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ON THE GROUP $\pi(\Sigma A \times B, X)$

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Introduction

We work in the homotopy category of pointed CW-complexes, and denote the suspension functor with Σ. The group $\pi(\Sigma A \times B, X)$ is not **abelian** in general under the usual multiplication but we use the notation "+" for it. In §1 we shall describe the multiplication of the group $\pi(\Sigma A \times B, X)$, and, as a by-product, obtain from the associativity of the multiplication that the bi-additivity of the generalized Whitehead product (GWP) does not hold in general (see Proposition 3.4 of [1]). In fact, as an example, we offer the following:

Let i_1 and i_2 be inclusion maps from CP^2 and S^n into $CP^2 \vee S^n$ respectively. Then we have that

$$[2\Sigma i_1, \Sigma i_2] \neq 2[\Sigma i_1, \Sigma i_2] \text{ and } [\Sigma i_1, 2\Sigma i_2] = 2[\Sigma i_1, \Sigma i_2]$$

in $\pi(\Sigma CP^2 \wedge S^n, \Sigma CP^2 \vee S^n)$.

In §2 we investigate the group $\pi(\Sigma X, \Sigma A \times B)$ and show that any element of this group can be determined by 4-components under some assumptions. In §3 we apply §2 to the case of $X = A \times B$, i.e. the group of self-maps of the space $\Sigma A \times B$.

Specially we are interested in describing the composition of two elements with their components and give the special case of $A = B = S^n$ as an example.

1. $\pi(\Sigma A \times B, X)$

Let i_1 and i_2 be inclusion maps: $A, B \to A \times B$, and let P_A and P_B be projections: $A \times B \to A, B$ respectively.

Lemma 1.1. *Let $\pi\colon A \times B \to A \wedge B$ be the projection. Any element $f \in \pi(\Sigma A \times B, X)$ can be uniquely represented by the form*

$$f = \alpha\Sigma P_A + \beta\Sigma P_B + \gamma\Sigma\pi$$

for $\alpha \in \pi(\Sigma A, X)$, $\beta \in \pi(\Sigma B, X)$ and $\gamma \in \pi(\Sigma A \wedge B, X)$.

Proof. In fact a representation can be obtained from a part of Puppe exact sequence of the cofibering: $A \vee B \to A \times B$. Then clearly we have $\alpha = f\Sigma i_A$, $\beta = f\Sigma i_B$ and moreover the uniqueness of γ follows from the injectivity of $(\Sigma\pi)^*$. □

Here we give a brief account of GWP of [1]. For two maps $\alpha\colon \Sigma A \to X$ and $\beta\colon \Sigma B \to X$, GWP $[\alpha,\beta] \in \pi(\Sigma A \wedge B, X)$ is defined by

$$\alpha\Sigma P_A + \beta\Sigma P_B = \beta\Sigma P_B + \alpha\Sigma P_A + [\alpha,\beta]\Sigma\pi.$$

Proposition 1.2. GWP *has following properties*:

(1) *Let us be* $\langle\alpha,\beta\rangle$ *the commutator of* α *and* β $(\in \pi(\Sigma A, X))$ *then we have*
$$\langle\alpha,\beta\rangle = [\alpha,\beta]\Sigma d_A,$$
where d_A *is the diagonal map*: $A \to A \wedge A$.

(2) *If* $f \in \pi(X,Y)$ *then* $f[\alpha,\beta] = [f\alpha, f\beta]$.

(3) *If* $\sigma_k \in \pi(\Sigma Y_k, Z)$ *and* $f_k \in \pi(X_k, Y_k)$ *for* $k = 1,2$ *then it holds that*
$$[\sigma_1\Sigma f_1, \sigma_2\Sigma f_2] = [\sigma_1,\sigma_2]\Sigma f_1 \wedge f_2.$$

(4) *For four maps* $\Sigma f \in \pi(\Sigma Y, \Sigma A)$, $\Sigma g \in \pi(\Sigma Y, \Sigma B)$, $\sigma \in \pi(\Sigma A, X)$ *and* $\tau \in \pi(\Sigma B, X)$ *we have*
$$\sigma\Sigma f + \tau\Sigma g = \tau\Sigma g + \sigma\Sigma f + [\sigma,\tau]\Sigma(f \wedge g)\Sigma d_Y.$$

(5) *If* X *is a suspension* (*i.e.* $X = \Sigma X^*$) *then* $d_X = 0$.

