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CERTAIN METRICS ON $R^4{}_+$ (II)

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Abstract. This paper is a continuation of the one with the same title ([4]), in which we obtained a special solution for a system of differential equations on metric tensors on $R^4{}_+$ satisfying the Einstein condition for Case I generalizing the Ot-metric:

$$ds^2 = \frac{1}{x_4x_4}\left\{\sum_{b,c=1}^{3}\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dx_4\right\}$$

in Theorem 1, and proved that there exist no solutions for Case II in Theorem 2. In this work, we shall show that we can obtain more general solutions for Case I which depend on the latitude parameter. We use the results in [4], so the section numbers start from 5.

5. Search for general solutions for Case I

We consider solutions $g_{ij} = F_{ij}/u_4u_4$ which satisfy the Einstein condition under the restriction (1.3):

$$F_{\alpha\beta} = F_{\alpha\beta}(u_1, u_2),\ F_{\lambda\mu} = F_{\lambda\mu}(u_1, u_2, u_4),\ F_{12} = F_{\alpha\lambda} = 0,$$

where $\alpha, \beta, \gamma, \ldots = 1, 2$ and $\lambda, \mu, \nu, \ldots = 3, 4$ in the paper and (2.6):

$$\frac{\partial F_{33}}{\partial u_4} = \frac{\partial F_{34}}{\partial u_4} = 0.$$

Then the solutions are divided into two cases as follows:

$$\text{Case I} : Y_1 = Y_2 = 0 \quad \text{and} \quad \text{Case II} : (Y_1, Y_2) \neq (0, 0),$$

where

$$Y_\alpha = \frac{1}{\Delta}\left(F_{33}\frac{\partial F_{34}}{\partial u_\alpha} - F_{34}\frac{\partial F_{33}}{\partial u_\alpha}\right), \quad \Delta = F_{33}F_{44} - F_{34}F_{34}.$$

For Case I, we obtain relations (3.2):

$$
\tag{5.1}
\begin{aligned}
&F_{34} = bF_{33},\ F_{44} = b^2F_{33} + \phi,\ \Delta = \phi F_{33},\\
&F^{33} = \frac{F_{44}}{\Delta} = \frac{1}{F_{33}} + \frac{b^2}{\phi},\ F^{34} = -\frac{F_{34}}{\Delta} = -\frac{b}{\phi},\ F^{44} = \frac{F_{33}}{\Delta} = \frac{1}{\phi},\\
&\phi = \phi(u_4),\ b = \text{constant}.
\end{aligned}
$$

We found some special solution in §3 under the restriction (3.14):

$$
\frac{\partial F_{11}}{\partial u_2} = \frac{\partial F_{22}}{\partial u_2} = 0 \text{ and } F_{33} = \psi(u_1)\sin^2 u_2,
$$

from which we get the relation (3.19): $\psi(u_1) = cF_{22}(u_1)$, $c =$ constant and so $F_{33}/F_{22} = c\sin^2 u_2$.

Now, we consider Case I without assuming (3.14) and so F_{ij} are determined by the conditions $(3.10')\sim(3.13')$ from which we see that

$$
-\frac{1}{u_4\phi^2}\frac{d\phi}{du_4} = \frac{1}{u_4}\frac{d}{du_4}\frac{1}{\phi} = 2c_1
$$

and so

$$
\tag{5.2}
\frac{1}{\phi} = c_1u_4u_4 + c_0, \quad c_0, c_1 = \text{constants}.
$$

$(3.10')$, $(3.11')$, $(3.12')$ and $(3.13')$ can be written respectively as follows.

$$
\tag{5.3}
\begin{aligned}
&\frac{1}{F_{11}}\left(\frac{\partial^2 \log F_{33}}{\partial u_1\partial u_1} + \frac{1}{2}\frac{\partial \log F_{33}}{\partial u_1}\frac{\partial}{\partial u_1}\log\left(\frac{F_{33}F_{22}}{F_{11}}\right)\right)\\
&+\frac{1}{F_{22}}\left(\frac{\partial^2 \log F_{33}}{\partial u_2\partial u_2} + \frac{1}{2}\frac{\partial \log F_{33}}{\partial u_2}\frac{\partial}{\partial u_2}\log\left(\frac{F_{33}F_{11}}{F_{22}}\right)\right) + 4c_1 = 0,
\end{aligned}
$$

$$
\tag{5.4}
\begin{aligned}
&\frac{\partial^2 \log F_{33}}{\partial u_1\partial u_2} + \frac{1}{2}\left(\frac{\partial \log F_{33}}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_2}\right.\\
&\qquad\left. - \frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_2} - \frac{\partial \log F_{11}}{\partial u_2}\frac{\partial \log F_{33}}{\partial u_1}\right) = 0,
\end{aligned}
$$

$$
\tag{5.5}
\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2 \log F_{33}}{\partial u_1\partial u_1} + \frac{\partial^2 \log F_{22}}{\partial u_1\partial u_1}\right.\\
&\qquad\left. + \frac{1}{2}\left(\frac{\partial \log F_{33}}{\partial u_1}\frac{\partial}{\partial u_1}\log\left(\frac{F_{33}}{F_{11}}\right) + \frac{\partial \log F_{22}}{\partial u_1}\frac{\partial}{\partial u_1}\log\left(\frac{F_{22}}{F_{11}}\right)\right)\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2 \log F_{11}}{\partial u_2\partial u_2} + \frac{1}{2}\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial}{\partial u_2}\log\left(\frac{F_{33}F_{11}}{F_{22}}\right)\right\} + 4c_1 = 0,
\end{aligned}
$$

and

$$\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2\log F_{22}}{\partial u_1\partial u_1}+\frac{1}{2}\frac{\partial\log F_{22}}{\partial u_1}\frac{\partial}{\partial u_1}\log\Big(\frac{F_{33}F_{22}}{F_{11}}\Big)\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2\log F_{33}}{\partial u_2\partial u_2}+\frac{\partial^2\log F_{11}}{\partial u_2\partial u_2}+\frac{1}{2}\Big(\frac{\partial\log F_{33}}{\partial u_2}\frac{\partial}{\partial u_2}\log\Big(\frac{F_{33}}{F_{22}}\Big)\right.\\
&\quad\left.+\frac{1}{2}\frac{\partial\log F_{11}}{\partial u_2}\frac{\partial}{\partial u_2}\log\Big(\frac{F_{11}}{F_{22}}\Big)\Big)\right\}+4c_1=0.
\end{aligned}\tag{5.6}$$

We obtain by (5.5)−(5.3) the equation:

$$\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2\log F_{22}}{\partial u_1\partial u_1}+\frac{1}{2}\frac{\partial\log F_{22}}{\partial u_1}\frac{\partial}{\partial u_1}\log\Big(\frac{F_{22}}{F_{11}F_{33}}\Big)\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2\log F_{11}}{\partial u_2\partial u_2}-\frac{\partial^2\log F_{33}}{\partial u_2\partial u_2}\right.\\
&\left.+\frac{1}{2}\frac{\partial}{\partial u_2}\log\Big(\frac{F_{11}}{F_{33}}\Big)\frac{\partial}{\partial u_2}\log\Big(\frac{F_{33}F_{11}}{F_{22}}\Big)\right\}=0
\end{aligned}\tag{5.7}$$

and by (5.6)−(5.3) the equation:

$$\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2\log F_{22}}{\partial u_1\partial u_1}-\frac{\partial^2\log F_{33}}{\partial u_1\partial u_1}+\frac{1}{2}\frac{\partial}{\partial u_1}\log\Big(\frac{F_{22}}{F_{33}}\Big)\frac{\partial}{\partial u_1}\log\Big(\frac{F_{33}F_{22}}{F_{11}}\Big)\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2\log F_{11}}{\partial u_2\partial u_2}+\frac{1}{2}\frac{\partial\log F_{11}}{\partial u_2}\frac{\partial}{\partial u_2}\log\Big(\frac{F_{11}}{F_{22}F_{33}}\Big)\right\}=0.
\end{aligned}\tag{5.8}$$

The system of equations (5.3), (5.4), (5.5), (5.6) is equivalent to the system of equations (5.3), (5.4), (5.7), (5.8). Now setting $F_{33}/F_{22}=p$, the above equations are written as follows. First, (5.3) is written as

$$\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2\log F_{22}}{\partial u_1\partial u_1}+\frac{\partial^2\log p}{\partial u_1\partial u_1}+\frac{1}{2}\Big(\frac{\partial\log F_{22}}{\partial u_1}+\frac{\partial\log p}{\partial u_1}\Big)\frac{\partial}{\partial u_1}\log\Big(\frac{F_{22}F_{22}p}{F_{11}}\Big)\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2\log F_{22}}{\partial u_2\partial u_2}+\frac{\partial^2\log p}{\partial u_2\partial u_2}+\frac{1}{2}\Big(\frac{\partial\log F_{22}}{\partial u_2}+\frac{\partial\log p}{\partial u_2}\Big)\frac{\partial}{\partial u_2}\log(F_{11}p)\right\}\\
&+4c_1=0,
\end{aligned}$$

i.e.

$$
\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2 \log F_{22}}{\partial u_1 \partial u_1}+\frac{\partial^2 \log p}{\partial u_1 \partial u_1}+\frac{1}{2}\left(\frac{\partial \log F_{22}}{\partial u_1}+\frac{\partial \log p}{\partial u_1}\right)\left(2\frac{\partial \log F_{22}}{\partial u_1}\right.\right.\\
&\qquad\left.\left.-\frac{\partial \log F_{11}}{\partial u_1}+\frac{\partial \log p}{\partial u_1}\right)\right\}+\frac{1}{F_{22}}\left\{\frac{\partial^2 \log F_{22}}{\partial u_2 \partial u_2}+\frac{\partial^2 \log p}{\partial u_2 \partial u_2}\right.\\
&\qquad\left.+\frac{1}{2}\left(\frac{\partial \log F_{22}}{\partial u_2}+\frac{\partial \log p}{\partial u_2}\right)\left(\frac{\partial \log F_{11}}{\partial u_2}+\frac{\partial \log p}{\partial u_2}\right)\right\}+4c_1=0.
\end{aligned}
\tag{5.3'}
$$

Second, (5.4) is written as

$$
\begin{aligned}
&\frac{\partial^2 \log F_{22}}{\partial u_1 \partial u_2}+\frac{\partial^2 \log p}{\partial u_1 \partial u_2}+\frac{1}{2}\left(\frac{\partial \log F_{22}}{\partial u_2}+\frac{\partial \log p}{\partial u_2}\right)\frac{\partial \log p}{\partial u_1}\\
&-\frac{1}{2}\frac{\partial \log F_{11}}{\partial u_2}\left(\frac{\partial \log F_{22}}{\partial u_1}+\frac{\partial \log p}{\partial u_1}\right)=0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\frac{\partial^2 \log F_{22}}{\partial u_1 \partial u_2}-\frac{1}{2}\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial \log F_{22}}{\partial u_1}+\frac{\partial^2 \log p}{\partial u_1 \partial u_2}\\
&+\frac{1}{2}\frac{\partial \log p}{\partial u_1}\frac{\partial \log p}{\partial u_2}+\frac{1}{2}\frac{\partial}{\partial u_2}\log\left(\frac{F_{22}}{F_{11}}\right)\frac{\partial \log p}{\partial u_1}=0.
\end{aligned}
\tag{5.4'}
$$

Third, (5.7) is written as

$$
\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2 \log F_{22}}{\partial u_1 \partial u_1}-\frac{1}{2}\frac{\partial \log F_{22}}{\partial u_1}\left(\frac{\partial \log F_{11}}{\partial u_1}+\frac{\partial \log p}{\partial u_1}\right)\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2 \log F_{11}}{\partial u_2 \partial u_2}-\frac{\partial^2 \log F_{22}}{\partial u_2 \partial u_2}-\frac{\partial^2 \log p}{\partial u_2 \partial u_2}\right.\\
&\quad\left.+\frac{1}{2}\left(\frac{\partial}{\partial u_2}\log\left(\frac{F_{11}}{F_{22}}\right)-\frac{\partial \log p}{\partial u_2}\right)\left(\frac{\partial \log F_{11}}{\partial u_2}+\frac{\partial \log p}{\partial u_2}\right)\right\}=0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2 \log F_{22}}{\partial u_1 \partial u_1}-\frac{1}{2}\frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log F_{11}}{\partial u_1}-\frac{1}{2}\frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log p}{\partial u_1}\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2}{\partial u_2 \partial u_2}\log\left(\frac{F_{11}}{F_{22}}\right)+\frac{1}{2}\frac{\partial}{\partial u_2}\log\left(\frac{F_{11}}{F_{22}}\right)\frac{\partial \log F_{11}}{\partial u_2}\right.\\
&\quad\left.-\frac{\partial^2 \log p}{\partial u_2 \partial u_2}-\frac{1}{2}\frac{\partial \log p}{\partial u_2}\frac{\partial \log p}{\partial u_2}-\frac{1}{2}\frac{\partial \log F_{22}}{\partial u_2}\frac{\partial \log p}{\partial u_2}\right\}=0.
\end{aligned}
\tag{5.7'}
$$

Fourth, (5.8) is written as

$$
\begin{aligned}
&-\frac{1}{F_{11}}\left\{\frac{\partial^2\log p}{\partial u_1\partial u_1}+\frac{1}{2}\frac{\partial\log p}{\partial u_1}\left(2\frac{\partial\log F_{22}}{\partial u_1}-\frac{\partial\log F_{11}}{\partial u_1}+\frac{\partial\log p}{\partial u_1}\right)\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2\log F_{11}}{\partial u_2\partial u_2}\right.\\
&\left.-\frac{1}{2}\frac{\partial\log F_{11}}{\partial u_2}\left(2\frac{\partial\log F_{22}}{\partial u_2}-\frac{\partial\log F_{11}}{\partial u_2}+\frac{\partial\log p}{\partial u_2}\right)\right\}=0.
\end{aligned}\tag{5.8$'$}
$$

Now, we put a restriction given by

$$
p=\sin^2 u_2\times\text{constant}.\tag{5.9}
$$

Then,(5.3′), (5.4′), (5.7′) and (5.8′) turn respectively into the following equations. From (5.3′) we obtain

$$
\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2\log F_{22}}{\partial u_1\partial u_1}+\frac{\partial\log F_{22}}{\partial u_1}\frac{\partial\log F_{22}}{\partial u_1}-\frac{1}{2}\frac{\partial\log F_{22}}{\partial u_1}\frac{\partial\log F_{11}}{\partial u_1}\right\}\\
&+\frac{1}{F_{22}}\left\{\frac{\partial^2\log F_{22}}{\partial u_2\partial u_2}+\frac{1}{2}\frac{\partial\log F_{11}}{\partial u_2}\frac{\partial\log F_{22}}{\partial u_2}\right.\\
&\left.+\frac{\cos u_2}{\sin u_2}\frac{\partial}{\partial u_2}\log(F_{11}F_{22})-2\right\}+4c_1=0.
\end{aligned}\tag{5.3*}
$$

