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STABLE MODULE THEORY WITH KERNELS

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1. Introduction

Auslander and Bridger introduced the notion of projective stabilization $\underline{\mathrm{mod}}\, R$ of a category of finite modules. The category $\underline{\mathrm{mod}}\, R$ is known to be non-abelian. But realistically, $\underline{\mathrm{mod}}\, R$ is almost abelian. It fails to be abelian because of the lack of kernel and cokernel. In fact, each morphism has a pseudo-kernel and a pseudo-cokernel (see §3). On the other hand, a pseudo-kernel of a monomorphism does not necessarily vanish. In this paper we focus on how $\underline{\mathrm{mod}}\, R$ is similar or dissimilar to an abelian category (§4). What is a monomorphism? Which object makes monomorphisms split? One reason for similarity is that $\underline{\mathrm{mod}}\, R$ is closely related to the homotopy category of complexes. We discuss the functor from $\underline{\mathrm{mod}}\, R$ to homotopy category (§2). The method we use already produced important results in representation theory on commutative rings [2], [5].

Throughout the paper, R is a commutative semiperfect ring, equivalently a finite direct sum of local rings; that is, each finite module has a projective cover (see [4] for semiperfect rings). The category of finitely generated R-modules is denoted by $\mathrm{mod}\, R$, and the category of finite projective R-modules is denoted by $\mathrm{proj}\, R$. For an abelian category $\mathcal{A}$, $\mathsf{K}(\mathcal{A})$ stands for the homotopy category of complexes where a complex is denoted as

$$F^\bullet : \cdots \to F^{n-1} \xrightarrow{{d_F}^{n-1}} F^n \xrightarrow{{d_F}^n} F^{n+1} \to \cdots .$$

A morphism in $\mathsf{K}(\mathcal{A})$ is a homotopy equivalence class of chain maps. A degree-shifting T is an autofunctor on $\mathsf{K}(\mathrm{mod}\, R)$;

$$(TF)^n = F^{n+1}, \quad {d_{TF}}^n = {d_F}^{n+1},$$

$\tau_{\leq n} F^\bullet$, $\tau_{\geq n} F^\bullet$ are truncations;

$$\tau_{\leq n} F^\bullet : \cdots \to F^{n-2} \to F^{n-1} \to F^n \to 0 \to 0 \to \cdots ,$$
$$\tau_{\geq n} F^\bullet : \cdots \to 0 \to 0 \to F^n \to F^{n+1} \to F^{n+2} \to \cdots ,$$

and $F^*_\bullet$ is the cocomplex such as $F^*_n = (F_n)^*$, ${d^{F^*}}_n = ({d_F}^{n-1})^*$ where * means $\mathrm{Hom}_R(\quad, R)$. The projective stabilization $\underline{\mathrm{mod}}\, R$ is defined as follows:

- Each object of $\underline{\mathrm{mod}}\, R$ is an object of $\mathrm{mod}\, R$.
- For $A, B \in \mathrm{mod}\, R$, a set of morphisms from A to B is

$$\mathrm{Hom}_R(A, B)/\mathcal{P}(A, B),$$

where $\mathcal{P}(A,B) := \{f \in \mathrm{Hom}_R(A,B) \mid f$ factors through some projective module$\}$. Each element is denoted as $\underline{f} = f \bmod \mathcal{P}(A,B)$. If $A, B \in \mathrm{mod}\, R$ are isomorphic in $\underline{\mathrm{mod}\, R}$, we write $A \overset{\mathrm{st}}{\cong} B$.

For an R-module M, define a transpose $\mathrm{Tr}\, M$ of M to be $\mathrm{Cok}\, \delta^*$ where $P \xrightarrow{\delta} Q \to M \to 0$ is a projective presentation of M. The transpose of M is uniquely determined as an object of $\underline{\mathrm{mod}\, R}$. If $\underline{f} \in \underline{\mathrm{Hom}}_R(M,N)$, then f induces a map $\mathrm{Tr}\, N \to \mathrm{Tr}\, M$, which represents a morphism $\mathrm{Tr}\, \underline{f} \in \underline{\mathrm{Hom}}_R(\mathrm{Tr}\, N, \mathrm{Tr}\, M)$. Hence Tr is an autofunctor on $\underline{\mathrm{mod}\, R}$.

