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SKEW GROUP ALGEBRAS AND THEIR YONEDA ALGEBRAS

Dedicated to Helmut Lenzing on the occasion of the 60-th birthday

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Abstract. Skew group algebras appear in connection with the study of singularities [1], [2]. It was proved in [4], [6], [10] the preprojective algebra of an Euclidean diagram is Morita equivalent to a skew group algebra of a polynomial algebra. In [7] we investigated the Yoneda algebra of a selfinjective Koszul algebra and proved they have properties analogous to the commutative regular algebras, we call such algebras generalized Auslander regular. The aim of the paper is to prove that given a positively graded locally finite K-algebra $\Lambda = \sum_{i \geq 0} \Lambda_i$ and a finite grading preserving group G of automorphisms of Λ, with characteristic K not dividing the order of G, then G acts naturally on the Yoneda algebra $\Gamma = \bigoplus_{k \geq 0} \mathrm{Ext}^k_\Lambda(\Lambda_0, \Lambda_0)$ and the skew group algebra $\Gamma * G$ is isomorphic to the Yoneda algebra $\Lambda * G = \bigoplus_{k \geq 0} \mathrm{Ext}^k_{\Lambda * G}(\Lambda_0 * G, \Lambda_0 * G)$. As an application we prove Λ is generalized Auslander regular if and only if $\Lambda * G$ is generalized Auslander regular and Λ is Koszul if and only if $\Lambda * G$ is so.

It was proved by H. Lenzing the indecomposable modules over a quiver algebra KQ with K a field and Q an Euclidean diagram, are parametrized by Klein curve singularities $P_1(C)/G$ arising from the action of polyhedral groups on projective line.

The polyhedral groups are the finite subgroups of $SL(2, C)$, [8] they act naturally on $C[x, y]$ sending homogeneous elements to homogeneous elements, for example the cyclic group

$$Z_n = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \,\middle|\, \xi \text{ is a complex primitive } n\text{-th root of unity} \right\}$$

acts on $C[x, y]$ by $x \mapsto \xi x$, $y \mapsto \xi^{-1} y$. The coordinate ring of $P_1(C)/G$ is $C[x, y]^G$.

In general, given a positively graded locally finite K-algebra $\Lambda = \bigoplus_{j \geq 0} \Lambda_j$, with Λ_0 semisimple, $\Lambda_i \Lambda_j = \Lambda_{i+j}$ and G a finite group of grading preserving K-automorphisms of Λ, we associate to them two K-algebras, the fixed ring $\Lambda^G = \{\lambda \in \Lambda \mid \lambda^g = \lambda \text{ for all } g \in G\}$ and the skew group algebra $\Lambda * G$ defined as follows:

As a vector space $\Lambda * G = \Lambda \bigotimes\limits_K KG$. For $\lambda \in \Lambda$ and $g \in G$, we write λg instead of $\lambda \bigotimes\limits_K g$, and multiplication is given by $g\lambda = \lambda^g g$, where the element $\lambda^g \in \Lambda$ denotes the image of λ under the automorphism g.

The algebra Λ^G is the endomorphism ring of a finitely generated projective $\Lambda * G$-module, explicitly: given the idempotent $e = 1/|G| \sum\limits_{g \in G} g$ of $\Lambda * G$ there exists an isomorphisms $e(\Lambda * G)e \cong \Lambda^G$ [3].

Given a quiver Q and a field K the preprojective algebra is the K-algebra $K\hat{Q}/I$ with quiver $\hat{Q}$ with vertices $\hat{Q}_0 = Q_0$ and arrows $\hat{Q}_1 = Q_1 \cup Q_1^{\mathrm{op}}$, where Q^{op} denotes the opposite quiver of Q. For any arrow $\alpha \in Q_1$ write $\hat{\alpha}$ for the corresponding arrow in the opposite quiver. The ideal I is generated by relations $\sum \alpha_i \hat{\alpha}_i$ and $\sum \hat{\alpha}_i \alpha_i$.

We have the following:

Theorem 1 (Lenzing [6], Reiten and Van den Bergh [10], Crawley -Boevey [4])**.** *The preprojective algebras $\Lambda = C\hat{Q}/I$ corresponding to an Euclidean quiver Q are Morita equivalent to the skew group algebras $C[x, y] * G$, with G a polyhedral group.*

In the example of the cyclic group Z_n acting on $C[x, y]$ given above the skew group algebra $C[x, y] * Z_n$ is the Mckay quiver [8] obtained as follows:

Let $\{S_1, S_2, \ldots, S_n\}$ be the irreducible representations of Z_n, put a vertex v_i for each simple S_i and m_{ij} arrows from v_i to v_j if $V \bigotimes\limits_C S_i = \bigoplus\limits_j m_{ij} S_j$, where V is the two dimensional representation given by $(x, y)/(x, y)^2$. The Mckay quiver is:

$$\begin{tikzcd} 1 \arrow[r, shift left] \arrow[rrrrrr, bend right] \arrow[rrrrrr, bend right=40] & 2 \arrow[l, shift left] \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n-1 \arrow[l, shift left] \arrow[r, shift left] & n. \arrow[l, shift left] \end{tikzcd}$$

In this paper we consider generalized Auslander regular algebras, they constitute non commutative versions of the regular algebras and contain the preprojective algebras. We will prove that given a positively graded algebra Λ as above and a finite a group of automorphisms G, the skew group algebra $\Lambda * G$ is generalized Auslander regular if and only if Λ is so.

Let M be a $\Lambda * G$-module and V a KG-module. The vector space $M \bigotimes\limits_K V$ is a $\Lambda * G$-module with action given by: $\lambda.(m \otimes v) = \lambda m \otimes v$ and $g(m \otimes v) = gm \otimes gv$, for $\lambda \in \Lambda$, $m \in M$ and $g \in G$.

Lemma 2. *Let $\Lambda = \bigoplus\limits_{j \geq 0} \Lambda_j$ be a positively graded, locally finite K-algebra with Λ_0 semisimple, $\Lambda_i \Lambda_j = \Lambda_{i+j}$ and K an algebraically closed field. Let*

G be a finite grading preserving group of automorphisms of Λ such that the characteristic of K does not divide the order of G and $\Lambda * G$ the skew group algebra. Then the $\Lambda * G$-simple modules are of the form: $S \bigotimes_K V$ with S a simple $\Lambda * G$ submodule of Λ_0 and V an irreducible KG-module.

