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ADJOINT ORBIT TYPES OF COMPACT EXCEPTIONAL LIE GROUP G_2 IN ITS LIE ALGEBRA

Takashi MIYASAKA

Introduction

A Lie group G naturally acts on its Lie algebra $\mathfrak{g}$, called the adjoint action. In this paper, we determine the orbit types of the compact exceptional Lie group G_2 in its Lie algebra $\mathfrak{g}_2$. As results, the group G_2 has four orbit types in the Lie algebra $\mathfrak{g}_2$ as

$$G_2/G_2, \qquad G_2/(U(1) \times U(1)),$$
$$G_2/((Sp(1) \times U(1))/\boldsymbol{Z}_2), \qquad G_2/((U(1) \times Sp(1))/\boldsymbol{Z}_2).$$

These orbits, especially the last two orbits, are not equivalent, that is, there exists no G_2-equivariant homeomorphism among them. Finally, the author would like to thank Professor Ichiro Yokota for his earnest guidance, useful advice and constant encouragement.

0. Preliminaries and notation

(1) For a group G and an element s of G, $\widetilde{s}$ denotes the inner automorphism induced by s:

$$\widetilde{s}(g) = sgs^{-1}, \ g \in G,$$

then $G^{\widetilde{s}} = \{g \in G \mid sg = gs\}$. Hereafter $G^{\widetilde{s}}$ is briefly written by G^s.

(2) For a transformation group G of a space M, the isotropy subgroup of G at a point $m \in M$ is denoted by G_m:

$$G_m = \{g \in G \mid gm = m\}.$$

(3) As mentioned in the introduction, a Lie group G acts on its Lie algebra $\mathfrak{g}$. When G is a compact Lie group, any element $X \in \mathfrak{g}$ is transformed to some element D of a fixed Cartan subalgebra $\mathfrak{h}$. Hence, to determine the conjugate classes of isotropy subgroups G_X, it suffices to consider the case of $X = D \in \mathfrak{h}$.

1. The Cayley algebra $\mathfrak{C}$ and the group G_2

Let $\mathfrak{C}$ be the division Cayley algebra with the canonical basis $\{e_0 = 1, e_1, \ldots, e_7\}$ and in $\mathfrak{C}$ the conjugation $\overline{x}$, the inner product (x, y) and the length $|x|$ are naturally defined ([1], [3]). The Cayley algebra $\mathfrak{C}$ contains naturally the field $\boldsymbol{R}$ of real numbers, furthermore the field $\boldsymbol{C}$ of complex numbers and the field $\boldsymbol{H}$ of quaternion numbers:

$$\begin{aligned}\boldsymbol{R} &= \{x = x1 \mid x \in \boldsymbol{R}\},\\ \boldsymbol{C} &= \{x_0 + x_1e_1 \mid x_0, x_1 \in \boldsymbol{R}\},\\ \boldsymbol{H} &= \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \mid x_0, x_1, x_2, x_3 \in \boldsymbol{R}\}.\end{aligned}$$

The automorphism group G_2 of the Cayley algebra $\mathfrak{C}$:

$$G_2 = \{\alpha \in \mathrm{Iso}_R(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}$$

is the simply connected compact Lie group of type G_2.

Any element $x \in \mathfrak{C}$ can be expressed as $x = a + be_4$, $a, b \in \boldsymbol{H}$, and $\mathfrak{C}$ is isomorphic to the algebra $\boldsymbol{H} \oplus \boldsymbol{H}e_4$ with multiplication

$$(a + be_4)(c + de_4) = (ac - \overline{d}b) + (b\overline{c} + da)e_4.$$

We define an $\boldsymbol{R}$-linear transformation γ of $\mathfrak{C}$ by

$$\gamma(a + be_4) = a - be_4,\ a + be_4 \in \boldsymbol{H} \oplus \boldsymbol{H}e_4 = \mathfrak{C}.$$