From (5.4′) we obtain

$$
\frac{\partial^2\log F_{22}}{\partial u_1\partial u_2}-\frac{1}{2}\frac{\partial\log F_{11}}{\partial u_2}\frac{\partial\log F_{22}}{\partial u_1}=0.\tag{5.4*}
$$

From (5.7′) we obtain

$$
\begin{aligned}
&\frac{1}{F_{11}}\left\{\frac{\partial^2\log F_{22}}{\partial u_1\partial u_1}-\frac{1}{2}\frac{\partial\log F_{22}}{\partial u_1}\frac{\partial\log F_{11}}{\partial u_1}\right\}+\frac{1}{F_{22}}\left\{\frac{\partial^2}{\partial u_2\partial u_2}\log\left(\frac{F_{11}}{F_{22}}\right)\right.\\
&\left.+\frac{1}{2}\frac{\partial}{\partial u_2}\log\left(\frac{F_{11}}{F_{22}}\right)\frac{\partial\log F_{11}}{\partial u_2}+2-\frac{\cos u_2}{\sin u_2}\frac{\partial\log F_{22}}{\partial u_2}\right\}=0.
\end{aligned}\tag{5.7*}
$$

From (5.8′) we obtain

$$
\begin{aligned}
&\frac{1}{F_{22}}\left\{\frac{\partial^2\log F_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial\log F_{11}}{\partial u_2}\left(2\frac{\partial\log F_{22}}{\partial u_2}-\frac{\partial\log F_{11}}{\partial u_2}\right)\right.\\
&\left.-\frac{\cos u_2}{\sin u_2}\frac{\partial\log F_{11}}{\partial u_2}\right\}=0.
\end{aligned}\tag{5.8*}
$$

Since (5.4*) can be written as

$$\frac{\partial}{\partial u_2}\log\Big(\frac{\partial \log F_{22}}{\partial u_1}\Big) = \frac{1}{2}\frac{\partial \log F_{11}}{\partial u_2},$$

we see that

$$\Big(\frac{\partial \log F_{22}}{\partial u_1}\Big)^2 = F_{11}\sigma, \tag{5.10}$$

where $\sigma = \sigma(u_1)$ is an integral function depending only on u_1. (5.10) implies

$$\frac{\partial \log F_{11}}{\partial u_1} = 2\frac{\partial^2 \log F_{22}}{\partial u_1 \partial u_1} \Big/ \frac{\partial \log F_{22}}{\partial u_1} - \frac{\sigma'}{\sigma}. \tag{5.11}$$

Now, substituting (5.4*) and (5.11) into (5.3*) and multiplying $\dfrac{\partial \log F_{22}}{\partial u_1}$ we obtain

$$\begin{aligned}(\sigma + 4c_1)\frac{\partial \log F_{22}}{\partial u_1} + \frac{\sigma'}{2} + \frac{1}{F_{22}}\Big\{ & \frac{\partial}{\partial u_2}\Big(\frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log F_{22}}{\partial u_2}\Big) \\ & + \frac{\cos u_2}{\sin u_2}\Big(\frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log F_{22}}{\partial u_2} + 2\frac{\partial^2 \log F_{22}}{\partial u_1 \partial u_2}\Big) - 2\frac{\partial \log F_{22}}{\partial u_1}\Big\} = 0.\end{aligned} \tag{5.3**}$$

(5.3*)–(5.7*) and (5.10) give the equation:

$$\begin{aligned}\sigma + \frac{1}{F_{22}}\Big\{ & \frac{\partial^2}{\partial u_2 \partial u_2}\log\frac{(F_{22})^2}{F_{11}} + \frac{1}{2}\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial}{\partial u_2}\log\frac{(F_{22})^2}{F_{11}} \\ & + \frac{\cos u_2}{\sin u_2}\frac{\partial}{\partial u_2}\log(F_{11}(F_{22})^2) - 4\Big\} + 4c_1 = 0.\end{aligned} \tag{5.7**}$$

From (5.8*) we obtain

$$\frac{\partial}{\partial u_2}\log\frac{\partial \log F_{11}}{\partial u_2} + \frac{1}{2}\frac{\partial \log F_{11}}{\partial u_2} - \frac{\partial \log F_{22}}{\partial u_2} - \frac{\cos u_2}{\sin u_2} = 0,$$

and hence

$$\frac{\partial}{\partial u_2}\log\Big(\Big(\frac{\partial \log F_{11}}{\partial u_2}\Big)^2 F_{11}\frac{1}{(F_{22})^2}\frac{1}{\sin^2 u_2}\Big) = 0.$$

By integration we obtain

$$\Big(\frac{\partial \log F_{11}}{\partial u_2}\Big)^2 \frac{F_{11}}{(F_{22})^2 \sin^2 u_2} = \sigma_1,$$

where σ_1 is a function of u_1. Substituting (5.4*) and (5.10) into the above equation we obtain

$$4\Big(\frac{\partial^2 \log F_{22}}{\partial u_2 \partial u_1}\Big)^2 = \sigma\sigma_1(F_{22})^2 \sin^2 u_2.$$

Therefore we can replace (5.8*) by

$$\frac{\partial^2 \log F_{22}}{\partial u_2 \partial u_1} = \sigma_2 \sin u_2 F_{22}, \tag{5.8**}$$

where σ_2 is a function of u_1 only. This equation can be written as

$$\frac{\partial^2 F_{22}}{\partial u_2 \partial u_1} - \frac{1}{F_{22}} \frac{\partial F_{22}}{\partial u_2} \frac{\partial F_{22}}{\partial u_1} = \sigma_2 \sin u_2 (F_{22})^2,$$

from which we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial u_2 \partial u_1}\left(\frac{1}{F_{22}}\right) &= -\frac{1}{(F_{22})^2} \frac{\partial^2 F_{22}}{\partial u_2 \partial u_1} + \frac{2}{(F_{22})^3} \frac{\partial F_{22}}{\partial u_2} \frac{\partial F_{22}}{\partial u_1} \\
&= -\frac{1}{(F_{22})^2}\left(\frac{1}{F_{22}} \frac{\partial F_{22}}{\partial u_2} \frac{\partial F_{22}}{\partial u_1} + \sigma_2 \sin u_2 (F_{22})^2\right) + \frac{2}{(F_{22})^3} \frac{\partial F_{22}}{\partial u_2} \frac{\partial F_{22}}{\partial u_1} \\
&= \frac{1}{(F_{22})^3} \frac{\partial F_{22}}{\partial u_2} \frac{\partial F_{22}}{\partial u_1} - \sigma_2 \sin u_2.
\end{aligned}$$

Denoting $1/F_{22} = y$, the above equation becomes

$$\frac{\partial^2 y}{\partial u_2 \partial u_1} - \frac{1}{y} \frac{\partial y}{\partial u_2} \frac{\partial y}{\partial u_1} = -\sigma_2 \sin u_2. \tag{5.8''}$$

Now (5.3**) can be written as

$$\begin{aligned}
&-(\sigma + 4c_1) \frac{1}{y} \frac{\partial y}{\partial u_1} + \frac{\sigma'}{2} + y\left\{\frac{\partial}{\partial u_2}\left(\frac{1}{y^2} \frac{\partial y}{\partial u_1} \frac{\partial y}{\partial u_2}\right)\right. \\
&\qquad \left. + \frac{\cos u_2}{\sin u_2}\left(\frac{1}{y^2} \frac{\partial y}{\partial u_1} \frac{\partial y}{\partial u_2} + 2\sigma_2 \frac{\sin u_2}{y}\right) + \frac{2}{y} \frac{\partial y}{\partial u_1}\right\} = 0
\end{aligned}$$

by using (5.8**), from which we get

$$\begin{aligned}
&-(\sigma + 4c_1) \frac{\partial y}{\partial u_1} + \frac{\sigma'}{2} y + \frac{\partial^2 y}{\partial u_2 \partial u_1} \frac{\partial y}{\partial u_2} + \frac{\partial^2 y}{\partial u_2 \partial u_2} \frac{\partial y}{\partial u_1} \\
&- \frac{2}{y} \frac{\partial y}{\partial u_1} \frac{\partial y}{\partial u_2} \frac{\partial y}{\partial u_2} + \frac{\cos u_2}{\sin u_2} \frac{\partial y}{\partial u_1} \frac{\partial y}{\partial u_2} + 2\sigma_2 \cos u_2 y + 2y \frac{\partial y}{\partial u_1} = 0,
\end{aligned}$$

and using (5,8''), we obtain the equation

$$\begin{aligned}
&\frac{\partial y}{\partial u_1}\left\{-(\sigma + 4c_1) - \frac{1}{y} \frac{\partial y}{\partial u_2} \frac{\partial y}{\partial u_2} + \frac{\partial^2 y}{\partial u_2 \partial u_2} + \frac{\cos u_2}{\sin u_2} \frac{\partial y}{\partial u_2} + 2y\right\} \\
&- \sigma_2 \sin u_2 \frac{\partial y}{\partial u_2} + \left(\frac{\sigma'}{2} + 2\sigma_2 \cos u_2\right) y = 0.
\end{aligned} \tag{5.3''}$$

Next, (5.10) can be written as

$$F_{11} = \frac{1}{\sigma}\Big(\frac{\partial \log F_{22}}{\partial u_1}\Big)^2 = \frac{1}{\sigma y^2}\Big(\frac{\partial y}{\partial u_1}\Big)^2,$$

from which we obtain by (5.8″)

$$\begin{aligned}\frac{\partial \log F_{11}}{\partial u_2} &= -\frac{2}{y}\frac{\partial y}{\partial u_2} + \frac{2}{\partial y/\partial u_1}\frac{\partial^2 y}{\partial u_2 \partial u_1} \\ &= -\frac{2}{y}\frac{\partial y}{\partial u_2} + \frac{2}{\partial y/\partial u_1}\Big(\frac{1}{y}\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_1} - \sigma_2 \sin u_2\Big) \\ &= -2\sigma_2 \frac{\sin u_2}{\partial y/\partial u_1},\end{aligned}$$

and so

$$\frac{\partial}{\partial u_2}(2\log F_{22} \mp \log F_{11}) = -\frac{2}{y}\frac{\partial y}{\partial u_2} \pm 2\sigma_2 \sin u_2 \frac{1}{\partial y/\partial u_1}.$$

(5.7**) can be written as

$$\begin{aligned}&\sigma + \frac{1}{F_{22}}\Big\{\frac{\partial^2}{\partial u_2 \partial u_2}(2\log F_{22} - \log F_{11}) + \frac{1}{2}\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial}{\partial u_2}(2\log F_{22} - \log F_{11}) \\ &\quad + \frac{\cos u_2}{\sin u_2}\frac{\partial}{\partial u_2}(2\log F_{22} + \log F_{11}) - 4\Big\} + 4c_1 \\ &= \sigma + 4c_1 + y\Big\{\frac{\partial}{\partial u_2}\Big(-\frac{2}{y}\frac{\partial y}{\partial u_2} + 2\sigma_2 \frac{\sin u_2}{\partial y/\partial u_1}\Big) + \frac{1}{2}\Big(-2\sigma_2\frac{\sin u_2}{\partial y/\partial u_1}\Big) \\ &\quad \times \Big(-\frac{2}{y}\frac{\partial y}{\partial u_2} + 2\sigma_2 \frac{\sin u_2}{\partial y/\partial u_1}\Big) + \frac{\cos u_2}{\sin u_2}\Big(-\frac{2}{y}\frac{\partial y}{\partial u_2} - 2\sigma_2\frac{\sin u_2}{\partial y/\partial u_1}\Big) - 4\Big\} \\ &= \sigma + 4c_1 + y\Big\{-\frac{2}{y}\frac{\partial^2 y}{\partial u_2 \partial u_2} + \frac{2}{y^2}\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_2} + 2\sigma_2 \frac{\cos u_2}{\partial y/\partial u_1} \\ &\quad - 2\sigma_2 \sin u_2 \frac{1}{(\partial y/\partial u_1)^2}\frac{\partial^2 y}{\partial u_2 \partial u_1} - 2\sigma_2\frac{\sin u_2}{\partial y/\partial u_1}\Big(-\frac{1}{y}\frac{\partial y}{\partial u_2} + \sigma_2\frac{\sin u_2}{\partial y/\partial u_1}\Big) \\ &\quad - 2\frac{\cos u_2}{\sin u_2}\Big(\frac{1}{y}\frac{\partial y}{\partial u_2} + \sigma_2 \frac{\sin u_2}{\partial y/\partial u_1}\Big) - 4\Big\} \\ &= \sigma + 4c_1 - 2\frac{\partial^2 y}{\partial u_2 \partial u_2} + \frac{2}{y}\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_2} - 2\sigma_2 \sin u_2 \frac{y}{(\partial y/\partial u_1)^2}\frac{\partial^2 y}{\partial u_2 \partial u_1} \\ &\quad + 2\sigma_2 \frac{\sin u_2}{\partial y/\partial u_1}\frac{\partial y}{\partial u_2} - 2y\Big(\sigma_2 \frac{\sin u_2}{\partial y/\partial u_1}\Big)^2 - 2\frac{\cos u_2}{\sin u_2}\frac{\partial y}{\partial u_2} - 4y = 0,\end{aligned}$$

into which substituting (5.8″) we obtain the equation

$$\frac{\partial^2 y}{\partial u_2 \partial u_2} - \frac{1}{y}\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_2} + \frac{\cos u_2}{\sin u_2}\frac{\partial y}{\partial u_2} + 2y - \frac{\sigma}{2} - 2c_1 = 0. \tag{5.12}$$

Substituting (5.12) into (5.3″), we obtain the equation

$$(\sigma + 4c_1)\frac{\partial y}{\partial u_1} + 2\sigma_2 \sin u_2 \frac{\partial y}{\partial u_2} - (\sigma' + 4\sigma_2 \cos u_2)y = 0. \tag{5.13}$$

Thus, we obtain the following conclusion.

Proposition 1. *In order to solve the system of the equations* (5.3′), (5.4′), (5.7′) *and* (5.8′) *under the restriction* (5.9), *it is sufficient to solve the system of the equations* (5.8″), (5.12) *and* (5.13) *on* y *with* $y = 1/F_{22}$ *and* F_{11} *is given by* (5.10).