2. A functor to the homotopy category

Let $\mathcal{L}$ be a full subcategory of $\mathsf{K}(\mathrm{mod}\, R)$ defined as

$$\mathcal{L} = \{F^\bullet \in \mathsf{K}(\mathrm{proj}\, R) \mid H^i(F) = 0 \ (i < 0),\ H_j(F_\bullet^*) = 0 \ (j \geq 0)\}.$$

Lemma 2.1. *For a morphism $f^\bullet$ in $\mathcal{L}$, $f^\bullet = 0$ in $\mathsf{K}(\mathrm{mod}\, R)$ if and only if $\underline{H^0(\tau_{\leq 0} f)} = 0$ in $\underline{\mathrm{mod}\, R}$.*

Proof. Let $f^\bullet : A^\bullet \to B^\bullet$ be a chain map with $A^\bullet, B^\bullet \in \mathcal{L}$ such that $\underline{H^0(\tau_{\leq 0} f^\bullet)} = 0$. Then there exists $g \in \mathrm{Hom}_R(H^0(\tau_{\leq 0} A^\bullet), B^0)$ that satisfies $H^0(\tau_{\leq 0} f^\bullet) = \rho \circ g$ where $\rho : B_0 \to H^0(\tau_{\leq 0} B^\bullet)$ is the natural projective cover. We get chain maps $\rho^\bullet \in \mathrm{Hom}_R(B_0, \tau_{\leq 0} B^\bullet)$ and $g^\bullet \in \mathrm{Hom}_R(\tau_{\leq 0} A^\bullet, B_0)$ such as $H^0(\rho^\bullet) = \rho$ and $H^0(g^\bullet) = g$. From the assumption, $\tau_{\leq 0} f^\bullet$ is homotopic to $\rho^\bullet \circ g^\bullet$, which implies

$$f^i = h^{i+1} \circ {d_A}^i + {d_B}^{i-1} \circ h^i$$

with some $h^{i+1} : A^{i+1} \to B^i$ for $i \leq -1$. Similarly, since $\underline{H^0(\tau_{\leq 0} f^\bullet)^*} = 0$, we have

$$f^j = h^{j+1} \circ {d_A}^j + {d_B}^{j-1} \circ h^j$$

with some $h^j : A^j \to B^{j-1}$ for $j \geq 2$. Therefore as a morphism in $\mathcal{L}$, we may assume $f^i = 0$ $(i \neq 0, 1)$. Moreover, we may assume $f^i = 0$ $(i \neq 1)$; since ${d_A}^{-1*} \circ f^{0*} = 0$, we get $s^1 : A^1 \to B^0$ with $f^0 = s^1 \circ {d_A}^0$. Finally, to see $f^\bullet = 0$, observe ${d_A}^{0*} \circ f^{1*} = 0$, then we get $u^2 : A^2 \to B^1$ with $f^1 = u^2 \circ {d_A}^1$. Since ${d_A}^{1*} \circ u^{2*} \circ {d_B}^{1*} = f^{1*} \circ {d_B}^{1*} = 0$, there exits a map $u^3 : A^3 \to B^2$ such that

$${d_B}^1 \circ u^2 + u^3 \circ {d_A}^2 = 0.$$

Thus we obtain a homotopy map $u : A^\bullet \to T^{-1} B^\bullet$ which shows that $f^\bullet$ is homotopic to zero.

The "only if" part comes from a more general result Lemma 2.2. □

Lemma 2.2 ([5])**.** *Let $f^\bullet$ be a chain map between two projective complexes. If $f^\bullet$ is homotopic to zero, then $\underline{H^n(\tau_{\leq n} f^\bullet)} = 0$ for every $n \in \mathbf{Z}$.*

For the proof of Lemma 2.2, the argument in [5, p. 246] completely works so we omit the proof here.

Lemma 2.1 is a key lemma and we obtain the following results as corollaries.