Next, to an element

$$x = a + m_1e_2 + m_2e_4 + m_3e_6,\ a, m_1, m_2, m_3 \in \boldsymbol{C}$$

of $\mathfrak{C}$, we associate an element

$$a + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

of the algebra $\boldsymbol{C} \oplus \boldsymbol{C}^3$ with the multiplication

$$(a + \boldsymbol{m})(b + \boldsymbol{n}) = (ab - \langle \boldsymbol{m}, \boldsymbol{n}\rangle) + (a\boldsymbol{n} + \overline{b}\boldsymbol{m} - \overline{\boldsymbol{m} \times \boldsymbol{n}}),$$

where $\langle \boldsymbol{m}, \boldsymbol{n}\rangle = {}^t\boldsymbol{m}\overline{\boldsymbol{n}}$ and $\boldsymbol{m} \times \boldsymbol{n} \in \boldsymbol{C}^3$ is the exterior product of $\boldsymbol{m}$, $\boldsymbol{n}$. Note that $\boldsymbol{C} \oplus \boldsymbol{C}^3$ is a left $\boldsymbol{C}$-module. Hereafter we identify $\mathfrak{C}$ with $\boldsymbol{H} \oplus \boldsymbol{H}e_4$ and $\boldsymbol{C} \oplus \boldsymbol{C}^3$:

$$\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4, \quad \mathfrak{C} = \boldsymbol{C} \oplus \boldsymbol{C}^3.$$

We define an $\boldsymbol{R}$-linear transformation $\gamma_{\boldsymbol{c}}$ of $\mathfrak{C}$ by

$$\gamma_{\boldsymbol{c}}(a + \boldsymbol{m}) = \overline{a} + \overline{\boldsymbol{m}},\ a + \boldsymbol{m} \in \boldsymbol{C} \oplus \boldsymbol{C}^3 = \mathfrak{C}.$$

Then, $\gamma, \gamma_{\boldsymbol{c}} \in G_2$, $\gamma^2 = {\gamma_{\boldsymbol{c}}}^2 = 1$ and $\gamma, \gamma_{\boldsymbol{c}}$ are conjugate in G_2 ([1]).

2. Subgroups $(Sp(1) \times Sp(1))/\boldsymbol{Z}_2$ and $SU(3)$ of G_2

Proposition 1 ([3]). $(G_2)^\gamma \cong (Sp(1) \times Sp(1))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1,1), (-1,-1)\}$.

Proof. Let $Sp(1) = \{p \in \boldsymbol{H} \mid p\overline{p} = 1\}$ and we define a map $\varphi : Sp(1) \times Sp(1) \to (G_2)^\gamma$ by

$$\varphi(p,q)(a + be_4) = qa\overline{q} + (pb\overline{q})e_4,\ a + be_4 \in \boldsymbol{H} \oplus \boldsymbol{H}e_4 = \mathfrak{C}.$$

φ is well-defined: $\varphi(p,q) \in (G_2)^\gamma$ and φ is a homomorphism. We shall show that φ is onto. Let $\alpha \in (G_2)^\gamma$. Note that

$$\mathfrak{C}_\gamma = \{x \in \mathfrak{C} \mid \gamma x = x\} = \boldsymbol{H}, \quad \mathfrak{C}_{-\gamma} = \{x \in \mathfrak{C} \mid \gamma x = -x\} = \boldsymbol{H}e_4$$

and these spaces are invariant under the action of the group $(G_2)^\gamma$. Now, since $\alpha \in (G_2)^\gamma$ induces an automorphism of $\mathfrak{C}_\gamma = \boldsymbol{H}$, there exists $q \in Sp(1)$ such that

$$\alpha a = qa\overline{q},\ a \in \boldsymbol{H}.$$

Let $\beta = \varphi(1,q)^{-1}\alpha$. Then $\beta \in (G_2)^\gamma$ and $\beta|\boldsymbol{H} = 1$. Since β induces an endomorphism of $\boldsymbol{H}e_4$, there exists $p \in \boldsymbol{H}$ such that $\beta e_4 = pe_4$. From $|p| = |pe_4| = |\beta e_4| = |e_4| = 1$, we see that $p \in Sp(1)$. Then

$$\beta(a + be_4) = \beta a + (\beta b)(\beta e_4) = a + b(pe_4) = a + (pb)e_4 = \varphi(p,1)(a + be_4).$$