6. Treatment of the system of (5.8″), (5.12) and (5.13)

Noticing of (5.13) which is linear on y, $\partial y/\partial u_1$ and $\partial y/\partial u_2$, we put

$$y = \sum_{m=0}^{\infty} P_m(u_1) \sin^m u_2. \tag{6.1}$$

Then we have from (5.13)

$$\begin{aligned} &\sum_{m=0}^{\infty}((\sigma + 4c_1){P_m}' - \sigma' P_m) \sin^m u_2 \\ &+ 2\sigma_2 \cos u_2 \sum_{m=0}^{\infty}(m-2)P_m \sin^m u_2 = 0. \end{aligned} \tag{6.2}$$

Setting $u_2 = 0$, we obtain

$$(\sigma + 4c_1){P_0}' - \sigma' P_0 - 4\sigma_2 P_0 = 0.$$

Hence we have by integration

$$P_0 = (\sigma + 4c_1)\rho^2 b_0 \tag{6.3}$$

where $b_0 =$ constant and

$$\rho = \exp\left(\int \frac{2\sigma_2}{\sigma + 4c_1} du_1\right), \text{ with } \rho(0) = 1. \tag{6.4}$$

Using (6.3), we obtain

$$\begin{aligned} 4\sigma_2 P_0(1 - \cos u_2) &+ \sum_{m=1}^{\infty}((\sigma + 4c_1){P_m}' - \sigma' P_m) \sin^m u_2 \\ &+ \cos u_2 \sum_{m=1}^{\infty} 2(m-2)\sigma_2 P_m \sin^m u_2 = 0, \end{aligned}$$

i.e.

$$4\sigma_2 P_0 \frac{1-\cos u_2}{\sin u_2} + \sum_{m=1}^{\infty}((\sigma+4c_1){P_m}' - \sigma' P_m)\sin^{m-1} u_2$$
$$+ \cos u_2 \sum_{m=1}^{\infty} 2(m-2)\sigma_2 P_m \sin^{m-1} u_2 = 0,$$

from which by setting $u_2 = 0$, we obtain the relation

$$(\sigma+4c_1){P_1}' - \sigma' P_1 - 2\sigma_2 P_1 = 0.$$

Hence we obtain

$$P_1 = (\sigma+4c_1)\rho b_1, \quad b_1 = \text{constant}. \tag{6.5}$$

Noticing these facts, we set

$$P_m = (\sigma+4c_1)Q_m, \quad m = 0,1,2,\ldots. \tag{6.6}$$

Then, we have

$$(\sigma+4c_1){P_m}' - \sigma' P_m = (\sigma+4c_1)\{(\sigma+4c_1){Q_m}' + \sigma' Q_m\} - \sigma'(\sigma+4c_1)Q_m$$
$$= (\sigma+4c_1)^2 {Q_m}'.$$

Hence, from (6.2) we have

$$(\sigma+4c_1)\sum_{m=0}^{\infty} {Q_m}' \sin^m u_2 + 2\sigma_2 \cos u_2 \sum_{m=0}^{\infty}(m-2)Q_m \sin^m u_2 = 0. \tag{6.2$'$}$$

Since we have

$$(1-X)^{\frac{1}{2}} = 1 - \sum_{m=1}^{\infty} k_m X^m,$$

where $k_1 = \frac{1}{2}$ and

$$k_m = \frac{(2m-3)!!}{2^m m!}, \quad m = 2,3,\ldots, \tag{6.7}$$

we have from (6.2$'$)

$$\begin{aligned}(\sigma+4c_1)&\sum_{m=0}^{\infty} {Q_m}' \sin^m u_2 \\ &= -2\sigma_2 \sum_{m=0}^{\infty}(m-2)Q_m \sin^m u_2 \left(1 - \sum_{m=1}^{\infty} k_m \sin^{2m} u_2\right).\end{aligned} \tag{6.2$''$}$$

From the coefficients of $\sin^m u_2$, $m = 0,1$, we obtain (6.3) and (6.5), and hence $Q_0 = \rho^2 b_0$ and $Q_1 = \rho b_1$. From the coefficients of $\sin^2 u_2$, we have the relation

$$(\sigma+4c_1){Q_2}' = -4\sigma_2 Q_0 k_1 = -2\sigma_2 Q_0 = -2\sigma_2 \rho^2 b_0,$$

from which we have

$$Q_2{}' = -\frac{2\sigma_2}{\sigma+4c_1}\rho^2 b_0 = -\frac{\rho'}{\rho}\rho^2 b_0 = -\rho\rho' b_0$$

and by integration we obtain

$$Q_2 = -\frac{\rho^2}{2}b_0 + b_2, \quad b_2 = \text{constant}. \tag{6.8}$$

Next, regarding $\sin^3 u_2$ we have the relation

$$(\sigma+4c_1)Q_3{}' = -2\sigma_2(Q_3 + Q_1 k_1) = -2\sigma_2\Big(Q_3 + \frac{1}{2}Q_1\Big),$$

from which we have

$$Q_3{}' + \frac{2\sigma_2}{\sigma+4c_1}Q_3 = -\frac{2\sigma_2}{\sigma+4c_1}\frac{Q_1}{2} = -\frac{\rho'}{\rho}\frac{1}{2}\rho b_1 = -\frac{\rho'}{2}b_1$$

and hence

$$(\rho Q_3)' = -\frac{1}{2}\rho\rho' b_1, \quad \text{i.e. } \rho Q_3 = -\frac{\rho^2}{4}b_1 + b_3,$$

that is

$$Q_3 = -\frac{1}{4}\rho b_1 + \frac{1}{\rho}b_3, \quad b_3 = \text{constant}. \tag{6.9}$$

Next, regarding $\sin^4 u_2$ we have the relation

$$(\sigma+4c_1)Q_4{}' = -2\sigma_2(2Q_4 + 2Q_0 k_2) = -2\sigma_2\left(2Q_4 + \frac{1}{4}Q_0\right),$$

that is

$$Q_4{}' + \frac{4\sigma_2}{\sigma+4c_1}Q_4 = -\frac{2\sigma_2}{\sigma+4c_1}\frac{Q_0}{4} = -\frac{\rho'}{\rho}\frac{\rho^2}{4}b_0,$$

hence we have

$$Q_4{}' + \frac{2\rho'}{\rho}Q_4 = -\frac{\rho\rho'}{4}b_0,$$

from which by integration we obtain

$$Q_4 = -\frac{\rho^2}{16}b_0 + \frac{1}{\rho^2}b_4, \quad b_4 = \text{constant}. \tag{6.10}$$

In general we obtain from (6.2″) the relations as follows. For $\sin^{2m} u_2, m > 2$, we have

$$(\sigma+4c_1)Q_{2m}{}' = -2\sigma_2\left\{2(m-1)Q_{2m} + 2Q_0 k_m - 2\sum_{h=2}^{m-1}(h-1)Q_{2h}k_{m-h}\right\},$$

that is

$$
\begin{aligned}
&Q_{2m}{}' + \frac{4(m-1)\sigma_2}{\sigma+4c_1}Q_{2m} \\
&= -\frac{4\sigma_2}{\sigma+4c_1}Q_0k_m + \frac{4\sigma_2}{\sigma+4c_1}\sum_{h=2}^{m-1}(h-1)Q_{2h}k_{m-h} \\
&= \frac{4\sigma_2}{\sigma+4c_1}\sum_{h=0}^{m-1}(h-1)Q_{2h}k_{m-h},
\end{aligned} \tag{6.11}
$$

and for $\sin^{2m+1}u_2$, $m \geq 2$, we have

$$
\begin{aligned}
&(\sigma+4c_1)Q_{2m+1}{}' \\
&= -2\sigma_2\{Q_1k_m - Q_3k_{m-1} - 3Q_5k_{m-2} - \cdots + (2m-1)Q_{2m+1}\} \\
&= -2\sigma_2\left\{(2m-1)Q_{2m+1} - \sum_{h=0}^{m-1}(2h-1)Q_{2h+1}k_{m-h}\right\},
\end{aligned}
$$

that is

$$
Q_{2m+1}{}' + \frac{2\sigma_2(2m-1)}{\sigma+4c_1}Q_{2m+1} = \frac{2\sigma_2}{\sigma+4c_1}\sum_{h=0}^{m-1}(2h-1)Q_{2h+1}k_{m-h}. \tag{6.12}
$$

By means of (6.4), (6.11) and (6.12) can be written respectively as

$$
(\rho^{2m-2}Q_{2m})' = 2\rho^{2m-3}\rho'\sum_{h=0}^{m-1}(h-1)Q_{2h}k_{m-h}, \quad m > 2 \tag{6.11$'$}
$$

and

$$
(\rho^{2m-1}Q_{2m+1})' = \rho^{2m-2}\rho'\sum_{h=0}^{m-1}(2h-1)Q_{2h+1}k_{m-h}, \quad m \geq 2. \tag{6.12$'$}
$$

From these considerations, we may put

$$
Q_m = R_m(\rho) + \frac{b_m}{\rho^{m-2}}, \quad b_m = \text{constant} \tag{6.13}
$$

where R_m is a polynomial of ρ with negative powers. By means of (6.3), (6.5), (6.6), (6.8), (6.9), (6.10), we have

$$
R_0 = R_1 = 0,\ R_2 = -\frac{b_0}{2}\rho^2,\ R_3 = -\frac{b_1}{4}\rho,\ R_4 = -\frac{b_0}{16}\rho^2. \tag{6.14}
$$

From (6.11$'$) and (6.12$'$) we obtain the relations as follows.

$$
(\rho^{2m-2}R_{2m})^* = 2\rho^{2m-3}\sum_{h=0}^{m-1}(h-1)\left(R_{2h} + \frac{b_{2h}}{\rho^{2h-2}}\right)k_{m-h}, \tag{6.15}
$$

and

$$(\rho^{2m-1}R_{2m+1})^* = \rho^{2m-2}\sum_{h=0}^{m-1}(2h-1)\Big(R_{2h+1}+\frac{b_{2h+1}}{\rho^{2h-1}}\Big)k_{m-h}, \tag{6.16}$$

where "*" denotes the derivative with respect to ρ. We compute R_5, R_6, R_7 by the above formula. First, from (6.16) with $m=2$, we have

$$\begin{aligned}(\rho^3R_5)^* &= \sum_{h=0}^{1}(2h-1)(\rho^2R_{2h+1}+\rho^{3-2h}b_{2h+1})k_{2-h}\\ &= -(\rho^2R_1+\rho^3b_1)k_2+(\rho^2R_3+\rho b_3)k_1\\ &= -\frac{b_1}{8}\rho^3+\frac{1}{2}(-\frac{b_1}{4}\rho^3+b_3\rho)\\ &= -\frac{b_1}{4}\rho^3+\frac{b_3}{2}\rho,\end{aligned}$$

from which we obtain

$$R_5 = -\frac{b_1}{16}\rho+\frac{b_3}{4\rho}. \tag{6.17}$$

Next, from (6.15) with $m=3$, we have

$$\begin{aligned}(\rho^4R_6)^* &= 2\sum_{h=0}^{2}(h-1)(\rho^3R_{2h}+\rho^{5-2h}b_{2h})k_{3-h}\\ &= -2(\rho^3R_0+\rho^5b_0)k_3+2(\rho^3R_4+\rho b_4)k_1\\ &= -\frac{b_0}{8}\rho^5+\Big(\rho^3\Big(-\frac{b_0}{16}\rho^2\Big)+b_4\rho\Big)\\ &= -\frac{3b_0}{16}\rho^5+b_4\rho.\end{aligned}$$

from which we obtain

$$R_6 = -\frac{b_0}{32}\rho^2+\frac{b_4}{2\rho^2}. \tag{6.17$_2$}$$

Then, from (6.16) with $m=3$, we have

$$\begin{aligned}(\rho^5R_7)^* &= \sum_{h=0}^{2}(2h-1)(\rho^4R_{2h+1}+\rho^{5-2h}b_{2h+1})k_{3-h}\\ &= -(\rho^4R_1+\rho^5b_1)k_3+(\rho^4R_3+\rho^3b_3)k_2+3(\rho^4R_5+\rho b_5)k_1\\ &= -\frac{b_1}{16}\rho^5+\frac{1}{8}\Big(\rho^4\Big(-\frac{b_1}{4}\rho\Big)+b_3\rho^3\Big)+\frac{3}{2}\Big(\rho^4\Big(-\frac{b_1}{16}\rho+\frac{b_3}{4\rho}\Big)+b_5\rho\Big)\\ &= -\frac{3b_1}{16}\rho^5+\frac{b_3}{2}\rho^3+\frac{3b_5}{2}\rho,\end{aligned}$$

from which we obtain

$$R_7 = -\frac{b_1}{32}\rho + \frac{b_3}{8\rho} + \frac{3b_5}{4\rho^3}. \tag{6.17$_3$}$$

Now, we consider the condition (5.12), or

$$y\Big(\frac{\partial^2 y}{\partial u_2 \partial u_2} + \frac{\cos u_2}{\sin u_2}\frac{\partial y}{\partial u_2} + 2y - \frac{1}{2}(\sigma + 4c_1)\Big) = \Big(\frac{\partial y}{\partial u_2}\Big)^2. \tag{6.18}$$