Proof. The radical of $\Lambda * G$ is $J * G$ and $J * G(S \bigotimes_K V) = JS \bigotimes_K V = 0$, then $S \bigotimes_K V$ is a $\Lambda * G/J * G = \Lambda_0 * G$-module. By [9], $\Lambda_0 * G$ is semisimple, therefore: $S \bigotimes_K V$ is semisimple. We need to prove $S \bigotimes_K V$ is indecomposable. We have natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\Lambda * G}(S \bigotimes_K V, S \bigotimes_K V) &\cong \mathrm{Hom}_{\Lambda}(S \bigotimes_K V, S \bigotimes_K V)^G \\
&\cong \mathrm{Hom}_K(V, \mathrm{Hom}_{\Lambda}(S, S \bigotimes_K V))^G \\
&\cong \mathrm{Hom}_{KG}(V, \mathrm{Hom}_{\Lambda}(S, S \bigotimes_K V)).
\end{aligned}$$

We have also natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\Lambda}(S, S \bigotimes_K V) &\cong \mathrm{Hom}_{\Lambda/J}(S, S \bigotimes_K V) \\
&\cong \mathrm{Hom}_{\Lambda/J}(S, \Lambda/J) \bigotimes_{\Lambda/J} S \bigotimes_K V \\
&\cong \mathrm{Hom}_{\Lambda/J}(S, S) \bigotimes_K V \\
&\cong \mathrm{Hom}_{\Lambda}(S, S) \bigotimes_K V.
\end{aligned}$$

Then we have isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{KG}(V, \mathrm{Hom}_{\Lambda}(S, S \bigotimes_K V)) &\cong \mathrm{Hom}_{KG}(V, V) \bigotimes_K \mathrm{Hom}_{\Lambda}(S, S) \\
&\cong K \otimes K \cong K.
\end{aligned}$$

If S is a $\Lambda * G$ simple, then S is a $\Lambda * G/J * G \cong \Lambda_0 * G$-module. Therefore: S is isomorphic to a summand of $\Lambda_0 * G = \Lambda_0 \otimes KG$. Decomposing $\Lambda_0 = \bigoplus_{i=1}^{t} S_i$, $KG = \bigoplus_{j=1}^{m} V_j$ we obtain S is isomorphic to some module $S_i \bigotimes_K V_j$. □

Lemma 3. *Let K be a field, G a finite group with characteristic of K not dividing the order of G. Let M be a K-vector space with G-action. Denote by $M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$ the set of fixed points. Then the fixed point functor $(\quad)^G : \mathrm{Mod}_{KG} \to \mathrm{Mod}_{KG}$ is exact.*

Proof. Let $t : M \to M$ be a K-linear map given by: $t(M) = 1/|G| \sum\limits_{g \in G} gm$. Then it is clear $t(M) = M^G$.

Let $0 \to L \xrightarrow{j} M \xrightarrow{\pi} N \to 0$ be an exact sequence of G-modules and G-maps. Then we have an exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "t"] & M \arrow[r] \arrow[d, "t"] & N \arrow[r] \arrow[d, "t"] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0,
\end{tikzcd}$$

which induces an exact diagram:

$$\begin{tikzcd}
M \arrow[r, "\pi"] \arrow[d, "t"'] & N \arrow[r] \arrow[d, "t"'] & 0 \\
M^G \arrow[r, "\operatorname{res}\pi"] \arrow[d] & N^G \arrow[d] & \\
0 & 0. &
\end{tikzcd}$$

Therefore, $\operatorname{res}\pi$ is a map onto N^G. It is clear $M^G \cap L = L^G$. Hence; $0 \to L^G \to M^G \to N^G \to 0$ is exact. $\square$

Lemma 4. *Let Λ be a positively graded K-algebra, G a finite grading preserving group of automorphisms of Λ. Let L, M and N be $\Lambda * G$-modules. Then the following statements hold*:

i) *The abelian group $\mathrm{Hom}_\Lambda(M, N)$ is a G-module with action $g*f(m) = gf(g^{-1}m)$.*

ii) *There is an equality $\mathrm{Hom}_\Lambda(M, N)^G = \mathrm{Hom}_{\Lambda * G}(M, N)$.*

iii) *For all $k \geq 0$ there is a natural action of G on $\mathrm{Ext}^k_\Lambda(M, N)$. If $x \in \mathrm{Ext}^k_\Lambda(M, N)$ and $y \in \mathrm{Ext}^s_\Lambda(N, L)$, then $g(y.x) = (gy).(gx)$.*

Proof. i) Let g_1, g_2 be elements of G and $f \in \mathrm{Hom}_\Lambda(M, N)$. We have identities: $(g_1.g_2) * f(m) = g_1 g_2 f(g_2^{-1} g_1^{-1} m) = g_1(g_2 f(g_2^{-1}))(g_1^{-1} m) = g_1 * (g_2 * f(m))$.

ii) If f is an element of $\mathrm{Hom}_\Lambda(M, N)^G$, then $g * f = f$ for all $g \in G$, in particular: $g^{-1} * f = f$. It follows $gf(m) = f(gm)$ and $f \in \mathrm{Hom}_{\Lambda * G}(M, N)$. If f is in $\mathrm{Hom}_{\Lambda * G}(M, N)$, then $g^{-1} f(m) = f(g^{-1} m)$. Therefore $f(m) = gf(g^{-1} m)$, this is $g * f = f$ for all $g \in G$, hence; $f \in \mathrm{Hom}_\Lambda(M, N)^G$.

iii) Let g be an element of G and M a Λ-module. Define the Λ-module $M^{g^{-1}}$ as follows: $M^{g^{-1}} = M$ as K-vector space and for $\lambda \in \Lambda$ and $m \in M$ we have $\lambda * m = \lambda^{g^{-1}} m$.

If M is a G-module, then we have an isomorphism $\phi_{g^{-1}} : M \to M^{g^{-1}}$ given by $\phi_{g^{-1}}(m) = g^{-1}m$. Then $\phi_{g^{-1}}(\lambda m) = g^{-1}\lambda m = \lambda^{g^{-1}} g^{-1} m = \lambda * \phi_{g^{-1}}(m)$.