Hence, $\beta = \varphi(p,1)$ and $\alpha = \varphi(1,q)\varphi(p,1) = \varphi(p,q)$ which shows that φ is onto. $\ker\varphi = \{(1,1), (-1,-1)\} = \boldsymbol{Z}_2$. Thus, we have $(G_2)^\gamma \cong (Sp(1) \times Sp(1))/\boldsymbol{Z}_2$. □

Proposition 2 ([1], [3]). $(G_2)_{e_1} \cong SU(3)$.

Proof. Let $SU(3) = \{A \in M(3, \boldsymbol{C}) \mid A^*A = E,\ \det A = 1\}$ and we define a map $\psi : SU(3) \to (G_2)_{e_1}$ by

$$\psi(A)(a + \boldsymbol{m}) = a + A\boldsymbol{m},\ a + \boldsymbol{m} \in \boldsymbol{C} \oplus \boldsymbol{C}^3 = \mathfrak{C}.$$

ψ is well-defined: $\psi(A) \in (G_2)_{e_1}$. ψ is injective and a homomorphism. We shall show that ψ is onto. Let $\alpha \in (G_2)_{e_1}$. Note that α induces a $\boldsymbol{C}$-linear transformation of $\boldsymbol{C}^3$. Now let

$$\alpha e_2 = \boldsymbol{a}_1,\ \alpha e_4 = \boldsymbol{a}_2,\ \alpha e_6 = \boldsymbol{a}_3$$

and consider a matrix $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) \in M(3, \boldsymbol{C})$. From $(\alpha e_2)(\alpha e_4) = \alpha(e_2 e_4) = -\alpha(e_6)$, we have $\boldsymbol{a}_1\boldsymbol{a}_2 = -\boldsymbol{a}_3$, namely, $-\langle \boldsymbol{a}_1, \boldsymbol{a}_2\rangle - \overline{\boldsymbol{a}_1 \times \boldsymbol{a}_2} = -\boldsymbol{a}_3$, then

$$\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle = 0, \quad \boldsymbol{a}_3 = \overline{\boldsymbol{a}_1 \times \boldsymbol{a}_2}.$$

Similarly, we have $\langle \boldsymbol{a}_2, \boldsymbol{a}_3\rangle = \langle \boldsymbol{a}_3, \boldsymbol{a}_1 \rangle = 0$. Moreover, from $(\alpha e_k)(\alpha e_k) = \alpha(e_k e_k) = \alpha(-1) = -1$, we have $\langle \boldsymbol{a}_k, \boldsymbol{a}_k\rangle = 1$, hence, $A \in U(3) = \{A \in M(3, \boldsymbol{C}) \mid A^*A = E\}$. Finally, $\det A = (\boldsymbol{a}_3, \boldsymbol{a}_1 \times \boldsymbol{a}_2) = (\boldsymbol{a}_3, \overline{\boldsymbol{a}_3}) = \langle \boldsymbol{a}_3, \boldsymbol{a}_3 \rangle = 1$ (where $(\boldsymbol{a}, \boldsymbol{b})$ is the usual inner product in $\boldsymbol{C}^3$: $(\boldsymbol{a}, \boldsymbol{b}) = {}^t\boldsymbol{a}\boldsymbol{b}$). Hence, we have $A \in SU(3)$ and $\psi(A) = \alpha$ which shows that ψ is onto. Thus, we have $SU(3) \cong (G_2)_{e_1}$. $\square$

3. Adjoint orbit types of G_2 in the Lie algebra $\mathfrak{g}_2$

The Lie algebra $\mathfrak{g}_2$ of the group G_2 is given by

$$\mathfrak{g}_2 = \{D \in \mathrm{Hom}_R(\mathfrak{C}) \mid D(xy) = (Dx)y + x(Dy)\}.$$

The adjoint action of the group G_2 on the Lie algebra $\mathfrak{g}_2$ is given by

$$\mu : G_2 \times \mathfrak{g}_2 \to \mathfrak{g}_2,\ \mu(\alpha, D) = \alpha D \alpha^{-1}.$$