Since we have

$$\begin{aligned}
\frac{\partial y}{\partial u_2} &= \Big(\sum_{m=1}^{\infty} mP_m \sin^{m-1} u_2\Big)\cos u_2 \\
&= \Big(\sum_{m=0}^{\infty} (m+1)P_{m+1} \sin^{m} u_2\Big)\cos u_2, \\
\frac{\partial^2 y}{\partial u_2 \partial u_2} &= -\sin u_2 \Big(\sum_{m=1}^{\infty} mP_m \sin^{m-1} u_2\Big) \\
&\quad + \Big(\sum_{m=2}^{\infty} m(m-1)P_m \sin^{m-2} u_2\Big)\cos^2 u_2 \\
&= -\sum_{m=1}^{\infty} mP_m \sin^m u_2 + \sum_{m=0}^{\infty}(m+2)(m+1)P_{m+2}\sin^m u_2 \\
&\quad - \sum_{m=2}^{\infty} m(m-1)P_m \sin^m u_2 \\
&= 2P_2 + \sum_{m=1}^{\infty}((m+2)(m+1)P_{m+2} - m^2 P_m)\sin^m u_2,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{\partial^2 y}{\partial u_2 \partial u_2} + \frac{\cos u_2}{\sin u_2}\frac{\partial y}{\partial u_2} + 2y - \frac{1}{2}(\sigma + 4c_1) \\
&= 2P_2 + \sum_{m=1}^{\infty}((m+2)(m+1)P_{m+2} - m^2 P_m)\sin^m u_2 \\
&\quad + \Big(\sum_{m=1}^{\infty} mP_m \sin^{m-1} u_2\Big)\frac{1}{\sin u_2} - \sum_{m=1}^{\infty} mP_m \sin^m u_2 + 2\sum_{m=0}^{\infty} P_m \sin^m u_2 \\
&\quad - \frac{1}{2}(\sigma + 4c_1)
\end{aligned}$$

$$= 2P_0 + \frac{1}{\sin u_2}P_1 + 4P_2 + \sum_{m=1}^{\infty}\{(m+2)^2P_{m+2} - (m^2+m-2)P_m\}\sin^m u_2$$
$$- \frac{1}{2}(\sigma + 4c_1)$$
$$= (\sigma + 4c_1)\Bigg\{2Q_0 + \frac{1}{\sin u_2}Q_1 + 4Q_2 - \frac{1}{2}$$
$$+ \sum_{m=1}^{\infty}((m+2)^2Q_{m+2} - (m+2)(m-1)Q_m)\sin^m u_2\Bigg\}$$

by (6.6), and

$$\begin{aligned}\left(\frac{\partial y}{\partial u_2}\right)^2 &= \left(\sum_{m=1}^{\infty} mP_m \sin^{m-1} u_2\right)^2 (1-\sin^2 u_2)\\ &= \left(\sum_{m=0}^{\infty}(m+1)P_{m+1}\sin^m u_2\right)^2 - \left(\sum_{m=1}^{\infty} mP_m \sin^m u_2\right)^2\\ &= (\sigma+4c_1)^2\Bigg\{\left(\sum_{m=0}^{\infty}(m+1)Q_{m+1}\sin^m u_2\right)^2\\ &\quad - \left(\sum_{m=1}^{\infty} mQ_m \sin^m u_2\right)^2\Bigg\}.\end{aligned}$$

Therefore (6.18) is equivalent to

$$\begin{aligned}&\left(\sum_{m=0}^{\infty} Q_m \sin^m u_2\right)\Bigg\{2Q_0 + \frac{1}{\sin u_2}Q_1 + 4Q_2 - \frac{1}{2}\\ &\quad + \sum_{m=1}^{\infty}\Big((m+2)^2Q_{m+2} - (m+2)(m-1)Q_m\Big)\sin^m u_2\Bigg\}\\ &= \left(\sum_{m=0}^{\infty}(m+1)Q_{m+1}\sin^m u_2\right)^2 - \left(\sum_{m=1}^{\infty} mQ_m \sin^m u_2\right)^2,\end{aligned} \tag{6.19}$$

from which we see that it must be

$$Q_0Q_1 = \rho^3 b_0 b_1 = 0$$

looking at the coefficient of $1/\sin u_2$.

In the following, first we shall discuss the case $b_0 \neq 0$, $b_1 = 0$. Then we have

$$Q_0 = b_0\rho^2,\ Q_1 = b_1\rho = 0,\ Q_2 = -\frac{b_0}{2}\rho^2 + b_2,$$
$$Q_3 = -\frac{b_1}{4}\rho + \frac{b_3}{\rho} = \frac{b_3}{\rho},\ Q_4 = -\frac{b_0}{16}\rho^2 + \frac{b_4}{\rho^2}$$

by (6.8), (6.10). Considering the constant term on $\sin u_2$, we have $Q_0(2Q_0 + 4Q_2 - 1/2) = 0$ and $2Q_0 + 4Q_2 - 1/2 = 4b_2 - 1/2 = 0$, that is $b_2 = 1/8$. Then, (6.19) can be written as

$$\begin{aligned} &\left(Q_0 + \sum_{m=2}^{\infty} Q_m \sin^m u_2\right)\left(\sum_{m=1}^{\infty}((m+2)^2 Q_{m+2}\right. \\ &\qquad \left. - (m+2)(m-1)Q_m)\sin^m u_2\right) \\ &= \left(\sum_{m=1}^{\infty}(m+1)Q_{m+1}\sin^m u_2\right)^2 - \left(\sum_{m=2}^{\infty} mQ_m \sin^m u_2\right)^2. \end{aligned} \tag{6.19'}$$

From the coefficient of $\sin u_2$, we obtain $Q_0 \times 9Q_3 = 0$, which implies $Q_3 = 0$ and so $b_3 = 0$.

Lemma 1. *We have $Q_{2m+1} = 0$ for $m = 1, 2, 3, \ldots$.*

Proof. We suppose that

$$Q_{2h+1} = 0, \quad h = 0, 1, \ldots, m\ (m \geq 1).$$

From the coefficients of $\sin^{2m+1} u_2$ of (6.19′), we obtain the relation

$$\begin{aligned} &Q_0((2m+3)^2 Q_{2m+3} - (2m+3)2mQ_{2m+1}) \\ &\quad + \sum_{p=2}^{2m} Q_p((2m+3-p)^2 Q_{2m+3-p} - (2m+3-p)(2m-p)Q_{2m+1-p}) \\ &= \sum_{p=1}^{2m}(p+1)(2m+2-p)Q_{p+1}Q_{2m+2-p} - \sum_{p=2}^{2m-1} p(2m+1-p)Q_pQ_{2m+1-p}, \end{aligned}$$

which is reduced to

$$(2m+3)^2 Q_0 Q_{2m+3} = 0$$

by the above supposition, hence we have $Q_{2m+3} = 0$. Thus we obtain the relation

$$Q_{2m+1} = 0, \quad m = 0, 1, 2, 3, \ldots. \tag{6.20}$$

□

Then, from the coefficients of $\sin^{2m} u_2$, we obtain from (6.19′)

$$\begin{aligned} &Q_0((2m+2)^2 Q_{2m+2} - (2m+2)(2m-1)Q_{2m}) \\ &\quad + \sum_{h=1}^{m-1} Q_{2h}((2m+2-2h)^2 Q_{2m-2h+2} \\ &\quad - (2m+2-2h)(2m-1-2h)Q_{2m-2h}) \\ &= \sum_{h=0}^{m-1} 4(h+1)(m-h)Q_{2h+2}Q_{2m-2h} - \sum_{h=1}^{m-1} 4h(m-h)Q_{2h}Q_{2m-2h}. \end{aligned} \tag{6.21}$$

Now, regarding (6.20), we have the following.

Lemma 2. *We have*

$$b_{2m+1} = 0, \quad m = 0, 1, 2, \ldots. \tag{6.20′}$$

Proof. We knew that $b_1 = b_3 = 0$ and from (6.17) $R_5 = 0$ and so $b_5 = 0$ by $Q_5 = 0$. From (6.17$_3$) we have $R_7 = 0$ and so $b_7 = 0$ by $Q_7 = 0$. Now, we suppose that $b_1 = b_3 = \cdots = b_{2m-1} = 0$ ($m \geq 4$), then we have

$$R_1 = R_3 = \cdots = R_{2m-1} = 0$$

by (6.20). From (6.12′) we have

$$(\rho^{2m-1} R_{2m+1})^* = \rho^{2m-2} \sum_{h=0}^{m-1} (2h-1)Q_{2h+1}k_{m-h} = 0,$$

and hence $R_{2m+1} = -\frac{b_{2m+1}}{\rho^{2m-1}}$ by $Q_{2m+1} = 0$. Since R_{2m+1} is a polynomial of ρ, for which b_{2m+1} does not relate. Hence we obtain $b_{2m+1} = 0$. □

In the following, we investigate the relation of $b_0, b_2, b_4, \ldots$. First, from the coefficients of $\sin^2 u_2$ of (6.19′) and (6.21) we have the relation

$$Q_0(16Q_4 - 4Q_2) = (2Q_2)^2,$$

which is written by (6.8), (6.10) as

$$b_0\rho^2\left(-b_0\rho^2 + \frac{16b_4}{\rho^2} + 2b_0\rho^2 - 4b_2\right) = (-b_0\rho^2 + 2b_2)^2,$$

and which is equivalent to

$$4b_0b_4 = b_2b_2. \tag{6.22}$$

Next, from the coefficients of $\sin^4 u_2$ of (6.19′) and (6.21) we have the relation

$$Q_0(36Q_6 - 18Q_4) + Q_2(16Q_4 - 4Q_2) = 16Q_2Q_4 - 4Q_2Q_2,$$

i.e.

$$18Q_0(2Q_6 - Q_4) = 0.$$

Since this is equivalent to

$$2Q_6 - Q_4 = \left(-\frac{b_0}{16}\rho^2 + \frac{b_4}{\rho^2} + \frac{2b_6}{\rho^4}\right) + \left(\frac{b_0}{16}\rho^2 - \frac{b_4}{\rho^2}\right) = \frac{2b_6}{\rho^4} = 0,$$

we have

$$b_6 = 0. \tag{6.23}$$

Next, from the coefficients of $\sin^6 u_2$ of (6.19′) and (6.21) we have the relation

$$\begin{aligned} &Q_0(64Q_8 - 40Q_6) + Q_2(36Q_6 - 18Q_4) + Q_4(16Q_4 - 4Q_2) \\ &= 24Q_2Q_6 + 16Q_4Q_4 - 16Q_2Q_4, \end{aligned}$$

i.e.

$$64Q_0Q_8 - 40Q_0Q_6 + 12Q_2Q_6 - 6Q_2Q_4 = 0. \tag{6.24}$$

Here, we compute $Q_8 = R_8 + \frac{b_8}{\rho^6}$. By means of (6.15), (6.8), (6.10) we have

$$\begin{aligned} (\rho^6 R_8)^* &= 2\rho^5 \sum_{h=0}^{3} (h-1)\left(R_{2h} + \frac{b_{2h}}{\rho^{2h-2}}\right)k_{4-h} \\ &= 2\rho^5 \left\{-(R_0 + b_0\rho^2)\frac{5}{128} + \left(R_4 + \frac{b_4}{\rho^2}\right)\frac{1}{8} + 2\left(R_6 + \frac{b_6}{\rho^4}\right)\frac{1}{2}\right\} \\ &= 2\rho^5 \left\{-\frac{5}{128}b_0\rho^2 + \frac{1}{8}\left(-\frac{b_0}{16}\rho^2 + \frac{b_4}{\rho^2}\right) + \left(-\frac{b_0}{32}\rho^2 + \frac{b_4}{2\rho^2} + \frac{b_6}{\rho^4}\right)\right\} \\ &= -\frac{5}{32}b_0\rho^7 + \frac{5}{4}b_4\rho^3 + 2b_6\rho \end{aligned}$$

and by integration we obtain

$$R_8 = -\frac{5}{256}b_0\rho^2 + \frac{5b_4}{16\rho^2} + \frac{b_6}{\rho^4}. \tag{6.25}$$

Then, we obtain from (5.24)

$$\begin{aligned} &64Q_0Q_8 - 40Q_0Q_6 + 12Q_2Q_6 - 6Q_2Q_4 \\ &= 8Q_0(8Q_8 - 5Q_6) + 6Q_2(2Q_6 - Q_4) \\ &= 8b_0\rho^2\left\{8\left(-\frac{5}{256}b_0\rho^2 + \frac{5b_4}{16\rho^2} + \frac{b_6}{\rho^4} + \frac{b_8}{\rho^6}\right) - 5\left(-\frac{1}{32}b_0\rho^2 + \frac{b_4}{2\rho^2} + \frac{b_6}{\rho^4}\right)\right\} \\ &\quad + 6\left(-\frac{1}{2}b_0\rho^2 + b_2\right)\left\{2\left(-\frac{1}{32}b_0\rho^2 + \frac{b_4}{2\rho^2} + \frac{b_6}{\rho^4}\right) - \left(-\frac{1}{16}b_0\rho^2 + \frac{b_4}{\rho^2}\right)\right\} \end{aligned}$$

$$= 8b_0\rho^2\left(\frac{3b_6}{\rho^4} + \frac{8b_8}{\rho^6}\right) + 6\left(-\frac{1}{2}b_0\rho^2 + b_2\right)\frac{2b_6}{\rho^4}$$
$$= \frac{18}{\rho^2}b_0b_6 + \frac{4}{\rho^4}(16b_0b_8 + 3b_2b_6) = 0,$$

and hence it must hold $b_0b_6 = 0$ and $16b_0b_8 + 3b_2b_6 = 0$. By means of (6.23) and $b_0 \neq 0$, we obtain

$$b_8 = 0. \tag{6.26}$$

Next, we compute $Q_{10} = R_{10} + \frac{b_{10}}{\rho^8}$. By means of (6.15) and using (6.23) and (6.26), we have

$$(\rho^8 R_{10})^* = 2\rho^7 \sum_{h=0}^{4}(h-1)\left(R_{2h} + \frac{b_{2h}}{\rho^{2h-2}}\right)k_{5-h}$$
$$= 2\rho^7\left\{-(R_0 + b_0\rho^2)k_5 + \left(R_4 + \frac{b_4}{\rho^2}\right)k_3 + 2\left(R_6 + \frac{b_6}{\rho^4}\right)k_2 + 3\left(R_8 + \frac{b_8}{\rho^6}\right)k_1\right\}$$
$$= 2\rho^7\left\{-\frac{7}{256}b_0\rho^2 + \frac{1}{16}\left(-\frac{1}{16}b_0\rho^2 + \frac{b_4}{\rho^2}\right) + \frac{1}{4}\left(-\frac{1}{32}b_0\rho^2 + \frac{b_4}{2\rho^2}\right) + \frac{3}{2}\left(-\frac{5}{256}b_0\rho^2 + \frac{5b_4}{16\rho^2}\right)\right\}$$
$$= -\frac{35}{256}b_0\rho^9 + \frac{21}{16}b_4\rho^5,$$

and by integration we obtain

$$R_{10} = -\frac{7}{512}b_0\rho^2 + \frac{7b_4}{32\rho^2}. \tag{6.27}$$

Then, from the coefficients of $\sin^8 u_2$ of (6.19′) and (6.21), we obtain

$$Q_0(100Q_{10} - 70Q_8) + \sum_{h=1}^{3} Q_{2h}((10-2h)^2 Q_{10-2h} - (10-2h)(7-2h)Q_{8-2h})$$
$$= \sum_{h=0}^{3} 4(h+1)(4-h)Q_{2h+2}Q_{8-2h} - \sum_{h=1}^{3} 4h(4-h)Q_{2h}Q_{8-2h},$$