Proposition 2.3. *For $A \in \underline{\mathrm{mod}\, R}$, there exists ${F_A}^\bullet \in \mathcal{L}$ that satisfies*

$$H^0(\tau_{\leq 0}{F_A}^\bullet) \overset{\mathrm{st}}{\cong} A.$$

Such an ${F_A}^\bullet$ is uniquely determined by A up to isomorphisms. We fix the notation ${F_A}^\bullet$ and call this a standard resolution of A.

Proof. First take a projective resolution ${P_A}^\bullet$ of A:

$$\cdots \to {P_A}^{-2} \to {P_A}^{-1} \to {P_A}^{0} \to A \to 0,$$

and then a projective resolution ${P_{\mathrm{Tr}\, A}}^\bullet$ of $\mathrm{Tr}\, A = \mathrm{Cok}\, {d_{F_A}}^{-1*}$ as

$$0 \leftarrow \mathrm{Tr}\, A \leftarrow {P_A}^{-1*} \leftarrow {P_A}^{0*} \leftarrow {P_{\mathrm{Tr}\, A}}^{-2*} \leftarrow \cdots .$$

Define a complex ${F_A}^\bullet$ as

$${F_A}^i = \begin{cases} {P_A}^i & (i \leq -1), \\ {P_{\mathrm{Tr}\, A}}^{-1-i*} & (i \geq 0), \end{cases} \qquad {d_{F_A}}^i = \begin{cases} {d_{P_A}}^i & (i \leq -1), \\ {d_{P_{TrA}}}^{-2-i*} & (i \geq 0). \end{cases}$$

We easily see ${F_A}^\bullet \in \mathcal{L}$ and

$$H^0(\tau_{\leq 0}{F_A}^\bullet) \overset{\mathrm{st}}{\cong} A.$$

Suppose both ${F_A}^\bullet$ and ${F'_A}^\bullet$ have this property. Adding some trivial complex $P^\bullet$ of projective modules

$$P^\bullet : \cdots \to 0 \to P^0 = P^1 \to 0 \to \cdots$$

if necessary, we may assume that $H^0(\tau_{\leq 0}{F_A}^\bullet) \cong H^0(\tau_{\leq 0}{F'_A}^\bullet) \cong A$ in $\mathrm{mod}\, R$. Then there are chain maps $\varphi^\bullet : \tau_{\leq 0}{F'_A}^\bullet \to \tau_{\leq 0}{F_A}^\bullet$ and $\gamma^\bullet : \tau_{\leq 0}{F_A}^\bullet \to \tau_{\leq 0}{F'_A}^\bullet$ such that $\varphi^\bullet \circ \gamma^\bullet = 1_{\tau_{\leq 0}{F'_A}^\bullet}$ and $\gamma^\bullet \circ \varphi^\bullet = 1_{\tau_{\leq 0}{F_A}^\bullet}$. As $\tau_{\geq -1}{F_A}^*_\bullet$ and $\tau_{\geq -1}{F'_A}^*_\bullet$ are acyclic, $H_{-1}(\tau_{\geq -1}\varphi^*_\bullet)$ induces a chain map $\tau_{\geq -1}{F'_A}^*_\bullet \to \tau_{\geq -1}{F_A}^*_\bullet$. With this map for the positive part, $\varphi^\bullet$ can be extended to a chain map $f^\bullet : {F_A}^\bullet \to {F'_A}^\bullet$ such that $\tau_{\leq 0}f^\bullet = \varphi^\bullet$. Similarly we get a chain map $g^\bullet : {F'_A}^\bullet \to {F_A}^\bullet$ such that $\tau_{\leq 0}g^\bullet = \gamma^\bullet$. It is easy to see $\underline{H^0(\tau_{\leq 0}(f^\bullet \circ g^\bullet))} = \underline{1_A}$ and $\underline{H^0(\tau_{\leq 0}(g^\bullet \circ f^\bullet))} = \underline{1_A}$. From Lemma 2.1, we have $f^\bullet \circ g^\bullet = 1_{{F'_A}^\bullet}$ and $g^\bullet \circ f^\bullet = 1_{{F_A}^\bullet}$. □

Proposition 2.4. *For $\underline{f} \in \underline{\mathrm{Hom}}_R(A, B)$, there exists*

$$f^\bullet \in \mathrm{Hom}_{\mathsf{K}(\mathrm{mod}\, R)}({F_A}^\bullet, {F_B}^\bullet)$$

that satisfies $\underline{H^0(\tau_{\leq 0} f^\bullet)} = \underline{f}$. *Such an* $f^\bullet$ *is uniquely determined by* $\underline{f}$ *up to isomorphisms, so we use the notation* $f^\bullet$ *to describe a chain map with this property for given* $\underline{f}$.