Now, we shall determine the adjoint orbit types of the group G_2 in $\mathfrak{g}_2$. Since the group G_2 contains the subgroup $SU(3)$ (Proposition 2), the Lie algebra $\mathfrak{g}_2$ also contains the subalgebra $\mathfrak{su}(3) = \{D \in M(3, \boldsymbol{C}) \mid D^* + D = 0,\ \mathrm{tr}(D) = 0\}$. We choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{su}(3)$ as

$$\mathfrak{h} = \left\{ D(r,s,t) = \begin{pmatrix} re_1 & 0 & 0 \\ 0 & se_1 & 0 \\ 0 & 0 & te_1 \end{pmatrix} \;\middle|\; r, s, t \in \boldsymbol{R},\ r + s + t = 0 \right\},$$

which is also a Cartan subalgebra of $\mathfrak{g}_2$. The order of r, s, t of $D(r,s,t)$ is arbitrarily exchanged by the action of G_2.

Note that between the induced mappings $\varphi_* : \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \mathfrak{g}_2$ and $\psi_* : \mathfrak{su}(3) \to \mathfrak{g}_2$ of φ and ψ, there exist the following relations:

$$\varphi_*(e_1, 0) = \psi_*(\mathrm{diag}(0, e_1, -e_1)), \quad \varphi_*(0, e_1) = \psi_*(\mathrm{diag}(2e_1, -e_1, -e_1)).$$

Theorem 3. *The orbit types in $\mathfrak{g}_2$ through $D(r,s,t) \in \mathfrak{h}$ under the adjoint action of the group G_2 are as follows*:

(1) *In the case*: $r = s = t = 0$, *the orbit through* $D(0,0,0)$ *is*

$$G_2/G_2.$$

(2) *In the case*: r, s, t *are non-zero and distinct, the orbit through* $D(r,s,t)$ *is*

$$G_2/(U(1) \times U(1)).$$

(3) *In the case*: r *is non-zero, the orbit through* $D(2r, -r, -r)$ *is*

$$G_2/((Sp(1) \times U(1))/\boldsymbol{Z}_2).$$

(4) *In the case*: r *is non-zero, the orbit through* $D(0, r, -r)$ *is*

$$G_2/((U(1) \times Sp(1))/\boldsymbol{Z}_2).$$

Proof. (1) trivial.

(2) Let $U(1) = \{a \in \boldsymbol{C} \mid a\overline{a} = 1\}$ and $S(U(1) \times U(1) \times U(1))$ be the diagonal subgroup of $SU(3)$ and $\psi : S(U(1) \times U(1) \times U(1)) \to (G_2)_{D(r,s,t)}$ be the restriction map ψ of Proposition 2. Then, ψ is well-defined and

injective. We shall show that ψ is onto. For this purpose, we first show that for $\alpha \in (G_2)_{D(r,s,t)}$, we have

$$\alpha e_1 = \pm e_1.$$

Indeed, let $\alpha e_1 = a + \boldsymbol{m} \in \boldsymbol{C} \oplus \boldsymbol{C}^3$. From the condition $\alpha D(r,s,t) = D(r,s,t)\alpha$,

$$0 = \alpha 0 = \alpha D(r,s,t)e_1 = D(r,s,t)\alpha e_1 = D(r,s,t)(a + \boldsymbol{m}) = D(r,s,t)\boldsymbol{m},$$

we have $\boldsymbol{m} = 0$. So $\alpha e_1 = a$. Since $|a| = |\alpha e_1| = |e_1| = 1$ and $\alpha 1 = 1$, we have $\alpha e_1 = \pm e_1$. In the case of $\alpha e_1 = -e_1$, consider $\gamma_{\boldsymbol{c}} \in G_2$. Then $\alpha\gamma_{\boldsymbol{c}} e_1 = e_1$, so $\alpha\gamma_{\boldsymbol{c}} = \psi(A)$, $A = (a_{kl}) \in SU(3)$ (Proposition 2). Hence, $\alpha = \psi(A)\gamma_{\boldsymbol{c}}$. Then

$$A\gamma_{\boldsymbol{c}} D(r,s,t)\begin{pmatrix}1\\0\\0\end{pmatrix} = A\gamma_{\boldsymbol{c}}\begin{pmatrix}re_1\\0\\0\end{pmatrix} = A\begin{pmatrix}-re_1\\0\\0\end{pmatrix} = \begin{pmatrix}-ra_{11}e_1\\-ra_{21}e_1\\-ra_{31}e_1\end{pmatrix}.$$