that is

$$Q_0(100Q_{10} - 70Q_8) + Q_2(64Q_8 - 40Q_6) + Q_4(36Q_6 - 18Q_4) + Q_6(16Q_4 - 4Q_2)$$

$$= 16Q_2Q_8 + 24Q_4Q_6 + 24Q_6Q_4 + 16Q_8Q_2 \\ - 12Q_2Q_6 - 16Q_4Q_4 - 12Q_6Q_2,$$

which is arranged as

$$10Q_0(10Q_{10} - 7Q_8) + 4Q_2(8Q_8 - 5Q_6) + 4Q_4(Q_6 - \frac{1}{2}Q_4) = 0.$$

Since we have

$$\begin{aligned} 8Q_8 - 5Q_6 &= 8R_8 - 5R_6 \\ &= -\frac{40}{256}b_0\rho^2 + \frac{5b_4}{2\rho^2} - 5\left(-\frac{b_0}{32}\rho^2 + \frac{b_4}{2\rho^2}\right) = 0, \\ Q_6 - \frac{1}{2}Q_4 &= R_6 - \frac{1}{2}R_4 - \frac{b_4}{2\rho^2} \\ &= -\frac{b_0}{32}\rho^2 + \frac{b_4}{2\rho^2} - \frac{1}{2}\left(-\frac{b_0}{16}\rho^2\right) - \frac{b_4}{2\rho^2} = 0, \\ 10Q_{10} - 7Q_8 &= 10R_{10} + \frac{10b_{10}}{\rho^8} - 7R_8 \\ &= -\frac{35}{256}b_0\rho^2 + \frac{35b_4}{16\rho^2} + \frac{10b_{10}}{\rho^8} - 7\left(-\frac{5}{256}b_0\rho^2 + \frac{5b_4}{16\rho^2}\right) \\ &= \frac{10b_{10}}{\rho^8}, \end{aligned}$$

the above expression becomes

$$10b_0\rho^2 \times \frac{10b_{10}}{\rho^8} = \frac{100}{\rho^6}b_0b_{10} = 0,$$

which implies

$$b_{10} = 0. \tag{6.28}$$

Now, we suppose for $m > 5$ that

$$R_{2h} = -\lambda_h b_0\rho^2 + \mu_h\frac{b_4}{\rho^2}, \quad h = 6, 7, \dots, m, \tag{6.29}$$

and $b_{12} = b_{14} = \cdots = b_{2m} = 0$. We knew already that

$$R_0 = 0,\ R_2 = -\frac{1}{2}b_0\rho^2,\ R_4 = -\frac{1}{16}b_0\rho^2,\ R_6 = -\frac{1}{32}b_0\rho^2 + \frac{b_4}{2\rho^2},$$

$$R_8 = -\frac{5}{256}b_0\rho^2 + \frac{5b_4}{16\rho^2} = \frac{5}{8}R_6,\ R_{10} = -\frac{7}{512}b_0\rho^2 + \frac{7b_4}{32\rho^2} = \frac{7}{16}R_6.$$

and $b_6 = b_8 = b_{10} = 0$. From (6.15) we obtain

$$(\rho^{2m}R_{2m+2})^* = 2\rho^{2m-1}\sum_{h=0}^{m}(h-1)\left(R_{2h} + \frac{b_{2h}}{\rho^{2h-2}}\right)k_{m+1-h}$$

$$= 2\rho^{2m-1}\Bigg\{-b_0\rho^2 k_{m+1} + \left(R_4 + \frac{b_4}{\rho^2}\right)k_{m-1} + \sum_{h=3}^{m}(h-1)k_{m+1-h}\left(-\lambda_h b_0\rho^2 + \mu_h\frac{b_4}{\rho^2}\right)\Bigg\}$$

($\lambda_3, \ldots, \lambda_5$; $\mu_3, \ldots, \mu_5$ are suitably determined from the above expressions)

$$= 2\rho^{2m-1}\Bigg\{-b_0\rho^2\left(k_{m+1} + \frac{1}{16}k_{m-1} + \sum_{h=3}^{m}(h-1)k_{m+1-h}\lambda_h\right) + \frac{b_4}{\rho^2}\left(k_{m-1} + \sum_{h=3}^{m}(h-1)k_{m+1-h}\mu_h\right)\Bigg\},$$

from which by integration we obtain the equation

$$R_{2m+2} = -\frac{1}{m+1}b_0\rho^2\left(k_{m+1} + \frac{1}{16}k_{m-1} + \sum_{h=3}^{m}(h-1)k_{m+1-h}\lambda_h\right) + \frac{1}{m-1}\frac{b_4}{\rho^2}\left(k_{m-1} + \sum_{h=3}^{m}(h-1)k_{m+1-h}\mu_h\right).$$

It follows that

$$\lambda_{m+1} = \frac{1}{m+1}\left(k_{m+1} + \frac{1}{16}k_{m-1} + \sum_{h=3}^{m}(h-1)k_{m+1-h}\lambda_h\right), \tag{6.30}$$

$$\mu_{m+1} = \frac{1}{m-1}\left(k_{m-1} + \sum_{h=3}^{m}(h-1)k_{m+1-h}\mu_h\right). \tag{6.30$_2$}$$

Then, from (6.21) we have

$$b_0\rho^2\left(4(m+1)^2\left(R_{2m+2} + \frac{b_{2m+2}}{\rho^{2m}}\right) - 2(m+1)(2m-1)R_{2m}\right) + \sum_{h=1}^{m-1}\left(R_{2h} + \frac{b_{2h}}{\rho^{2h-2}}\right)\Bigg\{4(m+1-h)^2\left(R_{2m+2-2h} + \frac{b_{2m+2-2h}}{\rho^{2m-2h}}\right) - 2(m+1-h)(2m-1-2h)\left(R_{2m-2h} + \frac{b_{2m-2h}}{\rho^{2m-2h-2}}\right)\Bigg\}$$

$$= \sum_{h=0}^{m-1} 4(h+1)(m-h)\left(R_{2h+2} + \frac{b_{2h+2}}{\rho^{2h}}\right)\left(R_{2m-2h} + \frac{b_{2m-2h}}{\rho^{2m-2h-2}}\right)$$
$$- \sum_{h=1}^{m-1} 4h(m-h)\left(R_{2h} + \frac{b_{2h}}{\rho^{2h-2}}\right)\left(R_{2m-2h} + \frac{b_{2m-2h}}{\rho^{2m-2h-2}}\right)$$

and we see that from the coefficients of $1/\rho^{2m-2}$ it must be $b_0 b_{2m+2} = 0$, hence we have

$$b_{2m+2} = 0. \tag{6.31}$$

We see that

$$R_{2m} = -\lambda_m b_0 \rho^2 + \mu_m \frac{b_4}{\rho^2} = Q_{2m}, \quad m \geq 6$$

and λ_m, μ_m are determined inductively by (6.30), (6.30_2) respectively.

Since, from (6.30) and (6.30_2) we have

$$\lambda_6 = \frac{1}{6}\left(k_6 + \frac{1}{16}k_4 + 2k_3\lambda_3 + 3k_2\lambda_4 + 4k_1\lambda_5\right)$$
$$= \frac{1}{6}\left(\frac{21}{2^{10}} + \frac{1}{2^4} \times \frac{5}{2^7} + \frac{1}{2^3} \times \frac{1}{2^5} + \frac{3}{2^3} \times \frac{5}{2^8} + 2 \times \frac{7}{2^9}\right) = \frac{21}{2^{11}}$$

and

$$\mu_6 = \frac{1}{4}(k_4 + 2k_3\mu_3 + 3k_2\mu_4 + 4k_1\mu_5)$$
$$= \frac{1}{4}\left(\frac{5}{2^7} + \frac{1}{2^3} \times \frac{1}{2} + \frac{3}{2^3} \times \frac{5}{2^4} + 2 \times \frac{7}{2^5}\right) = \frac{21}{2^7},$$

we obtain

$$R_{12} = Q_{12} = -\frac{21}{2^{11}} b_0 \rho^2 + \frac{21}{2^7} \frac{b_4}{\rho^2} = \frac{21}{64} R_6. \tag{6.32}$$

Then, from (6.21) with $m = 5$ we have

$$Q_0(144Q_{12} - 108Q_{10})$$
$$+ \sum_{h=1}^{4} Q_{2h}(4(6-h)^2 Q_{12-2h} - 2(6-h)(9-2h)Q_{10-2h})$$
$$= \sum_{h=0}^{4} 4(h+1)(5-h)Q_{2h+2}Q_{10-2h} - \sum_{h=1}^{4} 4h(5-h)Q_{2h}Q_{10-2h},$$

i.e.

$$Q_0(144Q_{12} - 108Q_{10}) + Q_2(100Q_{10} - 70Q_8) + Q_4(64Q_8 - 40Q_6)$$
$$+ Q_6(36Q_6 - 18Q_4) + Q_8(16Q_4 - 4Q_2)$$

$$= 40Q_2Q_{10} + 64Q_4Q_8 + 36Q_6Q_6 - 32Q_2Q_8 - 24Q_4Q_6,$$

which is arranged as

$$Q_0(144Q_{12} - 108Q_{10}) + 60Q_2Q_{10} + 16Q_4Q_8 - 42Q_2Q_8 - 10Q_4Q_6 = 0.$$

Substituting the relations on R_{2m}, the left-hand side becomes

$$\begin{aligned}
&144Q_0Q_{12} + (60Q_2 - 108Q_0)Q_{10} + (16Q_4 - 42Q_2)Q_8 - 10Q_4Q_6\\
&= 144 \times \frac{21}{64}Q_0R_6 + (60Q_2 - 108Q_0)\frac{7}{16}R_6 + (16Q_4 - 42Q_2)\frac{5}{8}R_6 - 10Q_4R_6\\
&= R_6\left(\frac{189}{4}Q_0 + \frac{7}{4}(15Q_2 - 27Q_0) + 10Q_4 - \frac{105}{4}Q_2 - 10Q_4\right) = 0.
\end{aligned}$$

Lemma 3. *For $m \geq 6$, we have*

$$Q_{2m} = R_{2m} = -\lambda_m b_0 \rho^2 + \mu_m \frac{b_4}{\rho^2}, \quad \lambda_m = \frac{1}{2}k_m$$

and $\mu_m = 16\lambda_m$ for $m = 3, 4, \ldots$.

Proof. The first part is clear by the above argument. As for the second part, we suppose that for $m \geq 6$

$$\mu_h = 16\lambda_h, \quad \lambda_h = \frac{1}{2}k_h \text{ for } h = 3, 4, \ldots, m.$$

Then we have by (6.30)

$$16\lambda_{m+1} = \frac{1}{m+1}\left(16k_{m+1} + k_{m-1} + 16\sum_{h=3}^{m}(h-1)k_{m+1-h}\lambda_h\right)$$

and

$$\mu_{m+1} = \frac{1}{m-1}\left(k_{m-1} + 16\sum_{h=3}^{m}(h-1)k_{m+1-h}\lambda_h\right).$$

Therefore, $\mu_{m+1} = 16\lambda_{m+1}$ is equivalent to the equation:

$$\begin{aligned}
&(m-1)\left(16k_{m+1} + k_{m-1} + 16\sum_{h=3}^{m}(h-1)k_{m+1-h}\lambda_h\right)\\
&= (m+1)\left(k_{m-1} + 16\sum_{h=3}^{m}(h-1)k_{m+1-h}\lambda_h\right)
\end{aligned}$$

which is reduced to

$$2k_{m-1} - 16(m-1)k_{m+1} + 32\sum_{h=3}^{m}(h-1)k_{m+1-h}\lambda_h = 0. \tag{6.33}$$

By the supposition, this is equivalent to

$$\sum_{h=2}^{m}(h-1)k_{m+1-h}k_h = (m-1)k_{m+1}, \tag{6.34}$$

which holds by the following Proposition.

Proposition 2. *Regarding k_h, the equation* (6.34) *holds for $m \geq 3$.*

Proof. We have for function $(1-x)^{\frac{1}{2}}$ the following equations

$$\frac{(1-x)^{\frac{1}{2}}-1}{x} = -k_1 - k_2x - k_3x^2 - \cdots - k_mx^{m-1} - \cdots,$$

$$-\left(\frac{(1-x)^{\frac{1}{2}}-1}{x}\right)' = k_2 + 2k_3x + \cdots + (h+1)k_{h+2}x^h + \cdots,$$

$$(1-x)^{\frac{1}{2}} = 1 - k_1x - k_2x^2 - \cdots - k_hx^h - \cdots,$$

$$\begin{aligned} -(1-x)^{\frac{1}{2}}\left(\frac{(1-x)^{\frac{1}{2}}-1}{x}\right)' &= (1-x)^{\frac{1}{2}}\left\{\frac{(1-x)^{\frac{1}{2}}-1}{x^2} + \frac{1}{2x}(1-x)^{-\frac{1}{2}}\right\} \\ &= \frac{1-x-(1-x)^{\frac{1}{2}}}{x^2} + \frac{1}{2x} \\ &= \frac{1}{x^2} - \frac{1}{2x} - \frac{1}{x^2}(1-x)^{\frac{1}{2}} \\ &= \frac{1}{x^2} - \frac{1}{2x} - \frac{1}{x^2}(1 - k_1x - k_2x^2 - k_3x^3 - \dots) \\ &= k_2 + k_3x + \cdots + k_{m+1}x^{m-1} + \cdots, \end{aligned}$$

from which we obtain the relation for $m \geq 2$

$$k_{m+1} = mk_{m+1} - \sum_{h=1}^{m-1} k_h(m-h)k_{m-h+1} = mk_{m+1} - \sum_{h=2}^{m} k_{m+1-h}(h-1)k_h,$$

hence we have the equality

$$\sum_{h=2}^{m}(h-1)k_{m+1-h}k_h = (m-1)k_{m+1}. \qquad \square$$

Corollary. *We have for $m \geq 2$*

$$\sum_{h=1}^{m} k_hk_{m+1-h} = 2k_{m+1}, \tag{6.35}$$

and

$$\sum_{h=1}^{m} hk_{m+1-h}k_h = (m+1)k_{m+1}. \tag{6.36}$$

Proof. From (6.34) we obtain immediately

$$\sum_{h=1}^{m-1}(m-h)k_hk_{m+1-h}=(m-1)k_{m+1},$$

whose left-hand side is written as

$$\sum_{h=2}^{m-1}(m-h)k_hk_{m+1-h}+(m-1)k_1k_m.$$

The left-hand side of (6.34) is written as

$$\sum_{h=2}^{m-1}(h-1)k_{m+1-h}k_h+(m-1)k_1k_m.$$

By adding these terms we obtain

$$(m-1)\sum_{h=2}^{m-1}k_hk_{m+1-h}+2(m-1)k_1k_m=2(m-1)k_{m+1}$$

and hence

$$\sum_{h=1}^{m}k_hk_{m+1-h}=2k_{m+1}.$$

(6.35) and (6.34) imply immediately (6.36). □

Now, returning to the proof of Lemma 3, by the supposition, (6.30) and (6.34) we have

$$\begin{aligned}\lambda_{m+1}&=\frac{1}{m+1}\left(k_{m+1}+\frac{1}{16}k_{m-1}+\sum_{h=3}^{m}(h-1)k_{m+1-h}\times\frac{1}{2}k_h\right)\\&=\frac{1}{m+1}\left(k_{m+1}+\frac{1}{2}\sum_{h=2}^{m}(h-1)k_{m+1-h}k_h\right)\\&=\frac{1}{m+1}\left(k_{m+1}+\frac{1}{2}(m-1)k_{m+1}\right)=\frac{1}{2}k_{m+1}.\end{aligned}$$ □