If M and N are G-modules and $f : M \to N$ is a Λ-map, then we have the following commutative diagram:

$$\begin{tikzcd} M^{g^{-1}} \arrow[r, "f^{g^{-1}}"] & N^{g^{-1}} \\ M \arrow[u, "\phi_{g^{-1}}"'] \arrow[r, "g*f"] & N, \arrow[u, "\phi_{g^{-1}}"'] \end{tikzcd}$$

where $f^{g^{-1}}(x) = f(x)$ and $f^{g^{-1}}(\lambda * x) = f(\lambda^{g^{-1}} x) = \lambda^{g^{-1}} f(x) = \lambda * f^{g^{-1}}(x)$. Then $\phi_g f^{g^{-1}} \phi_{g^{-1}}(m) = gf(g^{-1}m) = g * f(m)$.

Let $x \in \operatorname{Ext}^k_\Lambda(M, N)$ be the extension:

$$0 \to N \xrightarrow{j} E_k \xrightarrow{f_k} E_{k-1} \to \cdots \to E_1 \xrightarrow{f_1} M \to 0.$$

Define $g.x$ as the extension:

$$0 \to N \xrightarrow{j^{g^{-1}}\phi_{g^{-1}}} E_k^{g^{-1}} \xrightarrow{f_k^{g^{-1}}} E_{k-1}^{g^{-1}} \to \cdots \to E_1^{g^{-1}} \xrightarrow{\phi_g f_1^{g^{-1}}} M \to 0.$$

Since $(\quad)^g$ is an exact functor we have: $x \sim y$ if and only if $gx \sim gy$. We have the following commutative diagram:

$$\begin{tikzcd} N \arrow[r, "\phi_{h^{-1}}"] \arrow[dr, "\phi_{h^{-1}g^{-1}}"'] & N^{h^{-1}} \arrow[d, "\phi_{g^{-1}}"] \\ & N^{h^{-1}g^{-1}}. \end{tikzcd}$$

It follows $(hg)(x) = h(gx)$. Hence; G acts on $\operatorname{Ext}^k_\Lambda(M, N)$. It is clear that if $x \in \operatorname{Ext}^k_\Lambda(M, L)$ and $y \in \operatorname{Ext}^s_\Lambda(L, N)$, then $g(yx) = (gy)(gx)$. □

Corollary 5. *Let Λ be a positively graded K-algebra, G a finite grading preserving group of automorphisms of Λ such that the characteristic of K does not divide the order of the group G. Let $0 \to L \xrightarrow{f} M \xrightarrow{\pi} N \to 0$ be an exact sequence of G-modules and G-maps. Then for any G-module X the* *long exact sequences*:

$$
\begin{aligned}
&0 \to \mathrm{Hom}_\Lambda(X,L) \to \mathrm{Hom}_\Lambda(X,M) \to \mathrm{Hom}_\Lambda(X,N)\\
&\quad \to \mathrm{Ext}^1_\Lambda(X,L) \to \mathrm{Ext}^1_\Lambda(X,M) \to \cdots \to \mathrm{Ext}^k_\Lambda(X,M) \to \cdots\\
&0 \to \mathrm{Hom}_\Lambda(N,X) \to \mathrm{Hom}_\Lambda(M,X) \to \mathrm{Hom}_\Lambda(L,X)\\
&\quad \to \mathrm{Ext}^1_\Lambda(N,X) \to \mathrm{Ext}^1_\Lambda(M,X) \to \cdots \to \mathrm{Ext}^k_\Lambda(M,X) \to \cdots
\end{aligned}
$$

are exact sequences of G-modules and G-maps.

Proof. Let $0 \to L \xrightarrow{j} E_k \xrightarrow{t_k} E_{k-1} \to \cdots \to E_1 \xrightarrow{t_1} X \to 0$ be an exact sequence and $f : L \to M$ be a G-map. We have an induced exact sequence y obtained from the commutative diagram:

$$
\begin{tikzcd}
x: & 0 \arrow[r] & L \arrow[r, "j"] \arrow[d, "f"'] & E_k \arrow[r, "t_k"] \arrow[d] & E_{k-1} \to \cdots \to E_1 \arrow[r, "t_1"] & X \arrow[r] \arrow[d, "1"'] & 0\\
y: & 0 \arrow[r] & M \arrow[r, "i"] & F_k \arrow[r, "s_k"] & F_{k-1} \to \cdots \to F_1 \arrow[r, "s_1"] & X \arrow[r] & 0.
\end{tikzcd}
$$

Applying $(\quad)^{g^{-1}}$ and composing with the natural isomorphisms we have a commutative exact diagram:

$$
\begin{tikzcd}
gx: & 0 \arrow[r] & L \arrow[r, "j\phi_{g^{-1}}"] \arrow[d, "g*f"'] & E_k^{g^{-1}} \arrow[r] \arrow[d] & E_{k-1}^{g^{-1}} \to \cdots \to E_1^{g^{-1}} \arrow[r, "\phi_g t_1"] & X \arrow[r] \arrow[d, "1"'] & 0\\
gy: & 0 \arrow[r] & M \arrow[r] & F_k^{g^{-1}} \arrow[r] & F_{k-1}^{g^{-1}} \to \cdots \to F_1^{g^{-1}} \arrow[r, "\phi_g s_1"] & X \arrow[r] & 0.
\end{tikzcd}
$$

Then $\mathrm{Ext}^k_\Lambda(f,M)(x) = y$. We have $g\,\mathrm{Ext}^k_\Lambda(f,M)(x) = gy = \mathrm{Ext}^k_\Lambda(g*f,M)(gx)$. Since $g*f = f$, then $\mathrm{Ext}^k_\Lambda(f,M)$ is a G-map.

Let $\delta : \mathrm{Ext}^k_\Lambda(X,N) \to \mathrm{Ext}^{k+1}_\Lambda(X,L)$ be the connecting map and $x \in \mathrm{Ext}^k_\Lambda(X,N)$, where $x : 0 \to N \to E_k \to E_{k-1} \to \cdots \to E_1 \to X \to 0$ and z is the exact sequence:

$$0 \to L \xrightarrow{f} M \xrightarrow{\pi} N \to 0.$$

We have the following commutative exact diagram:

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\phi_{g^{-1}}"'] & M \arrow[r, "\pi"] \arrow[d, "\phi_{g^{-1}}"'] & N \arrow[r] \arrow[d, "\phi_{g^{-1}}"'] & 0\\
0 \arrow[r] & L^{g^{-1}} \arrow[r, "f"] & M^{g^{-1}} \arrow[r, "\pi"] & N^{g^{-1}} \arrow[r] & 0
\end{tikzcd}
$$

with $\phi_{g^{-1}}$ isomorphisms. This implies $gz = z$. Then $\delta(gx) = zgx = gzgx = g(zx) = g\delta(x)$. Hence; δ is a G-map.