Lemma 1.3. *For* $\alpha \in \pi(\Sigma A, X)$, $\beta \in \pi(\Sigma B, X)$ *and* $\gamma \in \pi(\Sigma A \wedge B, X)$ *we have the following*:

(1) $\alpha\Sigma P_A + \beta\Sigma P_B = \beta\Sigma P_B + \alpha\Sigma P_A + [\alpha,\beta]\Sigma\pi$.

(2) $\alpha\Sigma P_A + \gamma\Sigma\pi = \gamma\Sigma\pi + \alpha\Sigma P_A + [\alpha,\gamma]\Sigma\varphi_A\Sigma\pi$, *where* φ_A *is defined by* $\varphi_A(a \wedge b) = a \wedge a \wedge b$.

(3) $\beta\Sigma P_B + \gamma\Sigma\pi = \gamma\Sigma\pi + \beta\Sigma P_B + [\beta,\gamma]\Sigma\psi_B\Sigma\pi$, *where* ψ_B *is defined by* $\psi_B(a \wedge b) = b \wedge a \wedge b$.

Proof. (1) is just the definition of GWP. Next, by applying (1) and (4) of lemma 1.2 to the diagram:

$$\begin{tikzcd}
\Sigma A \times B \arrow[r, "\Sigma P_A"] \arrow[d, equal] & \Sigma A \arrow[r, "\alpha"] & X \arrow[d, equal] \\
\Sigma A \times B \arrow[r, "\Sigma\pi"] & \Sigma A \wedge B \arrow[r, "\gamma"] & X,
\end{tikzcd}$$

we have that

$$[\alpha\Sigma P_A, \gamma\Sigma\pi]\Sigma d_A \times 1_B = [\alpha,\gamma]\Sigma(P_A \wedge \pi)\Sigma d_A \times 1_B = [\alpha,\gamma]\Sigma\varphi_A\Sigma\pi.$$

The case (3) is analogous to the case (2). Thus the proof is completed. □

Now let us represent $f = \alpha\Sigma P_A + \beta\Sigma P_B + \gamma\Sigma\pi$ with the triple (α,β,γ).

Theorem 1.4. $(\alpha_1,\beta_1,\gamma_1) + (\alpha_2,\beta_2,\gamma_2) = (\alpha_1+\alpha_2, \beta_1+\beta_2, \delta+\gamma_2)$ *for* $\delta = \delta_1 - [\beta_2,\delta_1]\Sigma\psi_B$ *and* $\delta_1 = -[\alpha_2,\beta_1] + \gamma_1 - [\alpha_2,\gamma_1]\Sigma\varphi_A$.

Proof. For abbreviation we use notations: $\bar{\alpha} = \alpha\Sigma P_A$, $\bar{\beta} = \beta\Sigma P_B$, $\bar{\gamma} = \gamma\Sigma\pi$ and so on. Now by using lemma 1.3 we have equalities

$$\begin{aligned} f_1 + f_2 &= (\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) \\ &= \bar{\alpha}_1 + \bar{\beta}_1 + \bar{\gamma}_1 + \bar{\alpha}_2 + \bar{\beta}_2 + \bar{\gamma}_2 \\ &= \bar{\alpha}_1 + \bar{\beta}_1 + \bar{\alpha}_2 + \bar{\gamma}_1 - [\alpha_2, \gamma_1]\Sigma\varphi_A\Sigma\pi + \bar{\beta}_2 + \bar{\gamma}_2 \\ &= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 - \overline{[\alpha_2, \beta_1]} + \bar{\gamma}_1 - \overline{[\alpha_2, \gamma_1]\Sigma\varphi_A} + \bar{\beta}_2 + \bar{\gamma}_2 \\ &= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 + \bar{\delta}_1 + \bar{\beta}_2 + \bar{\gamma}_2 \\ &= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 + \bar{\beta}_2 + \bar{\delta}_1 - \overline{[\beta_2, \delta_1]\Sigma\psi_B} + \bar{\gamma}_2 \\ &= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 + \bar{\beta}_2 + \bar{\delta} + \bar{\gamma}_2 \\ &= (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \delta + \gamma_2). \end{aligned}$$