Finally, we have to check the equality (6.21). We knew that (6.21) holds identically for $m=2,3,4$. In the following, we suppose $m\geq 5$. We have

$$Q_{2h}=R_{2h}=\lambda_h\left(-b_0\rho^2+\frac{16b_4}{\rho^2}\right)=\frac{k_h}{2}\left(-b_0\rho^2+\frac{16b_4}{\rho^2}\right)=2^4k_hR_6\ (h\geq 3)$$

and

$$Q_4=-\frac{b_0}{16}\rho^2+\frac{b_4}{\rho^2}=2R_6.$$

To see (6.21) we put

$$
\begin{aligned}
\Omega_m :=\ & Q_0(4(m+1)^2 Q_{2m+2} - 2(m+1)(2m-1)Q_{2m}) \\
& + 4\left\{\sum_{h=1}^{m-1}(m+1-h)^2 Q_{2h}Q_{2m-2h+2} \right. \\
& \quad \left. - \sum_{h=0}^{m-1}(h+1)(m-h)Q_{2h+2}Q_{2m-2h}\right\} \\
& - 2\left\{\sum_{h=1}^{m-1}(m+1-h)(2m-1-2h)Q_{2h}Q_{2m-2h} \right. \\
& \quad \left. - \sum_{h=1}^{m-1} 2h(m-h)Q_{2h}Q_{2m-2h}\right\}.
\end{aligned} \tag{6.37}
$$

We show that Ω_m vanishes identically. Ω_m is written as

$$
\begin{aligned}
\Omega_m =\ & 2(m+1)\{2(m+1)\cdot 2^4 k_{m+1} - (2m-1)2^4 k_m\}R_6 \\
& + 4\left\{\sum_{h=1}^{m-1}(m+1-h)^2 Q_{2h}Q_{2m-2h+2} \right. \\
& \quad \left. - \sum_{h=1}^{m} h(m+1-h)Q_{2h}Q_{2m+2-2h}\right\} \\
& - 2\left\{\sum_{h=1}^{m-1}((m+1-h)(2m-1-2h) - 2h(m-h))Q_{2h}Q_{2m-2h}\right\}.
\end{aligned}
$$

Since $2(m+1)k_{m+1} = (2m-1)k_m$, Ω_m becomes

$$
\begin{aligned}
\Omega_m =\ & 4\left\{\sum_{h=2}^{m-1}(m+1-h)(m+1-2h)Q_{2h}Q_{2m-2h+2} + m(m-2)Q_2Q_{2m}\right\} \\
& - 2\left\{\sum_{h=1}^{m-1}(2m^2 - (6h-1)m + 4h^2 - h - 1)Q_{2h}Q_{2m-2h}\right\} \\
=\ & 4m(m-2)Q_2Q_{2m} + 4\sum_{h=2}^{m-1}(m+1-h)(m+1-2h)2^8 k_h k_{m-h+1} {R_6}^2 \\
& - 2(m-2)(2m-3)Q_2Q_{2m-2} \\
& - 2\sum_{h=2}^{m-2}(2m^2 - (6h-1)m + 4h^2 - h - 1)Q_{2h}Q_{2m-2h}
\end{aligned}
$$

$$
\begin{aligned}
&= 2(m-2)Q_2(2mR_{2m} - (2m-3)R_{2m-2}) \\
&\quad + 2^{10}\sum_{h=2}^{m-1}(m+1-h)(m+1-2h)k_h k_{m-h+1}{R_6}^2 \\
&\quad - 2^9\sum_{h=2}^{m-2}(2m^2-(6h-1)m+4h^2-h-1)k_h k_{m-h}{R_6}^2.
\end{aligned}
$$

Since we have

$$2mR_{2m} - (2m-3)R_{2m-2} = 2^4(2mk_m - (2m-3)k_{m-1})R_6 = 0$$

and

$$
\begin{aligned}
&2\sum_{h=2}^{m-1}(m+1-h)(m+1-2h)k_h k_{m-h+1} \\
&\quad - \sum_{h=2}^{m-2}(2m^2-(6h-1)m+4h^2-h-1)k_h k_{m-h} \\
&= \sum_{h=2}^{m-1}(m+1-2h)(2m-2h-1)k_h k_{m-h} \\
&\quad - \sum_{h=2}^{m-2}(2m^2-(6h-1)m+4h^2-h-1)k_h k_{m-h} \\
&= (3-m)k_{m-1}k_1 + \sum_{h=2}^{m-2}\{(m+1-2h)(2m-2h-1) - 2m^2 \\
&\quad + (6h-1)m - 4h^2 + h + 1\}k_h k_{m-h} \\
&= -(m-3)k_1 k_{m-1} + \sum_{h=2}^{m-2} h k_h k_{m-h} \\
&= -(m-3)k_1 k_{m-1} + \sum_{h=1}^{m-1} h k_h k_{m-h} - mk_1 k_{m-1} \\
&= -(2m-3)k_1 k_{m-1} + mk_m \\
&= -(2m-3)\frac{1}{2}k_{m-1} + \frac{1}{2}(2m-3)k_{m-1} = 0
\end{aligned}
$$

by (6.36), hence we obtain $\Omega_m \equiv 0$. Thus, we have the following conclusion.

Theorem 3. *The solution of the system of equations* (5.8″), (5.12) *and* (5.13) *on y with $y(0,0)\neq 0$ is given by*

$$y=(\sigma+4c_1)\left\{\frac{1}{2}b_0\rho^2+\frac{8}{\rho^2}b_4+\left(\frac{1}{2}b_0\rho^2-\frac{8}{\rho^2}b_4\right)\cos u_2\right.$$
$$\left.+\left(b_2-\frac{1}{4}b_0\rho^2-\frac{4}{\rho^2}b_4\right)\sin^2 u_2\right\},$$

where $b_0\neq 0$, b_2, b_4, and c_1 are constants such that $b_2=\frac{1}{8}$, $b_4=\frac{1}{256b_0}$ and σ and $\rho\neq 0$ are auxiliary functions depending on u_1 only, which is written as

$$y=\frac{\sigma+4c_1}{2}\left\{\left(b_0\rho^2+\frac{1}{16b_0\rho^2}\right)\left(1-\frac{1}{2}\sin^2 u_2\right)\right.$$
$$\left.+\frac{1}{4}\sin^2 u_2+\left(b_0\rho^2-\frac{1}{16b_0\rho^2}\right)\cos u_2\right\}. \tag{6.38}$$

Proof. By means of the argument of this section and (6.1), (6.6), (6.8), Lemma 1 and Lemma 3, we have

$$\begin{aligned}
y&=\sum_{m=0}^{\infty}P_m(u_1)\sin^m u_2=\sum_{m=0}^{\infty}P_{2m}(u_1)\sin^{2m}u_2\\
&=(\sigma+4c_1)\sum_{m=0}^{\infty}Q_{2m}\sin^{2m}u_2\\
&=(\sigma+4c_1)\left\{Q_0+Q_2\sin^2 u_2+\sum_{m=2}^{\infty}Q_{2m}\sin^{2m}u_2\right\}\\
&=(\sigma+4c_1)\left\{b_0\rho^2+\left(-\frac{1}{2}b_0\rho^2+b_2\right)\sin^2 u_2\right.\\
&\qquad\left.+\sum_{m=2}^{\infty}\frac{k_m}{2}\left(-b_0\rho^2+16\frac{b_4}{\rho^2}\right)\sin^{2m}u_2\right\}\\
&=(\sigma+4c_1)\left\{b_0\rho^2\left(1-k_1\sin^2 u_2-\frac{1}{2}\sum_{m=2}^{\infty}k_m\sin^{2m}u_2\right)\right.\\
&\qquad\left.+b_2\sin^2 u_2+\frac{8b_4}{\rho^2}\sum_{m=2}^{\infty}k_m\sin^{2m}u_2\right\}
\end{aligned}$$

and since

$$\cos u_2 = 1 - \sum_{m=1}^{\infty} k_m \sin^{2m} u_2,$$

the right-hand side of the above expression can be written as

$$\begin{aligned} &= (\sigma + 4c_1)\left\{\frac{1}{2}b_0\rho^2 \cos u_2 + \frac{1}{2}b_0\rho^2(1 - k_1 \sin^2 u_2)\right. \\ &\qquad \left. + b_2 \sin^2 u_2 + \frac{8}{\rho^2}b_4\left(1 - \cos u_2 - \frac{1}{2}\sin^2 u_2\right)\right\} \\ &= (\sigma + 4c_1)\left\{\frac{1}{2}b_0\rho^2 + \frac{8}{\rho^2}b_4 + \left(\frac{1}{2}b_0\rho^2 - \frac{8}{\rho^2}b_4\right)\cos u_2\right. \\ &\qquad \left. + \left(b_2 - \frac{1}{4}b_0\rho^2 - \frac{4}{\rho^2}b_4\right)\sin^2 u_2\right\}. \end{aligned}$$

From (6.19) we have $b_2 = \frac{1}{8}$ for $y(0,0) \neq 0$ and $4b_0b_4 = b_2b_2$ by (6.22), hence $b_4 = 1/256b_0$. Hence the last expressions becomes (6.38). □

Example 3. In (6.38), ρ is defined by (6.4) as

$$\rho = \exp\left(\int_0^{u_1} \frac{2\sigma_2}{\sigma + 4c_1} du_1\right), \text{ with } \rho(0) = 1.$$

We put $\sigma_2 = 0$, then ρ becomes 1. Then we have

$$\begin{aligned} y = \frac{\sigma + 4c_1}{2}&\left\{\left(b_0 + \frac{1}{16b_0}\right)\left(1 - \frac{1}{2}\sin^2 u_2\right) + \frac{1}{4}\sin^2 u_2\right. \\ &\left. + \left(b_0 - \frac{1}{16b_0}\right)\cos u_2\right\}. \end{aligned}$$

If we put $b_0 - \frac{1}{16b_0} = 0$, then we see easily

$$\partial F_{\alpha\beta}/\partial u_2 = 0,$$

which is treated in Section 3. Here, we put $b_0 - \frac{1}{16b_0} = \frac{3}{8}$, and so $b_0 = \frac{1}{2}$ or $-\frac{1}{8}$. When $b_0 = \frac{1}{2}$, (6.38) turns into

$$\begin{aligned} y &= \frac{\sigma + 4c_1}{2}\left\{\frac{5}{8}\left(1 - \frac{1}{2}\sin^2 u_2\right) + \frac{1}{4}\sin^2 u_2 + \frac{3}{8}\cos u_2\right\} \\ &= \frac{\sigma + 4c_1}{32}\{10 - \sin^2 u_2 + 6\cos u_2\} = \frac{\sigma + 4c_1}{32}(\cos u_2 + 3)^2. \end{aligned}$$

Hence we obtain

$$F_{22} = \frac{1}{y} = \frac{32}{(\sigma + 4c_1)(\cos u_2 + 3)^2}.$$

And by (5.10) we obtain

$$F_{11} = \frac{1}{\sigma}\left(\frac{\partial \log F_{22}}{\partial u_1}\right)^2 = \frac{1}{\sigma}\left(\frac{\sigma'}{\sigma + 4c_1}\right)^2.$$

By the restriction (5.9) we have

$$\begin{aligned} &F_{33} = cF_{22}\sin^2 u_2,\ F_{34} = bF_{33},\ F_{44} = b^2F_{33} + \frac{1}{c_0 + c_1u_4u_4}, \\ &F_{12} = 0,\ F_{\alpha\lambda} = 0, \end{aligned}$$

where b, c, c_0 are constants. If we take the auxiliary function $\sigma = \sigma(u_1)$ as

$$\sigma = \frac{4}{u_1^2} - 4c_1 = \frac{4(1 - c_1u_1u_1)}{u_1u_1},$$

then we obtain

$$\begin{aligned} F_{11} &= \frac{1}{1 - c_1u_1u_1},\ F_{22} = \frac{8u_1u_1}{(\cos u_2 + 3)^2},\ F_{33} = \frac{8cu_1u_1\sin^2 u_2}{(\cos u_2 + 3)^2}, \\ F_{34} &= \frac{8bcu_1u_1\sin^2 u_2}{(\cos u_2 + 3)^2},\ F_{44} = \frac{8b^2cu_1u_1\sin^2 u_2}{(\cos u_2 + 3)^2} + \frac{1}{c_0 + c_1u_4u_4}. \end{aligned}$$

Furthermore, setting $b = 0$, $c_0 = -1$, $c = 1$ and $c_1 = -a$ and $u_1 = r$, $u_2 = \vartheta$, $u_3 = \varphi$, $u_4 = t$ we obtain the metric

$$\begin{aligned} ds^2 = \frac{1}{t^2}\Bigg\{&\frac{1}{1 + ar^2}dr^2 + \frac{8r^2}{(\cos\vartheta + 3)^2}(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \\ &- \frac{1}{1 + at^2}dt^2\Bigg\}, \end{aligned} \tag{6.39}$$

which is an interesting one compared with the metric (1.1) as a simple form. □

7. Treatment of case $b_0 = 0$, $b_1 \neq 0$

Finally, we discus the remaining case: $b_0 = 0$, $b_1 \neq 0$. Then we have

$$\begin{aligned} &Q_0 = \rho^2 b_0 = 0,\ Q_1 = \rho b_1 \neq 0,\ Q_2 = -\frac{1}{2}\rho^2 b_0 + b_2 = b_2, \\ &Q_3 = -\frac{1}{4}\rho b_1 + \frac{1}{\rho}b_3,\ Q_4 = -\frac{1}{16}b_0\rho^2 + \frac{1}{\rho^2}b_4 = \frac{1}{\rho^2}b_4 \end{aligned} \tag{7.1}$$

by (6.3), (6.5), (6.8), (6.9), (6.10), respectively. The equation (6.19) becomes

$$\begin{aligned}&\left(\sum_{m=1}^{\infty} Q_m \sin^m u_2\right)\left\{\frac{1}{\sin u_2} Q_1 + \left(4Q_2 - \frac{1}{2}\right)\right.\\&\quad\left. + \sum_{m=1}^{\infty}((m+2)^2 Q_{m+2} - (m+2)(m-1)Q_m)\sin^m u_2\right\}\\&= \left(\sum_{m=0}^{\infty}(m+1)Q_{m+1}\sin^m u_2\right)^2 - \left(\sum_{m=1}^{\infty} mQ_m \sin^m u_2\right)^2.\end{aligned}$$