The proof for the second long exact sequence is by dual arguments. □

Lemma 6. *Let Λ be a positively graded K-algebra, G a finite grading preserving group of automorphisms of Λ such that the characteristic of K does not divide the order of the group G. Let X be a finitely generated graded $\Lambda * G$-module. Then X is projective if and only if X is projective as Λ-module.*

Proof. Assume the module X is projective as $\Lambda*G$-module, this implies there exists a graded $\Lambda * G$-module Q such that $X \bigoplus Q \cong (\Lambda * G)^n \cong (\bigoplus_{|G|} \Lambda)^n$. Therefore: X is a projective Λ-module.

Assume X is projective as Λ-module. Let $0 \to A \to B \to C \to 0$ be an exact sequence of $\Lambda * G$-modules. Then the sequence:

$$0 \to \mathrm{Hom}_\Lambda(X, A) \to \mathrm{Hom}_\Lambda(X, B) \to \mathrm{Hom}_\Lambda(X, C) \to 0$$

is an exact sequence of G-modules. Applying the fixed point functor we have an exact sequence:

$$0 \to \mathrm{Hom}_\Lambda(X, A)^G \to \mathrm{Hom}_\Lambda(X, B)^G \to \mathrm{Hom}_\Lambda(X, C)^G \to 0,$$

which is isomorphic to the sequence:

$$0 \to \mathrm{Hom}_{\Lambda*G}(X, A) \to \mathrm{Hom}_{\Lambda*G}(X, B) \to \mathrm{Hom}_{\Lambda*G}(X, C) \to 0.$$

It follows X is projective. □

Corollary 7. *If Λ and G are as in the lemma, then Λ is a projective $\Lambda * G$-module.*

Lemma 8. *Let G be a finite grading preserving group of automorphisms of the K-algebra Λ and assume the characteristic of Λ does not divide the order of the group G. Let P be a graded finitely generated projective $\Lambda * G$-module and N an arbitrary graded $\Lambda * G$-module. Then we have a natural isomorphism:* $\theta : \mathrm{Hom}_\Lambda(P, N) \bigotimes_K W \to \mathrm{Hom}_{\Lambda*G}(P \bigotimes_K KG, N \bigotimes_K W)$ *given by* $\theta(f \otimes w)(p \otimes g) = g * f(p) \otimes gw$.

Proof. We have a natural isomorphism of K-vector spaces:

$$\psi : \mathrm{Hom}_K(KG, \mathrm{Hom}_\Lambda(P, N \bigotimes_K W)) \to \mathrm{Hom}_\Lambda(P \bigotimes_K KG, N \bigotimes_K W)$$

given by $\psi(\gamma)(p \otimes g) = \gamma(g)(p)$. The map ψ is a G-map. We have equalities: $\psi(h * \gamma)(p \otimes g) = h * \gamma(g)(p)$. But $h * \gamma(g) = (h.\gamma)(h^{-1}g)$. Then,

$$\begin{aligned}[(h.\gamma)(h^{-1}g)](p) &= h(h^{-1}g)(h^{-1}p) = h(\psi(\gamma)(h^{-1}p \otimes h^{-1}g) \\ &= h(\psi(\gamma))(h^{-1}(p \otimes g) = (h * \psi(\gamma)(p \otimes g) \\ &= \psi(h \otimes \gamma)(p \otimes g).\end{aligned}$$

Hence $\psi(h*\gamma)=h*\psi(\gamma)$. It follows ψ induces an isomorphisms:

$$\psi:\mathrm{Hom}_K(KG,\mathrm{Hom}_\Lambda(P,N\bigotimes_K W))^G\to\mathrm{Hom}_\Lambda(P\bigotimes_K KG,N\bigotimes_K W)^G$$

Hence an isomorphism:

$$\psi:\mathrm{Hom}_{KG}(KG,\mathrm{Hom}_\Lambda(P,N\bigotimes_K W))\to\mathrm{Hom}_{\Lambda*G}(P\bigotimes_K KG,N\bigotimes_K W).$$

Consider the natural isomorphisms:

$$\sigma_1:\mathrm{Hom}_\Lambda(P,N)\bigotimes_K W\to\mathrm{Hom}_\Lambda(P,\Lambda)\bigotimes_\Lambda N\bigotimes_K W,$$
$$\sigma_2:\mathrm{Hom}_\Lambda(P,\Lambda)\bigotimes_\Lambda N\bigotimes_K W\to\mathrm{Hom}_\Lambda(P,N\bigotimes_K W),$$

where $f\in\mathrm{Hom}_\Lambda(P,N)$ we have the equality $f(p)=\sum f_i(p)n_i$ with $f_i\in\mathrm{Hom}_\Lambda(P,\Lambda)$ and $n_i\in N$. Then, $\sigma_1(f\otimes w)=\sum f_i\otimes n_i\otimes w$ and $\sigma_2(\sum f_i\otimes n_i\otimes w)(p)=\sum f_i(p)n_i\otimes w=(\sum f_i(p)n_i)\otimes w=f(p)\otimes w$. Hence; $\sigma(f\otimes w)(p)=\sigma_2\sigma_1(f\otimes w)(p)=f(p)\otimes w$. The map $\alpha:\mathrm{Hom}_\Lambda(P,N\bigotimes_K W)\to\mathrm{Hom}_{KG}(KG,\mathrm{Hom}_\Lambda(P,N\bigotimes_K W))$ is the isomorphism $\alpha(f)(h)=h*f$. The natural isomorphism θ is the composition $\psi\alpha\sigma$. Then we have a chain of equalities:

$$\begin{aligned}\theta(h\otimes w)(p\otimes g)&=\psi\alpha\sigma(h\otimes w)(p\otimes g)=\psi(\alpha\sigma(h\otimes w))(p\otimes g)\\&=\alpha\sigma(h\otimes w)(g)(p)=g*\sigma(h\otimes w)(g)(p)\\&=g\sigma(h\otimes w)(g^{-1}p)=g[h(g^{-1}p)\otimes w]\\&=ghg^{-1}p\otimes hw=g*h(p)\otimes gw.\end{aligned}$$