Thus the proof is completed. □

Corollary 1.5. *If A and B are both suspensions then it holds that*

$$(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, -[\alpha_2, \beta_1] + \gamma_1 + \gamma_2).$$

Proof. Since φ_A and ψ_B are trivial by (5) of lemma 1.2 the proof follows from Theorem 1.4. □

Corollary 1.6. *For $\alpha_1, \alpha_2 \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$ it holds that*

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_2, -[\alpha_1, \beta]]\Sigma\varphi_A + [\alpha_1, \beta] + [\alpha_2, \beta].$$

Proof.

$$\begin{aligned} &(0, \beta, 0) + \{(\alpha_1, 0, 0) + (\alpha_2, 0, 0)\} \\ &= (0, \beta, 0) + (\alpha_1 + \alpha_2, 0, 0) \\ &= (\alpha_1 + \alpha_2, \beta, -[\alpha_1 + \alpha_2, \beta]). \end{aligned}$$

On the other hand

$$\begin{aligned} &\{(0, \beta, 0) + (\alpha_1, 0, 0)\} + (\alpha_2, 0, 0) \\ &= (\alpha_1, \beta, -[\alpha_1, \beta]) + (\alpha_2, 0, 0) \\ &= (\alpha_1 + \alpha_2, \beta, -[\alpha_2, \beta] - [\alpha_1, \beta] - [\alpha_2, -[\alpha_1, \beta]]\Sigma\varphi_A). \end{aligned}$$

Thus the proof follows from the associativity of the addition of the group $\pi(\Sigma A \times B, X)$. □

By (5) of lemma 1.2 and the above corollary 1.6 it is easy to obtain the following:

Corollary 1.7 (Proposition 3.4 of [1])**.** *If A is a suspension then it holds that*

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta]$$

for $\alpha_1, \alpha_2 \in \pi(\Sigma A, X)$ *and* $\beta \in \pi(\Sigma B, X)$.

Now we consider a special case: $A = CP^2$, $B = S^n$ and $X = \Sigma CP^2 \vee S^n$. Let i_1 and i_2 be inclusions: $CP^2, S^n \to CP^2 \vee S^n$ respectively, and let us be $\alpha = \Sigma i_1$ and $\beta = \Sigma i_2$. Since $\varphi_A \colon CP^2 \wedge S^n \to CP^2 \wedge CP^2 \wedge S^n$ is defined by $\varphi_A(a \wedge b) = a \wedge a \wedge b$, φ_A can be regarded as $\Sigma^n d$ for the diagonal map $d \colon CP^2 \to CP^2 \wedge CP^2$.

Since $d^* \colon H^4(CP^2 \wedge CP^2) \to H^4(CP^2)$ is clearly an isomorphism $\Sigma^n d$ is non-trivial. On the other hand, accordingly to Hilton-Milnor Theorem ([3]) $[\alpha, [\alpha, \beta]]_*$ is injective. Hence $[\alpha, [\alpha, \beta]]\Sigma\varphi_A$ is non-trivial. Then these show

$$[2\Sigma i_1, \Sigma i_2] \neq 2[\Sigma i_1, \Sigma i_2] \text{ and } [\Sigma i_1, 2\Sigma i_2] = 2[\Sigma i_1, \Sigma i_2].$$

Remark. The second equality follows from Corollary 1.7.

2. On the group $\pi(\Sigma X, \Sigma A \times B)$

First by using lemma 1.1 we define $\xi_{AB} \in \pi(\Sigma A \wedge B, \Sigma A \times B)$ as follows:

$$1_{\Sigma A \times B} = \Sigma i_A \Sigma P_A + \Sigma i_B \Sigma P_B + \xi_{AB} \Sigma\pi.$$

Corollary 2.1. $\Sigma\pi\xi_{AB} = 1_{\Sigma A \wedge B}$.