From the coefficients of $\sin u_2$, we obtain the relation

$$Q_1(4Q_2 - \frac{1}{2}) + Q_2 Q_1 = Q_1 \cdot 2Q_2 \times 2,$$

which implies

$$Q_2 = b_2 = \frac{1}{2} \tag{7.2}$$

by means of $Q_1 \neq 0$. From the coefficients of $\sin^2 u_2$, we obtain the relation

$$Q_1 \cdot 9Q_3 + Q_2(4Q_2 - \frac{1}{2}) + Q_3 Q_1 = Q_1 \cdot 3Q_3 \times 2 + 2Q_2 \cdot 2Q_2 - Q_1 Q_1,$$

i.e.

$$4Q_1 Q_3 - \frac{1}{2}Q_2 + Q_1 Q_1 = 0.$$

Using (7.2), we obtain

$$Q_1(4Q_3 + Q_1) = \frac{1}{4},$$

which becomes by (7.1)

$$\begin{aligned}&\rho b_1(-\rho b_1 + \frac{4}{\rho} b_3 + \rho b_1) = 4b_1 b_3 = \frac{1}{4},\\&b_1 b_3 = \frac{1}{16} \text{ and } Q_3 = -\frac{1}{4} b_1 \rho + \frac{1}{16 b_1 \rho}.\end{aligned} \tag{7.3}$$

Hence (6.19) can be written as

$$\begin{aligned}&\left(\sum_{m=0}^{\infty} Q_{m+1} \sin^m u_2\right)\left\{Q_1 + \frac{3}{2}\sin u_2\right.\\&\quad\left. + \sum_{m=2}^{\infty}((m+1)^2 Q_{m+1} - (m+1)(m-2)Q_{m-1})\sin^m u_2\right\}\\&= \left(\sum_{m=0}^{\infty}(m+1)Q_{m+1}\sin^m u_2\right)^2 - \left(\sum_{m=1}^{\infty} mQ_m \sin^m u_2\right)^2.\end{aligned} \tag{7.4}$$

From the coefficients of $\sin^3 u_2$ of (7.4), we obtain

$$\begin{aligned} & Q_1 \cdot (16Q_4 - 4Q_2) + Q_2 \cdot 9Q_3 + Q_3 \cdot \frac{3}{2} + Q_4 \cdot Q_1 \\ & = Q_1 \cdot 8Q_4 + 2Q_2 \cdot 6Q_3 - Q_1 \cdot 4Q_2, \end{aligned}$$

that is

$$9Q_1Q_4 - 3Q_2Q_3 + \frac{3}{2}Q_3 = 9Q_1Q_4 = 0$$

by means of (7.2). Hence we obtain

$$Q_4 = 0. \tag{7.5}$$

Next, from the coefficients of $\sin^4 u_2$ and $\sin^5 u_2$ of (7.4) we obtain the relations

$$16Q_1Q_5 - 4Q_1Q_3 + \frac{3}{2}Q_4 = 0$$

and

$$25Q_1Q_6 + 9Q_2Q_5 + Q_3Q_4 - 10Q_1Q_4 - 2Q_2Q_3 - \frac{1}{2}Q_5 = 0,$$

from which we get by (7.5)

$$Q_5 = \frac{1}{4}Q_3 = 2k_2Q_3 \tag{7.6}$$

and so the second equation becomes

$$25Q_1Q_6 + \frac{9}{8}Q_3 - Q_3 - \frac{1}{8}Q_3 = 25Q_1Q_6 = 0,$$

hence it implies

$$Q_6 = 0. \tag{7.7}$$

From the coefficients of $\sin^6 u_2$ of (7.4), we obtain

$$36Q_1Q_7 + 12Q_2Q_6 + 4Q_3Q_5 + \frac{3}{2}Q_6 - 18Q_1Q_5 - Q_3Q_3 = 0,$$

from which we get

$$Q_7 = \frac{1}{2}Q_5 = \frac{1}{8}Q_3 = 2k_3Q_3 \tag{7.8}$$

by (7.5), (7.6) and (7.7). From the coefficients of $\sin^7 u_2$ of (7.4) we obtain the relation

$$49Q_1Q_8 + 12Q_7 - 6Q_5 = 49Q_1Q_8 = 0,$$

from which we get

$$Q_8 = 0 \tag{7.9}$$

by (7.5), (7.6), (7.7) and (7.8). From the coefficients of $\sin^8 u_2$ of (7.4) we obtain

$$64Q_1Q_9 + 16Q_3Q_7 - 40Q_1Q_7 - 8Q_3Q_5 = Q_1(64Q_9 - 40Q_7) = 0$$

by (7.5)$\sim$(7.9), which implies

$$Q_9 = \frac{40}{64} Q_7 = \frac{5}{64} Q_3 = 2k_4 Q_3. \tag{7.10}$$

Then, from the coefficients of $\sin^9 u_2$ of (7.4) we obtain

$$81Q_1Q_{10} + 48Q_2Q_9 - 30Q_2Q_7 = 0.$$

Since we have

$$48Q_9 - 30Q_7 = (96k_4 - 60k_3)Q_3 = (96 \times \frac{5}{128} - 60 \times \frac{1}{16})Q_3 = 0,$$

the above equality implies

$$Q_{10} = 0. \tag{7.11}$$

From the coefficients of $\sin^{10} u_2$ we obtain

$$100Q_1Q_{11} + 36Q_3Q_9 + 4Q_5Q_7 - 70Q_1Q_9 - 22Q_3Q_7 - 3Q_5Q_5 = 0.$$

By means of (7.6), (7.8) and (7.10), we have

$$\begin{aligned}
&36Q_3Q_9 + 4Q_5Q_7 - 22Q_3Q_7 - 3Q_5Q_5 \\
&= (72k_4 + 16k_2k_3 - 44k_3 - 12k_2k_2)Q_3Q_3 \\
&= \left(72 \times \frac{5}{128} + 16 \times \frac{1}{8} \times \frac{1}{16} - 44 \times \frac{1}{16} - 12 \times \frac{1}{64}\right) Q_3Q_3 = 0.
\end{aligned}$$

The above equality becomes

$$(100Q_{11} - 70Q_9)Q_1 = 0,$$

which implies

$$Q_{11} = \frac{7}{10} Q_9 = \frac{7}{10} \cdot 2k_4 Q_3 = 2k_5 Q_3. \tag{7.12}$$

Last, from the coefficients of $\sin^{11} u_2$ of (7.4) we obtain

$$121Q_1Q_{12} + 80Q_2Q_{11} - 56Q_2Q_9 = 0.$$

Since we have

$$80Q_{11} - 56Q_9 = (160k_5 - 112k_4)Q_3 = \left(160 \times \frac{7}{256} - 112 \times \frac{5}{128}\right) Q_3 = 0,$$

we obtain $121Q_1Q_{12} = 0$, which implies

$$Q_{12} = 0. \tag{7.13}$$

From the coefficients of $\sin^{12} u_2$ of (7.4) we obtain

$$144Q_1Q_{13} + 64Q_3Q_{11} + 16Q_5Q_9 - 108Q_1Q_{11} - 44Q_3Q_9 - 12Q_5Q_7 = 0.$$

By means of (5.6), (5.8) and (5.10) we have

$$64Q_3Q_{11} + 16Q_5Q_9 - 44Q_3Q_9 - 12Q_5Q_7$$

$$= (128k_5 + 64k_2k_4 - 88k_4 - 48k_2k_3)Q_3Q_3$$
$$= \left(128 \times \frac{7}{256} - 80 \times \frac{5}{128} - 6 \times \frac{1}{16}\right)Q_3Q_3 = 0,$$

and hence we obtain

$$Q_1(144Q_{13} - 108Q_{11}) = 0,$$

which implies

$$Q_{13} = \frac{108}{144}Q_{11} = \frac{18}{12}k_5Q_3 = 2k_6Q_3. \tag{7.14}$$

Collecting these results we have

$$\begin{cases} Q_{2h} = 0, & h = 2, 3, 4, 5, 6; \\ Q_{2h+1} = 2k_hQ_3, & h = 1, 2, 3, 4, 5, 6. \end{cases}$$

Now, for a fixed integer $m \geq 6$ we suppose that

$$Q_{2h} = 0 \text{ and } Q_{2h+1} = 2k_hQ_3, \quad h = 7, 8, \ldots, m.$$

From the coefficients of $\sin^{2m+1} u_2$ of (7.4) we obtain the relation

$$\begin{aligned} &Q_1(2m+2)^2Q_{2m+2} + Q_2((2m+1)^2Q_{2m+1} - (2m+1)(2m-2)Q_{2m-1}) \\ &\quad - Q_{11}(2m-8)(2m-11)Q_{2m-10} - Q_{2m-1}4Q_2 + \frac{3}{2}Q_{2m+1} + Q_{2m+2}Q_1 \\ &= \{Q_1(2m+2)Q_{2m+2} + 2Q_2(2m+1)Q_{2m+1}\} \times 2 \\ &\quad - \{2Q_2(2m-1)Q_{2m-1} + 11Q_{11}(2m-10)Q_{2m-10}\} \times \begin{cases} 2 & (\text{when } m > 6) \\ 1 & (\text{when } m = 6) \end{cases} \end{aligned}$$

which is arranged by the supposition as

$$(2m+1)^2Q_1Q_{2m+2} + 2m(2m-2)Q_2Q_{2m+1} - 2(2m^2 - 5m + 3)Q_2Q_{2m-1} = 0$$

when $m > 6$, and when $m = 6$ the expression becomes

$$13^2Q_{14}Q_1 + 120Q_{13}Q_2 - 90Q_{11}Q_2 = 0.$$

Since we have

$$120Q_{13} - 90Q_{11} = (240k_6 - 180k_5)Q_3 = 240(k_6 - \frac{3}{4}k_5)Q_3 = 0,$$

we obtain $Q_{14} = 0$. And we have also

$$\begin{aligned} &2m(2m-2)Q_{2m+1} - 2(2m^2 - 5m + 3)Q_{2m-1} \\ &= \{4m(2m-2)k_m - 4(2m^2 - 5m + 3)k_{m-1}\}Q_3 \\ &= 8m(m-1)\{k_m - \frac{2m-3}{2m}k_{m-1}\}Q_3 = 0, \end{aligned}$$

we obtain

$$Q_{2m+2} = 0. \tag{7.15}$$

Now, we go to prove the remaining claim. By means of the supposition and (7.15), from the coefficients of $\sin^{2m+2} u_2$ of (7.4) we obtain

$$\begin{aligned}
&Q_1((2m+3)^2 Q_{2m+3} - (2m+3)2mQ_{2m+1}) \\
&\quad + Q_3((2m+1)^2 Q_{2m+1} - (2m+1)(2m-2)Q_{2m-1}) \\
&\quad + \cdots + Q_{2h-1}((2m-2h+5)^2 Q_{2m-2h+5} \\
&\qquad - (2m-2h+5)(2m-2h+2)Q_{2m-2h+3}) \\
&\quad + \cdots + Q_{2m+1}(3^2 Q_3 - 0) + Q_{2m+3}Q_1 \\
&= (2m+3)Q_1Q_{2m+3} + 3(2m+1)Q_3Q_{2m+1} \\
&\quad + \cdots + (2h-1)(2m-2h+5)Q_{2h-1}Q_{2m-2h+5} \\
&\quad + \cdots + 5(2m-1)Q_{2m-1}Q_5 + (2m+1)3Q_{2m+1}Q_3 + (2m+3)Q_{2m+3}Q_1 \\
&\qquad - (2m+1)Q_1Q_{2m+1} - 3(2m-1)Q_3Q_{2m-1} \\
&\quad - \cdots - (2h-1)(2m-2h+3)Q_{2h-1}Q_{2m-2h+3} \\
&\quad + \cdots - (2m-1)3Q_{2m-1}Q_3 - (2m+1)Q_{2m+1}Q_1,
\end{aligned}$$

which is arranged as

$$\begin{aligned}
&\sum_{h=1}^{m+2} (2m-2h+5)^2 Q_{2h-1}Q_{2m-2h+5} \\
&- \sum_{h=1}^{m+2} (2h-1)(2m-2h+5)Q_{2h-1}Q_{2m-2h+5} \\
&- \sum_{h=1}^{m} (2m-2h+5)(2m-2h+2)Q_{2h-1}Q_{2m-2h+3} \\
&+ \sum_{h=1}^{m+1} (2h-1)(2m-2h+3)Q_{2h-1}Q_{2m-2h+3} = 0,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\sum_{h=1}^{m+2} (2m-2h+5)(2m-4h+6)Q_{2h-1}Q_{2m-2h+5} \\
&- \sum_{h=1}^{m+1} \{(2m-2h+5)(2m-2h+2) \\
&\qquad - (2h-1)(2m-2h+3)\}Q_{2h-1}Q_{2m-2h+3} = 0.
\end{aligned}$$

Since we have

$$\begin{aligned}&(2m-2h+5)(2m-2h+2)-(2h-1)(2m-2h+3)\\&=4(m-2h+3)(m-h+1)-2h+1,\end{aligned}$$

the above expression can be written as

$$\begin{aligned}
&\sum_{h=2}^{m+1}(2m-2h+5)(2m-4h+6)Q_{2h-1}Q_{2m-2h+5}\\
&\quad+4(m+1)^2Q_1Q_{2m+3}\\
&\quad-\sum_{h=2}^{m}\{4(m-2h+3)(m-h+1)-2h+1\}Q_{2h-1}Q_{2m-2h+3}\\
&\quad-(4m^2+2m-2)Q_1Q_{2m+1}\\
&=\{4(m+1)^2Q_{2m+3}-2(2m-1)(m+1)Q_{2m+1}\}Q_1\\
&\quad+\sum_{h=2}^{m+1}(2m-2h+5)(2m-4h+6)4k_{h-1}k_{m-h+2}Q_3Q_3\\
&\quad-\sum_{h=2}^{m}\{4(m-2h+3)(m-h+1)-2h+1\}4k_{h-1}k_{m-h+1}Q_3Q_3\\
&=2(m+1)\{2(m+1)Q_{2m+3}-(2m-1)Q_{2m+1}\}Q_1\\
&\quad+4\Bigg[2\sum_{h=2}^{m+1}(2m-2h+5)(m-2h+3)k_{h-1}k_{m-h+2}\\
&\qquad-\sum_{h=2}^{m}\{4(m-2h+3)(m-h+1)\\
&\qquad\quad-2h+1\}k_{h-1}k_{m-h+1}\Bigg]Q_3Q_3=0.
\end{aligned}\tag{7.16}$$