□

Proposition 9. *Let G be a finite group of grading preserving automorphisms of the K-algebra Λ such that the characteristic of K does not divide the order of G. Let M be a graded $\Lambda*G$-module with a graded projective resolution consisting of finitely generated modules, N a graded $\Lambda*G$-module and W a KG-module. Then for all $k\geq 0$ we have a natural isomorphism*:

$$\hat{\theta}:\mathrm{Ext}^k_\Lambda(M,N)\bigotimes_K W\to\mathrm{Ext}^k_{\Lambda*G}(M\bigotimes_K KG,N\bigotimes_K W).$$

Proof. Let $\cdots\to P_k\xrightarrow{f_k}P_{k-1}\to\cdots\to P_0\xrightarrow{f_0}M\to 0$ be a $\Lambda*G$-projective resolution of M. By lemma 6, each P_j is a finitely generated projective Λ-module. Tensoring with KG we have an exact sequence of $\Lambda*G$-modules:

$$\cdots\to P_k\bigotimes_K KG\xrightarrow{f_k\otimes 1}P_{k-1}\bigotimes_K KG\to\cdots\to P_0\bigotimes_K KG\xrightarrow{f_0\otimes 1}M\bigotimes_K KG\to 0.$$

Each $P_k \bigotimes\limits_K KG$ is isomorphic to $\bigoplus\limits_{|G|} P_k$ as Λ-module, hence projective as Λ-module. By lemma 6, $P_k \bigotimes\limits_K KG$ is a finitely generated graded projective $\Lambda * G$-module. We have a complex:

$$0 \to \mathrm{Hom}_\Lambda(P_0, N) \to \cdots \to \mathrm{Hom}_\Lambda(P_k, N) \to \mathrm{Hom}_\Lambda(P_{k+1}, N) \to \cdots .$$

Tensoring with W we obtain a complex:

$$0 \to \mathrm{Hom}_\Lambda(P_0, N) \bigotimes_K W \to \\ \cdots \to \mathrm{Hom}_\Lambda(P_k, N) \bigotimes_K W \to \mathrm{Hom}_\Lambda(P_{k+1}, N) \bigotimes_K W \to \cdots ,$$

whose k-th homology is $\mathrm{Ext}^k_\Lambda(M, N) \bigotimes\limits_K W$. By the lemma, we have an isomorphism of complexes:

$$\begin{array}{c} 0 \longrightarrow \mathrm{Hom}_\Lambda(P_0, N) \bigotimes\limits_K W \longrightarrow \\ \Big\downarrow \theta \\ 0 \to \mathrm{Hom}_\Lambda(P_0 \bigotimes\limits_K KG, N \bigotimes\limits_K W) \to \end{array}$$

$$\begin{array}{ccc} \cdots \longrightarrow \mathrm{Hom}_\Lambda(P_k, N) \bigotimes\limits_K W & \longrightarrow & \mathrm{Hom}_\Lambda(P_{k+1}, N) \bigotimes\limits_K W \longrightarrow \cdots \\ \Big\downarrow \theta & & \Big\downarrow \theta \\ \cdots \to \mathrm{Hom}_\Lambda(P_k \bigotimes\limits_K KG, N \bigotimes\limits_K W) & \longrightarrow & \mathrm{Hom}_\Lambda(P_{k+1} \bigotimes\limits_K KG, N \bigotimes\limits_K KW) \to \cdots , \end{array}$$

which induces an isomorphism $\hat{\theta}$ of the homologies, therefore an isomorphism:

$$\hat{\theta} : \mathrm{Ext}^k_\Lambda(M, N) \bigotimes_K W \to \mathrm{Ext}^k_{\Lambda * G}(M \bigotimes_K KG, N \bigotimes_K W). \qquad \square$$

Theorem 10. *Let Λ be a positively graded K-algebra, G a finite grading preserving group of automorphisms of Λ with characteristic of K do not dividing the order of G. Let M be a graded $\Lambda * G$-module with a graded projective resolution consisting of finitely generated modules $\Gamma = \bigoplus\limits_{k \geq 0} \mathrm{Ext}^k_\Lambda(M, M)$. Then the skew group algebra $\Gamma * G$ is isomorphic as graded algebra to $\hat{\Gamma} = \bigoplus\limits_{k \geq 0} \mathrm{Ext}^k_{\Lambda * G}(M \bigotimes\limits_K KG, M \bigotimes\limits_K KG)$.*

Proof. We need to prove isomorphism $\hat{\theta}$ preserves multiplication.

Let $x \in \mathrm{Ext}^k_\Lambda(M, N)$ and $y \in \mathrm{Ext}^s_\Lambda(M, N)$:

$$x = 0 \to M \to E_k \to E_{k-1} \to \cdots \to E_1 \to M \to 0,$$
$$y = 0 \to M \to F_s \to F_{s-1} \to \cdots \to F_1 \to M \to 0,$$

where the multiplication $x * y$ is the Yoneda product:

$$\begin{aligned} x * y = y.x = 0 \to M \to F_s \to F_{s-1} \to \cdots \to F_1 \\ \to E_k \to E_{k-1} \to \cdots \to E_1 \to M \to 0. \end{aligned}$$

Let $\cdots \to P_t \xrightarrow{\alpha_t} P_{t-1} \to \cdots \to P_0 \xrightarrow{\alpha_0} M \to 0$ be a $\Lambda * G$-projective resolution of M and $\Omega^t(M) = \operatorname{Ker}\alpha_t$.

To the extension x corresponds a map $f : \Omega^k(M) \to M$, to the extension y a map $h : \Omega^s(M) \to M$ and to the Yoneda product $y.x$ corresponds the composition $h.\Omega^k(f)$. We have the following commutative exact diagrams:

$$\begin{tikzcd} 0 \arrow[r] & \Omega^k(M) \arrow[r] \arrow[d, "f"'] & P_{k-1} \arrow[r] \arrow[d, "f_k"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "1"'] & 0 \\ 0 \arrow[r] & M \arrow[r] & E_k \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \Omega^{k-s}(M) \arrow[r] \arrow[d, "\Omega^s(f)"'] & P_{k+s-1} \arrow[r] \arrow[d, "f_k"'] & \cdots \arrow[r] & P_k \arrow[r] \arrow[d] & \Omega^k(M) \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & \Omega^s(M) \arrow[r] \arrow[d, "h"'] & P_{s-1} \arrow[r] \arrow[d, "h_s"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "h_1"'] & M \arrow[r] \arrow[d, "1"'] & 0 \\ 0 \arrow[r] & M \arrow[r] & F_s \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