Proof. Apply $(\Sigma\pi)_*$ to the above equality. Then we have

$$\begin{aligned} \Sigma\pi &= \Sigma\pi(\Sigma i_A \Sigma P_A + \Sigma i_B \Sigma P_B + \xi_{AB}\Sigma\pi) \\ &= 0 + 0 + \Sigma\pi\xi_{A\wedge B}\Sigma\pi. \end{aligned}$$

Since $(\Sigma\pi)^*$ is injective the proof is completed. □

Here we note that the representation of $f \in \pi(\Sigma A \times B, X)$ in lemma 1.1 is given by

$$f = f|_{\Sigma A}\Sigma P_A + f|_{\Sigma B}\Sigma P_B + f\xi_{AB}\Sigma\pi,$$

where $f|_K$ denotes the restriction of f on K.

For example if h is a map $A \times B \to X$ then $\Sigma h\xi_{AB}$ is essentially the Hopf-construction of f, i.e. $C(h)$ (see [2]) and we have a representation:

$$\Sigma h = \Sigma h_A + \Sigma h_B + C(h)\Sigma\pi,$$

where h is a map of type (h_A, h_B).

Secondly we define two maps $\varphi \in \pi(Y, \Sigma A \times B)$ and $\phi \in \pi(\Sigma A \times B, Y)$ for $Y = \Sigma A \vee \Sigma B \vee \Sigma A \wedge B$ by

$$\begin{aligned} \varphi &= \Sigma i_A \vee \Sigma i_B \vee \xi_{AB}, \\ \phi &= i_{\Sigma A}\Sigma P_A + i_{\Sigma B}\Sigma P_B + i_{\Sigma A\wedge B}\Sigma\pi. \end{aligned}$$

Lemma 2.2. *φ is a homotopy equivalence with ϕ as its inverse.*

Proof. Easy. □

In the following of this section we assume that

(1) A is a-connected and B is b-connected,
(2) $a \leq b$,
(3) $\dim X \leq 2a + b + 2$.

Then by Hilton-Milnor theorem, $f \in \pi(\Sigma X, \Sigma A \times B)$ can be represented as follows:

$$f = \Sigma i_A f_A + \Sigma i_B f_B + [\Sigma i_A, \Sigma i_B] f_{C1} + \xi_{AB} f_{C2},$$

where $f_* \in \pi(\Sigma X, \Sigma *)$ and $f_{C*} \in \pi(\Sigma X, \Sigma A \wedge B)$.

More precisely we have

Lemma 2.3. *$f_A = \Sigma P_A f$, $f_B = \Sigma P_B f$, $f_{C1} = \Sigma(P_A \wedge P_B)H(f)$ and $f_{C2} = \Sigma\pi f$, where $H(f)$ denotes Hopf-invariant of f.*

Proof. First we note that $[\gamma, \delta\Sigma\pi]H(f) = 0$ because this element is decomposed as follows:

$$\Sigma X \xrightarrow{H(f)} \Sigma(A \times B) \wedge (A \times B) \xrightarrow{\Sigma(1 \wedge \pi)} \Sigma(A \times B) \wedge (A \wedge B) \xrightarrow{[\gamma, \delta]} \Sigma A \times B.$$

Then the proof is deduced from our assumptions. Secondly apply f from the right to the equality. We obtain that

$$\begin{aligned} f &= (\Sigma i_A P_A + \Sigma i_B P_B + \xi_{AB}\Sigma\pi) f \\ &= (\Sigma i_A P_A + \Sigma i_B P_B) f + \xi_{AB}\Sigma\pi f \\ &= \Sigma i_A P_A f + \Sigma i_B P_B f + [\Sigma i_A, \Sigma i_B](\Sigma P_A \wedge \Sigma P_B) H(f) + \xi_{AB}\Sigma\pi f. \end{aligned}$$

Thus the proof is completed. □

3. On the group $\pi(\Sigma A \times B, \Sigma A \times B)$

In this section we assume that A and B are both n-connected, $\dim A \leq \dim B$ and $\dim A + \dim B \leq 3n + 2$.

Lemma 3.1. *Our* ***assumptions*** *contain*

(1) *A, B and $A \wedge B$ are all suspensions, so $\pi(\Sigma K, X)$ is* ***abelian*** *for $K = A$, B and $A \wedge B$,*
(2) $\dim B \leq 2n + 1$. *Hence $\pi(\Sigma *, \Sigma A \wedge B) = 0$ for $* = A$ or B,*
(3) $\Sigma\colon \pi(X, Y) \to \pi(\Sigma X, \Sigma Y)$ *is onto for any pair (X, Y) of $\{A, B\}$.*

Proof. Easy. □

In §1, $f \in \pi(\Sigma A \times B, \Sigma A \wedge B)$ has a representation:

$$f = f_A \Sigma P_A + f_B \Sigma P_B + f_C \Sigma\pi$$

for $f_* \in \pi(*, \Sigma A \times B)$ and $C = A \wedge B$.