On the other hand,

$$D(r,s,t)A\gamma_{\boldsymbol{c}}\begin{pmatrix}1\\0\\0\end{pmatrix} = D(r,s,t)A\begin{pmatrix}1\\0\\0\end{pmatrix} = D(r,s,t)\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix} = \begin{pmatrix}re_1a_{11}\\se_1a_{21}\\te_1a_{31}\end{pmatrix}.$$

From the condition $\alpha D(r,s,t) = D(r,s,t)\alpha$, we have

$${}^t(-ra_{11}e_1, -ra_{21}e_1, -ra_{31}e_1) = {}^t(re_1a_{11}, se_1a_{21}, te_1a_{31}).$$

But, it is easy to see that this is false. Hence, we have $\alpha e_1 = e_1$, so $\alpha = \psi(A)$, $A \in SU(3)$. From the condition $\alpha D(r,s,t) = D(r,s,t)\alpha$ again, we have $\alpha = \psi(A)$, $A \in S(U(1) \times U(1) \times U(1))$, which shows that ψ is onto. Thus we have $(G_2)_{D(r,s,t)} = S(U(1) \times U(1) \times U(1)) \cong U(1) \times U(1)$.

(3) Since

$$\begin{aligned}\exp(\pi D(2,-1,-1)) &= \exp(\pi\psi_*(\mathrm{diag}(2e_1, -e_1, -e_1)))\\ &= \exp(\pi\varphi_*(0, e_1)) = \gamma,\end{aligned}$$

if $\alpha \in G_2$ commutes with $D(2,-1,-1)$, then α also commutes with γ. Hence, $\alpha \in (G_2)^\gamma = \varphi(Sp(1) \times Sp(1))$ (Proposition 1). So there exist $p, q \in Sp(1)$ such that $\alpha = \varphi(p,q)$. Again from the commutativity with $\exp\Big(\dfrac{\pi}{2}D(2,-1,-1)\Big) = \exp\Big(\dfrac{\pi}{2}\psi_*(\mathrm{diag}(2e_1, -e_1, -e_1))\Big) = \varphi(1, e_1)$, we have $\varphi(p,q)\varphi(1,e_1) = \varphi(1,e_1)\varphi(p,q)$, hence, $\varphi(p, qe_1) = \varphi(p, e_1q)$. So $qe_1 = e_1q$, therefore $q \in \boldsymbol{C} \cap Sp(1) = U(1)$. Conversely, $\alpha = \varphi(p,q)$ ($p \in Sp(1)$, $q \in U(1)$) commutes with $\varphi(1, e^{te_1})$, $t \in \boldsymbol{R}$, so α also commutes with $\varphi_*(0, e_1) = D(2,-1,-1)$. Thus, we have $(G_2)_{D(2,-1,-1)} = \varphi(Sp(1) \times U(1)) = (Sp(1) \times U(1))/\boldsymbol{Z}_2$.

(4)

$$\begin{aligned}\exp(\pi D(0,1,-1)) &= \exp(\pi\psi_*(\mathrm{diag}(0,e_1,-e_1)))\\ &= \exp(\pi(\varphi_*(e_1,0)) = \gamma.\end{aligned}$$

Hence, $(G_2)_{D(0,r,-r)} = (U(1)\times Sp(1))/\boldsymbol{Z}_2$ is proved in a similar way to (3) above. □