We need to prove the following diagram:

$$\begin{array}{ccc} \operatorname{Ext}^k_\Lambda(M,M)\bigotimes\limits_K KG & \times & \operatorname{Ext}^s_\Lambda(M,M)\bigotimes\limits_K KG \\ \hat\theta\downarrow & & \hat\theta\downarrow \\ \operatorname{Ext}^k_{\Lambda*G}(M\bigotimes\limits_K KG, M\bigotimes\limits_K KG) & \times & \operatorname{Ext}^s_{\Lambda*G}(M\bigotimes\limits_K KG, M\bigotimes\limits_K KG) \\ & & \xrightarrow{\nu} \operatorname{Ext}^{k+s}_\Lambda(M,M)\bigotimes\limits_K KG \\ & & \hat\theta\downarrow \\ & & \xrightarrow{\mu} \operatorname{Ext}^{k+s}_{\Lambda*G}(M\bigotimes\limits_K KG, M\bigotimes\limits_K KG) \end{array}$$

with $\nu(x \otimes g, y \otimes t) = (x.g)(y.t)$ and μ the Yoneda product, commutes.

We have the following equalities:

$$(x.g)(y.t) = x.(gy.t) = x * .gy \otimes gt = gy.x \otimes gt$$

and the following correspondences under the natural isomorphisms:

$$x \otimes g \to f \otimes g, \quad y \otimes t \to h \otimes t.$$

Then the following correspondences: $\hat{\theta}(x\otimes g) \to \theta(f\otimes g)$, $\hat{\theta}(y\otimes t) \to \theta(h\otimes t)$. The maps $\theta(f\otimes g)$, $\theta(h\otimes t)$ induce exact commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^s(M)\bigotimes KG \arrow[r] \arrow[d, "\theta(h\otimes t)"'] & P_{s-1}\bigotimes KG \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0\bigotimes KG \arrow[r] \arrow[d] & M\bigotimes KG \arrow[r] \arrow[d, "1"'] & 0 \\
0 \arrow[r] & M\bigotimes KG \arrow[r] & \hat{F}_s \arrow[r] & \cdots \arrow[r] & \hat{F}_1 \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{k+s}(M)\bigotimes KG \arrow[r] \arrow[d, "\Omega^s\theta(f\otimes g)"'] & P_{k+s-1}\bigotimes KG \arrow[r] \arrow[d] & \cdots \arrow[r] & P_k\bigotimes KG \arrow[r] \arrow[d] & \Omega^k(M)\bigotimes KG \arrow[r] \arrow[d, "\theta(f\otimes g)"'] & 0 \\
0 \arrow[r] & \Omega^s M\bigotimes KG \arrow[r] & P_s\bigotimes KG \arrow[r] & \cdots \arrow[r] & P_0\bigotimes KG \arrow[r] & M\bigotimes KG \arrow[r] & 0,
\end{tikzcd}$$

where the bottom rows are $\hat{\theta}(y\otimes t)$ and $\hat{\theta}(x\otimes g)$. We have the following correspondence:

$$\hat{\theta}(y\otimes t)\hat{\theta}(x\otimes g) = \hat{\theta}(x\otimes g) * \hat{\theta}(y\otimes t) \to \theta(h\otimes t)\Omega^s\theta(f\otimes g).$$

We claim $\Omega^s\theta(f\otimes g) = \theta(\Omega^s f\otimes g)$. We have commutative squares:

$$\begin{tikzcd}
P_{k+s-i} \arrow[r, "\alpha_{k+s-i}"] \arrow[d, "f_{k+s-i}"'] & P_{k+s-i-1} \arrow[d, "f_{k+s-i-1}"'] \\
P_{s-i} \arrow[r, "\alpha_{s-i}"] & P_{s-i-1}.
\end{tikzcd}$$

It is enough to prove the following square commute:

$$\begin{tikzcd}
P_{k+s-i}\bigotimes KG \arrow[r, "\alpha_{k+s-i}\otimes 1"] \arrow[d, "\theta(f_{k+s-i}\otimes g)"'] & P_{k+s-i-1}\bigotimes KG \arrow[d, "\theta(f_{k+s-i-1}\otimes g)"'] \\
P_{s-i}\bigotimes KG \arrow[r, "\alpha_{s-i}\otimes 1"] & P_{s-i-1}\bigotimes KG.
\end{tikzcd}$$

But we have the following chain of equalities:

$$\begin{aligned}
&\theta(f_{k+s-i}\otimes g)\alpha_{k+s-i}\otimes 1(p\otimes t) = \theta(f_{k+s-i}\otimes g)(\alpha_{k+s-i}(p)\otimes t) \\
&= t * f_{k+s-i-1}(\alpha_{k+s-i}(p)\otimes gt) = tf_{k+s-i-1}(t^{-1}\alpha_{k+s-i}(p)\otimes gt \\
&= tf_{k+s-i-1}(\alpha_{k+s-i}(t^{-1}p)\otimes gt = t\alpha_{s-i}f_{k+s-i}(t^{-1}p)\otimes gt \\
&= \alpha_{s-i}(tf_{k+s-i}(t^{-1}p)\otimes gt = \alpha_{s-i}t * f_{k+s-i}(p)\otimes gt \\
&= (\alpha_{s-i}\otimes 1)(t * f_{k+s-i}(p)\otimes gt) = (\alpha_{s-i}\otimes 1)(\theta(f_{k+s-i}\otimes g)(p\otimes t).
\end{aligned}$$

We also have equalities:

$$\begin{aligned}
&\theta(h \otimes t)\Omega^s\theta(f \otimes g) = \theta(h \otimes t)\theta(\Omega^s f \otimes g)(m \otimes l)\\
&= \theta(h \otimes t)(l * \Omega^s f(m) \otimes lg) = lg \otimes h(l * \Omega^s f(m) \otimes lgt)\\
&= lg \otimes h(l\Omega^s f(l^{-1}m) \otimes lgt) = lgh(g^{-1}l^{-1}l\Omega^s f(l^{-1}m) \otimes lgt)\\
&= l(gh(g^{-1}\Omega^s f(l^{-1}m) \otimes lgt) = l((g * h)\Omega^s f(m) \otimes lgt)\\
&= \theta(g * h\Omega^s f \otimes gt)(m \otimes l).
\end{aligned}$$