Proof. As in the proof of Proposition 2.3, we obtain a chain map $f^\bullet$. Uniqueness follows from Lemma 2.1. □

Since the operation $H^0\tau_{\leq 0}$ commutes with composition, the next lemma is an immediate corollary of Proposition 2.4.

Lemma 2.5. *For* $\underline{f} \in \underline{\mathrm{Hom}}_R(A, B)$ *and* $\underline{g} \in \underline{\mathrm{Hom}}_R(B, C)$, *we have*

$$f^\bullet \circ g^\bullet = (f \circ g)^\bullet.$$

To sum up, we construct a functor.

Theorem 2.6. *The mapping* $A \mapsto F_A{}^\bullet$ *gives a functor from* $\underline{\mathrm{mod}\, R}$ *to* $\mathsf{K}(\mathrm{mod}\, R)$, *and this gives a category equivalence between* $\underline{\mathrm{mod}\, R}$ *and* $\mathcal{L}$.

Every short exact sequence of modules induces that of projective resolutions. But it does not necessarily induces an exact sequence of standard resolutions.

Lemma 2.7. *A short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathrm{mod}\, R$ *induces a short exact sequence of chain complexes*

$$0 \to F_A{}^\bullet \to F_B{}^\bullet \to F_C{}^\bullet \to 0$$

if and only if $0 \to C^* \to B^* \to A^* \to 0$ *is also exact.*

Proof. If $0 \to F_A{}^\bullet \to F_B{}^\bullet \to F_C{}^\bullet \to 0$ is exact, so is $0 \to \tau_{\geq 1} F_C{}^*_\bullet \to \tau_{\geq 1} F_B{}^*_\bullet \to \tau_{\geq 1} F_A{}^*_\bullet \to 0$, which induces an exact sequence of homology:

$$0 \to C^* \to B^* \to A^* \to 0.$$

With no assumption, we have a diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \tau_{\leq 0} F_A{}^\bullet \arrow[r] \arrow[d] & \tau_{\leq 0} F_B{}^\bullet \arrow[r] \arrow[d] & \tau_{\leq 0} F_C{}^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd} \tag{2.1}$$

If $0 \to C^* \to B^* \to A^* \to 0$ is exact, similarly we get a diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \tau_{\geq 1} F_C{}^*_\bullet \arrow[r] \arrow[d] & \tau_{\geq 1} F_B{}^*_\bullet \arrow[r] \arrow[d] & \tau_{\geq 1} F_A{}^*_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^* \arrow[r] & B^* \arrow[r] & A^* \arrow[r] & 0.
\end{tikzcd} \tag{2.2}$$

Applying $\mathrm{Hom}_R(\quad, R)$ to (2.2) and connecting the dualized diagram to (2.1), we get a desired exact sequence $0 \to F_A{}^\bullet \to F_B{}^\bullet \to F_C{}^\bullet \to 0$. □

3. Pseudo-kernels and pseudo-cokernels

For $A, B \in \operatorname{mod} R$, put $A^\bullet = F_A{}^\bullet$, $B^\bullet = F_B{}^\bullet$. For $f \in \mathrm{Hom}_R(A, B)$, consider the chain map $f^\bullet : A^\bullet \to B^\bullet$ with $\underline{H^0(\tau_{\le 0} f^\bullet)} = \underline{f}$. Putting $C^\bullet = C(f^\bullet)^\bullet$, we get a triangle

$$T^{-1}C^\bullet \xrightarrow{n^\bullet} A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{c^\bullet} C^\bullet. \tag{3.1}$$

In general, $C^\bullet$ does not belong to $\mathcal{L}$ any more but it satisfies the following:

$$H^i(C^\bullet) = 0 \ (i < -1), \quad H_j(C^*_\bullet) = 0 \ (j > -1).$$

Definition and Lemma 3.1. *As objects of* $\underline{\operatorname{mod} R}$, $\underline{\mathrm{Ker}}\ \underline{f} := H^{-1}(\tau_{\le -1} C^\bullet)$ *and* $\underline{\mathrm{Cok}}\ \underline{f} := H^0(\tau_{\le 0} C^\bullet)$ *are uniquely determined by* $\underline{f}$.

