, z]I = (0)$ for all $x, z \in I$, then $[x, z]RI = (0)$. Since $I \neq (0)$, we find that $[x, z] = 0$ for all $x, z \in I$, and hence again by Lemma 2.5, R is commutative. On the other hand if $d(x) = 0$ for all $x \in I$, then implies that $[d(x), x] = 0$ for all $x \in I$, and hence by Lemma 2.4, R is commutative. $\square$

Using similar arguments as used in the above theorem, we can prove the following.

Theorem 4.2. *Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation F with d such that $F(x \circ y) + x \circ y = 0$ holds for all $x, y \in I$, and if $F = 0$ or $d \neq 0$, then R is commutative.*

Remark. In view of the above results, it is an obvious question is whether these results can be extended to left multiplier (i.e. a generalized derivation with $d = 0$). Unfortunately, we are unable to extend these results to the case where F is a left multiplier. We leave as an open question whether or not these results can be extended in the setting of left multiplier.

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