We have the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Omega^s M & \longrightarrow & P_{s-1} & \to \cdots \longrightarrow & P_k & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \varphi_h^{-1}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi^{g^{-1}}} & & \\
0 & \longrightarrow & \Omega^s M^{g^{-1}} & \longrightarrow & P_{s-1}^{g-1} & \to \cdots \to & P_k^{g-1} & \longrightarrow & M^{g^{-1}} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & M^{g^{-1}} & \longrightarrow & F_s^{g-1} & \to \cdots \to & F_1^{g-1} & \longrightarrow & M^{g^{-1}} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \varphi_g} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi^g} & & \\
0 & \longrightarrow & M & \longrightarrow & F_s^{g-1} & \to \cdots \to & F_1^{g-1} & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

where the last raw is y^g. Since $\varphi_g h \varphi_{g^{-1}} = g * h$, then we have a correspondence $g * h \mapsto y^g$, and $\hat{\theta}(x \otimes g) * \hat{\theta}(y \otimes t) = \hat{\theta}(x * y^g, gt)$. □

Recall the following definitions from [7] and the references given there:

Definition 11. A positively graded K-algebra $\Lambda = \bigoplus_{j \geq 0} \Lambda_j$ such that each Λ_j is finite dimensional over the field K and for each pair of integers $i \neq j$ we have equalities $\Lambda_i\Lambda_j = \Lambda_{i+j}$, will be called a graded quiver algebra. By $\overline{J}$ we denote the graded Jacobson radical $\overline{J} = \bigoplus_{j \geq 1} \Lambda_j$.

Let M be a finitely generated graded Λ-module, generated in highest degree zero, we say M is a Koszul module if $F(M) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_\Lambda(M, \Lambda/\overline{J})$ is generated in highest degree zero. We say Λ is Koszul if all graded simples generated in degree zero are Koszul.

Definition 12. Let Λ be a graded quiver algebra, we say Λ is generalized Auslander regular if the following statements are true:

i) The algebra Λ has finite, graded, small, global dimension n.
ii) All graded simples have projective dimension n.
iii) For any graded simple S, $\mathrm{Ext}^k_\Lambda(S, \Lambda) = 0$ for $0 \leq k < n$.
iv) The functor $\mathrm{Ext}^n_\Lambda(-, \Lambda)$ induces a bijection between the Λ and the Λ^{op}-graded simples.

Lemma 13. *Let Λ be a positively graded K-algebra, G a finite group of grading preserving automorphisms of Λ with characteristic of K not dividing the order of G. Then $\Lambda * G$ is generalized Auslander regular if and only if Λ is generalized Auslander regular.*

Proof. Assume Λ is generalized Auslander regular, M a $\Lambda * G$-module semisimple. Then $\mathrm{Ext}^k_{\Lambda*G}(M \bigotimes_K KG, \Lambda \bigotimes_K KG) \cong \mathrm{Ext}^k_{\Lambda}(M, \Lambda) \bigotimes_K KG$ for all $k \neq n$. Then by hypothesis, $\mathrm{Ext}^n_{\Lambda}(M, \Lambda)$ is a semisimple $\Lambda * G$-module.

Assume $KG \cong \bigoplus_{i=1}^{n} V_i$, V_i an irreducible KG-module. Then

$$\mathrm{Ext}^n_{\Lambda*G}(M \bigotimes_K KG, \Lambda * G) = \bigoplus_{i=1}^{m} \mathrm{Ext}^n_{\Lambda*G}(M \bigotimes_K V_i, \Lambda * G)$$

and each $\mathrm{Ext}^n_{\Lambda*G}(M \bigotimes_K V_i, \Lambda * G)$ is a semisimple $\Lambda * G$-module.

Let T be a simple $\Lambda * G$-module with minimal projective resolution:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to T \to 0.$$

Dualizing by $(\quad)^* = \mathrm{Hom}_{\Lambda*G}(-, \Lambda * G)$ we have a complex:

$$(*) \qquad 0 \to P_0^* \to P_1^* \to \cdots \to P_n^* \to \mathrm{Ext}^n_{\Lambda*G}(T, \Lambda * G) \to 0$$

and $\mathrm{Ext}^k_{\Lambda*G}(T \bigotimes_K KG, \Lambda * G) = \mathrm{Ext}^k_{\Lambda}(T, \Lambda) \bigotimes KG = \bigoplus_{i=1}^{m} \mathrm{Ext}^k_{\Lambda*G}(T \bigotimes_K V_i, \Lambda * G) = 0$, imply $\mathrm{Ext}^k_{\Lambda*G}(T \bigotimes_K K, \Lambda * G) = 0$ for all $k \neq n$ and the sequence $(*)$ is exact. Let $S = \mathrm{Ext}^n_{\Lambda*G}(T, \Lambda * G)$. Dualizing, we have an exact diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \to \cdots \to & P_0 & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_n^{**} & \longrightarrow & P_{n-1}^{**} & \to \cdots \to & P_0^{**} & \longrightarrow & T & \longrightarrow & 0.
\end{array}$$

Therefore: $\mathrm{Ext}^k_{\Lambda*G}(S, \Lambda * G) = 0$ for all $k \neq n$. If $S_1 \bigoplus S_2 = S$, with S_1, S_2 non zero semisimple $\Lambda * G$-modules, then $\mathrm{Ext}^k_{\Lambda*G}(S_i, \Lambda * G) = 0$ for $i = 1, 2$ and all $k \neq n$. If $\mathrm{Ext}^n_{\Lambda*G}(S_i, \Lambda * G) = 0$, then $S_i = 0$, a contradiction. Therefore: $T \cong \mathrm{Ext}^n_{\Lambda*G}(S_1, \Lambda * G) \bigoplus \mathrm{Ext}^n_{\Lambda*G}(S_2, \Lambda * G)$, contradicting T is simple.

Now if T_1, T_2 are simple $\Lambda * G$-modules with $\mathrm{Ext}^n_{\Lambda*G}(T_1, \Lambda * G) \cong \mathrm{Ext}^n_{\Lambda*G}(T_2, \Lambda * G)$ and projective resolutions:

$$0 \to P'_n \to P'_{n-1} \to \cdots \to P'_0 \to T_1 \to 0,$$
$$0 \to P''_n \to P''_{n-1} \to \cdots \to P''_0 \to T_2 \to 0.$$

Dualizing, we have exact sequences:

$$0 \to (P_0')^* \to (P_1')^* \to \cdots \to (P_n')^* \to \operatorname{Ext}^n_{\Lambda * G}(T_1, \Lambda * G) \to 0,$$
$$0 \to (P_0'')^* \to (P_1'')^* \to \cdots \to (P_n'')^* \to \operatorname{Ext}^n_{\Lambda * G}(T_2, \Lambda * G) \to 0.$$

From the fact $\operatorname{Ext}^n_{\Lambda*G}(T_1, \Lambda*G) \cong \operatorname{Ext}^n_{\Lambda*G}(T_2, \Lambda*G)$ we have isomorphisms: $(P_j')^* \cong (P_j'')^*$, in particular $(P_0')^* \cong (P_0'')^*$. Then $P_0' \cong P_0''$ and $T_1 \cong T_2$.