Proof. Lemma 2.1 guarantees that $C^\bullet$ is uniquely determined in $\mathsf{K}(\operatorname{proj} R)$. Together with Lemma 2.2, we know that $H^n(\tau_{\le n} C^\bullet)$ are also uniquely determined by $\underline{f}$. □

Put

$$n_f := H^0(\tau_{\le 0} n^\bullet) : \underline{\mathrm{Ker}}\ \underline{f} \to A, \quad c_f := H^0(\tau_{\le 0} c^\bullet) : B \to \underline{\mathrm{Cok}}\ \underline{f}.$$

The triangle (3.1) gives an exact sequence of the following form:

$$0 \to \underline{\mathrm{Ker}}\ \underline{f} \xrightarrow{\binom{n_f}{q_f}} A \oplus P \xrightarrow{(f\ \rho)} B \to 0 \tag{3.2}$$

with some projective module P. In fact, $\underline{\mathrm{Ker}}\ \underline{f}$ is characterized with this property:

Proposition 3.2. *If an R-linear map $\rho' : P' \to B$ from a projective module P' makes $\tilde{f}' : A \oplus P' \xrightarrow{(f\ \rho')} B$ a surjective mapping, then* $\mathrm{Ker}\ \tilde{f}' \overset{\mathrm{st}}{\cong} \underline{\mathrm{Ker}}\ \underline{f}$.

Proof. It is easy to show that both of the composites $P' \xrightarrow{\rho'} B \to \mathrm{Cok} f$ and $P \xrightarrow{\rho} B \to \mathrm{Cok} f$ are projective covers of $\mathrm{Cok} f$. There exist $t \in \mathrm{Hom}_R(P_B, P')$ and $u \in \mathrm{Hom}_R(P_B, A)$ such that $\rho - \rho' \circ t = f \circ u$. If t is not an epimorphism, add some $s : Q \to P$ with $Q \in \operatorname{proj} R$ to make $P \oplus Q \xrightarrow{(t\ s)} P'$ surjective. From the diagram

$$\begin{tikzcd} A \oplus P \oplus Q \arrow[r, "(f\ \rho\ s\circ\rho')"] & B \\ A \oplus P \arrow[u, "\left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)"] \arrow[r, "(f\ \rho)"] & B, \arrow[u, equal] \end{tikzcd}$$

we get $\underline{\mathrm{Ker}}\ \underline{f} \overset{\mathrm{st}}{\cong} \mathrm{Ker}(f\ \rho\ s\circ\rho')$. Also we have a diagram

$$\begin{CD} A\oplus P\oplus Q @>{(f\ \rho\ s\circ\rho')}>> B \\ @V{\left(\begin{smallmatrix}1 & u & 0\\ 0 & t & s\end{smallmatrix}\right)}VV @| \\ A\oplus P' @>{(f\ \rho')}>> B, \end{CD}$$

hence $\mathrm{Ker}\,\tilde{f}' \overset{\mathrm{st}}{\cong} \mathrm{Ker}(f\ \rho\ s\circ\rho')$. □

Lemma 3.3. *With notation as above, we have the following*:

1) $\underline{f}\circ\underline{n_f} = \underline{0}$.
2) *If* $\underline{x}\in\underline{\mathrm{Hom}}_R(X,A)$ *satisfies* $\underline{f}\circ\underline{x}=\underline{0}$, *there exists*

$$\underline{h_x}\in\underline{\mathrm{Hom}}_R(X,\underline{\mathrm{Ker}}\ \underline{f})$$

such that $\underline{x}=\underline{n_f}\circ\underline{h_x}$.

The proof is straightforward from the definition. Strictly speaking, $\underline{\mathrm{Ker}}\ \underline{f}$ is not the kernel of $\underline{f}$. Because it lacks the uniqueness of $\underline{h_x}$ in 2) of Lemma 3.3. (See Example 3.4).