And moreover in §2, f_* has a representation:

$$f_* = \Sigma i_A f_{*1} + \Sigma i_B f_{*2} + \xi_{AB} f_{*3} + [\Sigma i_A, \Sigma i_B] f_{*4}$$

for $f_{*1} \in \pi(\Sigma *, \Sigma A)$, $f_{*2} \in \pi(\Sigma *, \Sigma B)$ and $f_{*3}, f_{*4} \in \pi(\Sigma *, \Sigma A \wedge B)$.

Thus $f \in \pi(\Sigma A \times B, \Sigma A \times B)$ has a form of a (3×4)-matrix (note lemma 3.1):

$$(f_{*k}) = \begin{pmatrix} f_{A1} & f_{A2} & 0 & 0 \\ f_{B1} & f_{B2} & 0 & 0 \\ f_{C1} & f_{C2} & f_{C3} & f_{C4} \end{pmatrix}.$$

We want to compute the composition gf for $f, g \in \pi(\Sigma A \times B, \Sigma A \times B)$. Since we have that $gf = (gf_A)\Sigma P_A + (gf_B)\Sigma P_B + (gf_C)\Sigma\pi$ it is sufficient for our purpose to compute $gf_* \in \pi(\Sigma *, \Sigma A \times B)$.

Lemma 3.2. $g_C = g\xi_{AB}$.

Proof. First we have

$$\begin{aligned} g &= g1_{\Sigma A \times B} \\ &= g(\Sigma i_A \Sigma P_A + g\Sigma i_B \Sigma P_B + g\xi_{AB}\Sigma\pi) \\ &= g|_{\Sigma A}\Sigma P_A + g|_{\Sigma B}\Sigma P_B + g\xi_{AB}\Sigma\pi. \end{aligned}$$

On the other hand $g = g_A\Sigma P_A + g_B \Sigma P_B + g_C\Sigma\pi$. Hence we have $g_C = g\xi_{AB}$. □

Now we proceed to gf_*:

$$\begin{aligned} gf_* &= g(\Sigma i_A f_{*1} + \Sigma i_B f_{*2} + \xi_{AB} f_{*3} + [\Sigma i_A, \Sigma i_B] f_{*4}) \\ &= g|_{\Sigma A} f_{*1} + g|_{\Sigma B} f_{*2} + g\xi_{AB} f_{*3} + [g|_{\Sigma A}, g|_{\Sigma B}] f_{*4} \\ &= g_A f_{*1} + g_B f_{*2} + g_C f_{*3} + [g_A, g_B] f_{*4}. \end{aligned}$$

Lemma 3.3.

$$\begin{aligned} [g_A, g_B] = &\ \Sigma i_A [g_{A1}, g_{B1}] + \Sigma i_B [g_{A2}, g_{B2}] \\ &+ [\Sigma i_A, \Sigma i_B](\Sigma g'_{A1} \wedge \Sigma g'_{B2} - \tau \Sigma g'_{A2} \wedge g'_{B1}), \end{aligned}$$

where $g_{*k} = \Sigma g'_{*k}$ for $* \in \{A, B\}$ and $k = 1, 2$.

Proof. Apply lemma 1.2 and Propositions 3.3, 3.4 of [1] to the equality:

$$[g_A, g_B] = [\Sigma i_A g_{A1} + \Sigma i_B g_{A2}, \Sigma i_A g_{B1} + \Sigma i_B g_{B2}],$$

then the proof is completed. □

Lemma 3.4. *We have*

$$g_A f_{C1} = \Sigma i_A (g_{A1} f_{C1}) + \Sigma i_B (g_{A2} f_{C1}) + [\Sigma i_A, \Sigma i_B] \Sigma g'_{A1} \wedge g'_{A2} H(f_{C,1}).$$