Proposition 4. *If r and s are non-zero, then the groups $(G_2)_{D(0,r,-r)} \cong (U(1)\times Sp(1))/\boldsymbol{Z}_2$ and $(G_2)_{D(2s,-s,-s)} \cong (Sp(1)\times U(1))/\boldsymbol{Z}_2$ are not conjugate in the group G_2.*

Proof. We shall prove that $D(0,r,-r)$ and $D(2s,-s,-s)$ are not conjugate under the action of the group G_2. Suppose that there exists $\alpha \in G_2$ such that

$$\alpha(D(0,r,-r)) = (D(2s,-s,-s))\alpha.$$

Let $\alpha e_2 = a_0 + \boldsymbol{m} \in \boldsymbol{C}\oplus\boldsymbol{C}^3$, $\boldsymbol{m} = {}^t(m_1,m_2,m_3)$. Then, from

$$\alpha\begin{pmatrix}0&0&0\\0&re_1&0\\0&0&-re_1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2se_1&0&0\\0&-se_1&0\\0&0&-se_1\end{pmatrix}\left(a_0+\begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix}\right),$$

we have $\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}2sm_1e_1\\-sm_2e_1\\-sm_3e_1\end{pmatrix}$, hence, $m_1=m_2=m_3=0$, so

$$\alpha e_2 = a_0 \in \boldsymbol{C}.$$

From $(\alpha e_2)(\alpha e_2) = \alpha(e_2e_2) = \alpha(-1) = -1$, we have $a_0a_0 = -1$, hence, $\alpha e_2 = a_0 = \pm e_1$. Similarly, we have $\alpha e_3 = \pm e_1$. Then $\alpha e_1 = (\alpha e_2)(\alpha e_3) = (\pm e_1)(\pm e_1) = \pm 1$, which is a contradiction. □

Theorem 5. *If r and s are non-zero, then the orbit spaces*

$X = \{\alpha(D(0,r,-r))\alpha^{-1} \mid \alpha\in G_2\}$ and $Y = \{\alpha(D(2s,-s,-s))\alpha^{-1} \mid \alpha\in G_2\}$

are not equivalent.

Proof. Suppose that there exists a G_2-equivariant homeomorphism $h : X \to Y$. Then, there exists $\alpha\in G_2$ such that

(1) $$h(D(0,r,-r)) = \alpha(D(2s,-s,-s))\alpha^{-1}.$$

For any $\delta \in (G_2)_{D(0,r,-r)}$, we have

$$\delta(h(D(0,r,-r)))\delta^{-1} = \delta(\alpha(D(2s,-s,-s))\alpha^{-1})\delta^{-1}.$$

Since h is a G_2-equivariant homeomorphism, we have $\widetilde{\delta}h = h\widetilde{\delta}$. Hence, we see

(2) $$h(D(0,r,-r)) = \delta(\alpha(D(2s,-s,-s))\alpha^{-1})\delta^{-1}.$$

From (1) and (2), we have

$$\delta(\alpha(D(2s,-s,-s))\alpha^{-1})\delta^{-1} = \alpha(D(2s,-s,-s))\alpha^{-1},$$

that is, $\alpha^{-1}(\delta(\alpha(D(2s,-s,-s))\alpha^{-1})\delta^{-1})\alpha = D(2s,-s,-s)$. This implies $\alpha^{-1}\delta\alpha \in (G_2)_{D(2s,-s,-s)}$. Thus, we have

$$\alpha^{-1}((G_2)_{D(0,r,-r)})\alpha \subset (G_2)_{D(2s,-s,-s)}.$$

Since $\dim((G_2)_{D(0,r,-r)}) = \dim((G_2)_{D(2s,-s,-s)})$ (Theorem 3 (3) and (4)), the inclusion above must be equal, that is,

$$\alpha^{-1}((G_2)_{D(0,r,-r)})\alpha = (G_2)_{D(2s,-s,-s)}.$$

This means that the groups $(G_2)_{D(0,r,-r)}$ and $(G_2)_{D(2s,-s,-s)}$ are conjugate in G_2, which contradicts Proposition 4. □

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Takashi Miyasaka
Takatoh High School
Obara, Takatoh 396-0293, Japan
e-mail address: coolkai@mac.com

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