Now assume $\Lambda * G$ is generalized Auslander regular, $S \subseteq \Lambda_0$ a simple Λ-module. The module T defined as $T = \sum\limits_{g\in G} gS$ is a simple $\Lambda * G$-module. Then $\operatorname{Ext}^k_{\Lambda*G}(T \bigotimes\limits_K KG, \Lambda * G) \cong \operatorname{Ext}^k_{\Lambda*G}(T, \Lambda) \bigotimes\limits_K KG$. Decomposing $KG = \bigoplus\limits_{i=1}^m V_i$ where each V_i is irreducible and using the fact $\operatorname{Ext}^k_{\Lambda}(T, \Lambda) \bigotimes KG = \bigoplus\limits_{i=1}^m \operatorname{Ext}^k_{\Lambda*G}(T \bigotimes\limits_K V_i, \Lambda * G)$, we obtain $\operatorname{Ext}^k_{\Lambda*G}(T \bigotimes\limits_K V_i, \Lambda * G) = 0$ if $k \neq n$. Therefore: $\operatorname{Ext}^k_{\Lambda}(T, \Lambda) = 0$ for all $k \neq n$.

Since $\operatorname{Ext}^n_{\Lambda}(T, \Lambda) \bigotimes KG = \bigoplus\limits_{i=1}^m \operatorname{Ext}^n_{\Lambda*G}(T \bigotimes\limits_K V_i, \Lambda * G)$, where $\operatorname{Ext}^n_{\Lambda*G}(T \bigotimes\limits_K V_i, \Lambda * G)$ is a semisimple $\Lambda * G$-module, then $\operatorname{Ext}^n_{\Lambda}(T, \Lambda) \bigotimes KG$ is a semisimple Λ-module. The module S is a submodule of the semisimple Λ-module T, hence; a summand. It follows $\operatorname{Ext}^n_{\Lambda}(S, \Lambda) \subseteq \operatorname{Ext}^n_{\Lambda}(T, \Lambda)$, hence $\operatorname{Ext}^n_{\Lambda}(S, \Lambda)$ is a semisimple Λ-module.

Let $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to T \to 0$ be a minimal projective resolution of the Λ-module S, dualizing with respect to Λ we have an exact sequence:

$$0 \to (P_0)^* \to (P_1)^* \to \cdots \to (P_n)^* \to \operatorname{Ext}^n_{\Lambda}(S, \Lambda) \to 0.$$

Dualizing again we have isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_n^{**} \arrow[r] & P_{n-1}^{**} \arrow[r] & \cdots \arrow[r] & P_0^{**} \arrow[r] & \operatorname{Ext}^n_{\Lambda}(\operatorname{Ext}^n_{\Lambda}(S, \Lambda), \Lambda) \arrow[r] & 0.
\end{tikzcd}$$

It follows $S \cong \operatorname{Ext}^n_{\Lambda}(\operatorname{Ext}^n_{\Lambda}(S, \Lambda), \Lambda)$. If S is simple, then $\operatorname{Ext}^n_{\Lambda}(S, \Lambda) \cong S_1 \bigoplus S_2$, $S_1 \neq 0 \neq S_2$. This implies $S \cong \operatorname{Ext}^n_{\Lambda}(S_1, \Lambda) \bigoplus \operatorname{Ext}^n_{\Lambda}(S_2, \Lambda)$. If $\operatorname{Ext}^n_{\Lambda}(S_i, \Lambda) = 0$, then $\operatorname{Ext}^k_{\Lambda}(S_i, \Lambda) = 0$ for all i, a contradiction. It follows $\operatorname{Ext}^n_{\Lambda}(S, \Lambda)$ is simple. □

Theorem 14. *Let G be a finite group of automorphisms of a graded K-algebra Λ. Assume characteristic of the field K does not divide the order of the group, let M be a finitely generated graded $\Lambda*G$-module. Then $M \bigotimes\limits_K KG$*

is a Koszul $\Lambda{*}G$-module if and only if M is a Koszul Λ-module. In particular Λ is Koszul if and only if $\Lambda * G$ is Koszul.

Proof. We have a natural isomorphisms:

$$\operatorname{Ext}^k_{\Lambda*G}(M \bigotimes_K KG, \Lambda * G) \cong \operatorname{Ext}^k_{\Lambda*G}(M, \Lambda) \bigotimes_K KG$$

and

$$\begin{aligned}\operatorname{Ext}^k_{\Lambda*G}(M \bigotimes_K KG, \Lambda_0 * G)\, \operatorname{Ext}^j_{\Lambda*G}(\Lambda_0 * G, \Lambda_0 * G) \\ \cong (\operatorname{Ext}^k_{\Lambda}(M, \Lambda_0) \bigotimes_K KG)(\operatorname{Ext}^j_{\Lambda}(\Lambda_0, \Lambda_0) \bigotimes_K KG) \\ \cong \operatorname{Ext}^k_{\Lambda}(M, \Lambda_0)\, \operatorname{Ext}^j_{\Lambda}(\Lambda_0, \Lambda_0) \bigotimes_K KG.\end{aligned}$$

Therefore, $\operatorname{Ext}^k_{\Lambda*G}(M \bigotimes_K KG, \Lambda_0 * G)\, \operatorname{Ext}^j_{\Lambda*G}(\Lambda_0 * G, \Lambda_0 * G) = \operatorname{Ext}^{k+j}_{\Lambda*G}(M \bigotimes_K KG, \Lambda_0 * G)$ if and only if $\operatorname{Ext}^k_{\Lambda}(M, \Lambda_0)\, \operatorname{Ext}^j_{\Lambda}(\Lambda_0, \Lambda_0) = \operatorname{Ext}^{k+j}_{\Lambda}(M, \Lambda_0)$. It follows M is Koszul if and only if $M \bigotimes_K KG$ is Koszul, in particular Λ is Koszul if and only if $\Lambda * G$ is Koszul. □

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