Proof. Apply the distributive law to the equality:

$$g_A f_{C1} = (\Sigma i_A g_{A1} + \Sigma i_B g_{A2}) f_{C1},$$

then the proof is completed. □

From these lemmas we have

Theorem 3.5. *If $f = (f_{*k})$ and $g = (g_{*k})$ then $gf = h = (h_{*k})$ is given by*

(1) *the case of $* = A$, or B,*

$$\begin{aligned} h_{*1} &= g_{A1} f_{*1} + g_{B1} f_{*2}, \\ h_{*2} &= g_{A2} f_{*1} + g_{B2} f_{*2}, \end{aligned}$$

(2) *the case of $* = C = A \wedge B$,*

$$\begin{aligned} h_{C1} &= g_{A1} f_{C1} + g_{B1} f_{C2} + [g_{A1}, g_{B1}] f_{C4} + g_{C1} f_{C3}, \\ h_{C2} &= g_{A2} f_{C1} + g_{B2} f_{C2} + [g_{A2}, g_{B2}] f_{C4} + g_{C2} f_{C3}, \\ h_{C3} &= g_{C3} f_{C3}, \\ h_{C4} &= \Sigma g'_{A1} \wedge g'_{A2} H(f_{C1}) + \Sigma g'_{B1} \wedge g'_{B2} H(f_{C2}) \\ &\quad + (\Sigma g'_{A1} \wedge g'_{B2} - \Sigma\tau\Sigma g'_{A2} \wedge g'_{B1}) f_{C4} + g_{C4} f_{C3}. \end{aligned}$$

As an example we take $A = B = S^n$. Let us be $f \in \pi(\Sigma S^n \times S^n, \Sigma S^n \times S^n)$ with its matrix:

$$\begin{pmatrix} f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ f_{31} & f_{32} & f_{33} & f_{34} \end{pmatrix},$$

where $f_{ij}(\{i,j\} = \{1,2\}), f_{33}, f_{34} \in \mathbf{Z}$ and $f_{31}, f_{32} \in \pi_{2n+1}(S^{n+1})$.

If $h = gf$ then h_{ij} is given by

$$\begin{aligned} h_{11} &= g_{11} f_{11} + g_{21} f_{12}, \\ h_{12} &= g_{12} f_{11} + g_{22} f_{12}, \\ h_{21} &= g_{11} f_{21} + g_{21} f_{22}, \\ h_{22} &= g_{12} f_{21} + g_{22} f_{22}, \\ h_{31} &= g_{11} \circ f_{31} + g_{21} \circ f_{32} + f_{33} g_{31} + f_{34} g_{12} g_{22} [\iota_{n+1}, \iota_{n+1}], \\ h_{32} &= g_{12} \circ f_{31} + g_{22} \circ f_{32} + f_{33} g_{32} + f_{34} g_{12} g_{22} [\iota_{n+1}, \iota_{n+1}], \\ h_{33} &= g_{33} f_{33}, \\ h_{34} &= (g_{11} g_{22} - (-1)^n g_{12} g_{21}) f_{34} + g_{34} f_{33} + g_{11} g_{12} H(f_{31}) + g_{21} g_{22} H(f_{32}), \end{aligned}$$

where $g_{**} \circ f_{**}$ denotes $(g_{**} \iota_{n+1}) f_{**}$.

Here we describe some results obtained from the above table of the composition.

1. f is a homotopy equivalence if and only if $f_{33} = \pm 1$ and
$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \pm 1.$$
2. We denote the group of self-homotopy equivalences of $\Sigma S^n \times S^n$ with $\varepsilon(G_n)$ and orientation-preserving self-homotopy equivalences with $\varepsilon_0(G_n)$, where **orientation-preserving** means that the above determinant is 1 and $f_{33} = 1$, then $\varepsilon_0(G_n)$ is a normal subgroup and the quotient is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.
3. $\varepsilon_0(G_n)$ contains a subgroup:
$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \end{pmatrix} \right\} \cong \mathbf{Z}.$$
4. $\varepsilon_0(G_n)$ contains a subgroup:
$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & \beta & 1 & 0 \end{pmatrix} \right\} \cong \pi_{2n+1}(S^{n+1}) \oplus \pi_{2n+1}(S^{n+1}).$$

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