We show that the expression in the brackets [ ] vanishes. First, we have

$$\begin{aligned}
&2\sum_{h=2}^{m+1}(2m-2h+5)(m-2h+3)k_{h-1}k_{m-h+2}\\
&=2\sum_{h=2}^{m+1}(2(m-h+1)+3)(m-1-2(h-2))k_{h-1}k_{m-h+2}
\end{aligned}$$

$$= 4(m-1)\sum_{h=2}^{m+1}(m-h+1)k_{h-1}k_{m-h+2} + 6(m-1)\sum_{h=2}^{m+1} k_{h-1}k_{m-h+2}$$
$$- 8\sum_{h=2}^{m+1}(m-h+1)(h-2)k_{h-1}k_{m-h+2} - 12\sum_{h=2}^{m+1}(h-2)k_{h-1}k_{m-h+2}.$$

Since we have

$$\sum_{h=2}^{m+1}(m-h+1)k_{h-1}k_{m-h+2} = \sum_{h=1}^{m}(m-h)k_h k_{m-h+1} = \sum_{h=1}^{m}(h-1)k_{m+1-h}k_h$$

and

$$\sum_{h=2}^{m+1} k_{h-1}k_{m-h+2} = \sum_{h=1}^{m} k_h k_{m-h+1},$$
$$\sum_{h=2}^{m+1}(h-2)k_{h-1}k_{m-h+2} = \sum_{h=1}^{m}(h-1)k_h k_{m-h+1},$$

the above expression becomes

$$4(m-1)\sum_{h=1}^{m}(h-1)k_{m+1-h}k_h + 6(m-1)\sum_{h=1}^{m} k_h k_{m-h+1}$$
$$- 8\sum_{h=2}^{m+1}(m-h+1)(h-2)k_{h-1}k_{m-h+2} - 12\sum_{h=1}^{m}(h-1)k_h k_{m-h+1}$$
$$= 4(m-4)\sum_{h=1}^{m}(h-1)k_h k_{m+1-h} + 6(m-1)\sum_{h=1}^{m} k_h k_{m-h+1}$$
$$- 8\sum_{h=2}^{m+1}(m-h+1)(h-2)k_{h-1}k_{m-h+2} \qquad \text{(by (6.34) and (6.35))}$$
$$= 4(m-4)(m-1)k_{m+1} + 6(m-1)2k_{m+1} - 8\sum_{h=1}^{m}(h-1)(m-h)k_h k_{m+1-h}$$
$$= 4(m-1)^2 k_{m+1} - 8\sum_{h=1}^{m}(h-1)(m-h)k_h k_{m+1-h}$$
$$= 4(m-1)^2 k_{m+1} - 8\sum_{h=2}^{m}(h-1)(m-h)k_h k_{m+1-h}$$
$$= 4(m-1)^2 k_{m+1} - 8\sum_{h=1}^{m-1}(m-h)(h-1)k_{m+1-h}k_h$$

$$
\begin{aligned}
&= 4(m-1)^2 k_{m+1} + 8m \sum_{h=1}^{m-1} k_h k_{m+1-h} \\
&\quad - 8(m+1) \sum_{h=1}^{m-1} h k_h k_{m+1-h} + 8 \sum_{h=1}^{m-1} h^2 k_h k_{m+1-h} \\
&= 4(m-1)^2 k_{m+1} + 8m(2k_{m+1} - k_m k_1) \\
&\quad - 8(m+1)((m+1)k_{m+1} - m k_m k_1) + 8 \sum_{h=1}^{m-1} h^2 k_h k_{m+1-h} \\
&= \{4(m-1)^2 + 16m - 8(m+1)^2\} k_{m+1} \\
&\quad - \{4m - 4m(m+1)\} k_m + 8 \sum_{h=1}^{m-1} h^2 k_h k_{m+1-h} \\
&= -4(m+1)^2 k_{m+1} + 4m^2 k_m + 8 \sum_{h=1}^{m-1} h^2 k_h k_{m+1-h}.
\end{aligned}
$$

Then, for the second part in [ ] of (7.16) we have

$$
\begin{aligned}
&\sum_{h=2}^{m} \{4(m-2h+3)(m-h+1) - 2h + 1\} k_{h-1} k_{m-h+1} \\
&= 4 \sum_{h=2}^{m} (m-2h+3)(m-h+1) k_{h-1} k_{m-h+1} - \sum_{h=2}^{m} (2h-1) k_{h-1} k_{m-h+1} \\
&= -4 \sum_{h=1}^{m-1} (m-1-2h) h k_{m-h} k_h - 2 \sum_{h=1}^{m-1} h k_h k_{m-h} - \sum_{h=1}^{m-1} k_h k_{m-h} \\
&= 8 \sum_{h=1}^{m-1} h^2 k_h k_{m-h} - 2(2m-1) \sum_{h=1}^{m-1} h k_h k_{m-h} - \sum_{h=1}^{m-1} k_h k_{m-h} \\
&= 8 \sum_{h=1}^{m-1} h^2 k_h k_{m-h} - 2(2m-1) m k_m - 2k_m.
\end{aligned}
$$

Therefore, the expression in [ ] of (7.16) becomes

$$
\begin{aligned}
&- 4(m+1)^2 k_{m+1} + 4m^2 k_m + 2m(2m-1) k_m + 2k_m \\
&\quad + 8 \left(\sum_{h=1}^{m-1} h^2 k_h k_{m+1-h} - \sum_{h=1}^{m-1} h^2 k_h k_{m-h} \right)
\end{aligned}
$$

$$= -4(m+1)^2 k_{m+1} + 2(4m^2 - m + 1)k_m + 8\left\{\sum_{h=1}^{m-1} h^2 k_h k_{m+1-h} - \sum_{h=1}^{m-1} h^2 k_h k_{m-h}\right\}.$$

Regarding the first term in $\{\quad\}$ of the above expression we have

$$\sum_{h=1}^{m-1} h^2 k_h k_{m+1-h} = \sum_{h=2}^{m-1} h^2 k_h k_{m+1-h} + k_1 k_m$$
$$= \sum_{h=2}^{m-1} h^2 \frac{2h-3}{2h} k_{h-1} k_{m+1-h} + k_1 k_m$$
$$= \sum_{h=2}^{m-1} h^2 k_{h-1} k_{m+1-h} - \frac{3}{2}\sum_{h=2}^{m-1} h k_{h-1} k_{m+1-h} + k_1 k_m$$
$$= \sum_{h=1}^{m-2} (h+1)^2 k_h k_{m-h} - \frac{3}{2}\sum_{h=2}^{m-1} (h-1) k_{h-1} k_{m+1-h} - \frac{3}{2}\sum_{h=2}^{m-1} k_{h-1} k_{m+1-h} + k_1 k_m$$
$$= \sum_{h=1}^{m-2} h^2 k_h k_{m-h} + 2\sum_{h=1}^{m-2} h k_h k_{m-h} + \sum_{h=1}^{m-2} k_h k_{m-h} - \frac{3}{2}\sum_{h=1}^{m-2} h k_h k_{m-h} - \frac{3}{2} sum_{h=1}^{m-2} k_h k_{m-h} + k_1 k_m$$
$$= \sum_{h=1}^{m-2} h^2 k_h k_{m-h} + \frac{1}{2}\sum_{h=1}^{m-2} h k_h k_{m-h} - \frac{1}{2}\sum_{h=1}^{m-2} k_h k_{m-h} + k_1 k_m$$
$$= \left(\sum_{h=1}^{m-1} h^2 k_h k_{m-h} - (m-1)^2 k_{m-1} k_1\right) + \frac{1}{2}\left(\sum_{h=1}^{m-1} h k_h k_{m-h} - (m-1) k_1 k_{m-1}\right) - \frac{1}{2}\left(\sum_{h=1}^{m-1} k_h k_{m-h} - k_1 k_{m-1}\right) + k_1 k_m$$
$$= \sum_{h=1}^{m-1} h^2 k_h k_{m-h} + \frac{1}{2}\sum_{h=1}^{m-1} h k_h k_{m-h} - \frac{1}{2}\sum_{h=1}^{m-1} k_h k_{m-h} - \left(\frac{(m-1)^2}{2} + \frac{m-1}{4} - \frac{1}{4}\right) k_{m-1} + k_1 k_m$$

$$= \sum_{h=1}^{m-1} h^2 k_h k_{m-h} + \frac{m}{2} k_m - k_m - \frac{m(2m-3)}{4} k_{m-1} + \frac{1}{2} k_m$$

(by means of (6.36) and (6.35))

$$= \sum_{h=1}^{m-1} h^2 k_h k_{m-h} + \frac{m-1}{2} k_m - \frac{m(2m-3)}{4} k_{m-1}$$

$$= \sum_{h=1}^{m-1} h^2 k_h k_{m-h} + \frac{m-1}{2} k_m - \frac{m^2}{2} k_m$$

$$= \sum_{h=1}^{m-1} h^2 k_h k_{m-h} - \frac{m^2 - m + 1}{2} k_m,$$

from which we obtain

$$\sum_{h=1}^{m} h^2 k_h k_{m+1-h} = \sum_{h=1}^{m-1} h^2 k_h k_{m-h} + m^2 k_m k_1 - \frac{m^2 - m + 1}{2} k_m$$

$$= \sum_{h=1}^{m-1} h^2 k_h k_{m-h} + \frac{m-1}{2} k_m,$$

that is

$$\sum_{h=1}^{m} h^2 k_h k_{m+1-h} = \sum_{h=1}^{m-1} h^2 k_h k_{m-h} + \frac{m-1}{2} k_m. \tag{7.17}$$

Using these facts, we obtain

$$\begin{aligned} & 2\sum_{h=2}^{m+1} (2m-2h+5)(m-2h+3) k_{h-1} k_{m-h+2} \\ & \quad - \sum_{h=2}^{m} \{4(m-2h+3)(m-h+1) - 2h + 1\} k_{h-1} k_{m-h+1} \\ & = -4(m+1)^2 k_{m+1} + 2(4m^2 - m + 1) k_m - 4(m^2 - m + 1) k_m \\ & = -2(m+1)\{2(m+1) k_{m+1} - (2m-1) k_m\} = 0. \end{aligned}$$

Therefore, from the expression (7.16) we see that

$$2(m+1)\{2(m+1) Q_{2m+3} - (2m-1) Q_{2m+1}\} Q_1 = 0,$$

hence from supposition we obtain

$$Q_{2m+3} = \frac{2m-1}{2(m+1)} Q_{2m+1} = \frac{2m-1}{2(m+1)} 2 k_m Q_3 = 2 k_{m+1} Q_3.$$

By these arguments, we obtain the following claim.

Lemma 4. *We have for the case* $b_0 = 0$*,* $b_1 \neq 0$

$$Q_0 = 0,\ Q_1 = \rho b_1,\ Q_2 = \frac{1}{2},\ Q_3 = -\frac{1}{4}b_1\rho + \frac{1}{16b_1\rho},$$
$$Q_{2m} = 0,\ Q_{2m+1} = 2k_m Q_3,\ k_m = \frac{(2m-3)!!}{2^m m!}, \quad m = 2, 3, 4, \dots.$$

Thus, we obtain a conclusion for the case $b_0 = 0$ and $b_1 \neq 0$.

Theorem 4. *The solution of the system of* (5.8″), (5.12) *and* (5.13) *on* y *with* $y(0,0) = 0$*, is given by*

$$y = (\sigma + 4c_1)\sin u_2 \left\{ \frac{1}{2}\sin u_2 + \frac{b_1\rho}{2}(1 + \cos u_2) + \frac{1}{8b_1\rho}(1 - \cos u_2) \right\}$$

where $b_1 \neq 0$ *and* σ *and* $\rho \neq 0$ *are auxiliary functions depending only on* u_1*.*

Proof. By means of Lemma 4, we have

$$\begin{aligned}
y &= \sum_{m=0}^{\infty} P_m(u_1)\sin^m u_2 \\
&= (\sigma + 4c_1)\sum_{m=0}^{\infty} Q_m(u_1)\sin^m u_2 \\
&= (\sigma + 4c_1)\left\{ \rho b_1 \sin u_2 + \frac{1}{2}\sin^2 u_2 + Q_3 \sin^3 u_2 + \sum_{m=2}^{\infty} 2k_m Q_3 \sin^{2m+1} u_2 \right\} \\
&= (\sigma + 4c_1)\sin u_2 \left\{ \rho b_1 + \frac{1}{2}\sin u_2 + Q_3\left(\sin^2 u_2 + 2\sum_{m=2}^{\infty} k_m \sin^{2m} u_2 \right) \right\} \\
&= (\sigma + 4c_1)\sin u_2 \left\{ \rho b_1 + \frac{1}{2}\sin u_2 + 2Q_3\left(k_1 \sin^2 u_2 + \sum_{m=2}^{\infty} k_m \sin^{2m} u_2 \right) \right\} \\
&= (\sigma + 4c_1)\sin u_2 \left\{ \rho b_1 + \frac{1}{2}\sin u_2 + 2Q_3(1 - (1 - \sin^2 u_2)^{\frac{1}{2}}) \right\} \\
&= (\sigma + 4c_1)\sin u_2 \left\{ \rho b_1 + \frac{1}{2}\sin u_2 + 2Q_3(1 - \cos u_2) \right\} \\
&= (\sigma + 4c_1)\sin u_2 \left\{ \rho b_1 + \frac{1}{2}\sin u_2 + \left(-\frac{b_1\rho}{2} + \frac{1}{8b_1\rho} \right)(1 - \cos u_2) \right\} \\
&= (\sigma + 4c_1)\sin u_2 \left\{ \frac{1}{2}\sin u_2 + \frac{b_1\rho}{2}(1 + \cos u_2) + \frac{1}{8b_1\rho}(1 - \cos u_2) \right\}. \qquad \square
\end{aligned}$$

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