Example 3.4. Let $R = k[[x,y,z]]/(x^2-yz)$, $A = R/(yz)$ and $B = R/(yz,y^2,z^2)$. Let $f: A\to B$ be the natural map induced from the inclusion $(yz)\subset(yz,y^2,z^2)$. Since f is surjective, $\underline{\mathrm{Ker}}\ \underline{f} \overset{\mathrm{st}}{\cong} \mathrm{Ker}\, f \cong R/(z)\oplus R/(y)$, and the sequence $0\to\mathrm{Ker}\, f\xrightarrow{n_f} A\xrightarrow{f} B\to 0$ is exact. Put $X=\mathrm{Tr}\, k$ and let $u\in\mathrm{Hom}_R(X,\mathrm{Ker}\, f)$ be as follows:

$$\begin{CD} 0 @>>> R @>{\left(\begin{smallmatrix}x\\ y\\ z\end{smallmatrix}\right)}>> R^3 @>>> X @>>> 0 \\ @. @V{\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)}VV @V{\left(\begin{smallmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{smallmatrix}\right)}VV @VV{u}V \\ 0 @>>> R^2 @>{\left(\begin{smallmatrix}z & 0\\ 0 & y\end{smallmatrix}\right)}>> R^2 @>>> \mathrm{Ker}\, f @>>> 0. \end{CD}$$

Easily we get $\underline{n_f}\circ\underline{u}=\underline{0}_A=\underline{u_f}\circ\underline{0}_K$ where $\underline{0}_A=\underline{0}\in\underline{\mathrm{Hom}}_R(X,A)$ and $\underline{0}_K=\underline{0}\in\underline{\mathrm{Hom}}_R(X,\mathrm{Ker}\, f)$. Also we have $\underline{u}\neq\underline{0}_K$ from this diagram. □

Dually, $(\underline{\mathrm{Cok}}\ \underline{f},\underline{c_f})$ satisfies the following, which comes from the observation

$$\underline{\mathrm{Cok}}\ \underline{f}=\mathrm{Tr}\,\underline{\mathrm{Ker}}\ \mathrm{Tr}\,\underline{f},\quad c_f=\mathrm{Tr}\,\underline{n_{\mathrm{Tr}\, f}}.$$

Lemma 3.5. 1) $\underline{c_f}\circ\underline{f}=\underline{0}$.
2) *If* $\underline{y}\in\underline{\mathrm{Hom}}_R(B,Y)$ *satisfies* $\underline{y}\circ\underline{f}=\underline{0}$, *there exists*

$$\underline{e_y}\in\underline{\mathrm{Hom}}_R(\underline{\mathrm{Cok}}\ \underline{f},Y)$$

such that $\underline{y}=\underline{e_y}\circ\underline{c_f}$.

Two modules $\operatorname{Ker} f$ and $\underline{\operatorname{Ker}}\ \underline{f}$ are not always stably isomorphic. But we get the following.

Lemma 3.6. 1) *There is an exact sequence* $0 \to L \to M \to N \to 0$ *such that* $L \overset{\mathrm{st}}{\cong} \operatorname{Ker} f$, $M \overset{\mathrm{st}}{\cong} \underline{\operatorname{Ker}}\ \underline{f}$ *and* $N \overset{\mathrm{st}}{\cong} \Omega^1_R(\operatorname{Cok} f)$.

2) *There is an exact sequence* $0 \to L' \to M' \to N' \to 0$ *such that* $M' \overset{\mathrm{st}}{\cong} \underline{\operatorname{Cok}}\ \underline{f}$, $N' \overset{\mathrm{st}}{\cong} \operatorname{Cok} f$ *and* $\Omega^1_R(L')$ *is the surjective image of* $\operatorname{Ker} f$.

Proof. 1) The claim easily follows from the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker} f \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\binom{1}{0}"'] & \operatorname{Im} f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\operatorname{Ker}}\ \underline{f} \arrow[r] \arrow[d] & A \oplus P \arrow[r, "(f\ \rho)"] \arrow[d, "\binom{0}{1}"'] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_R(\operatorname{Cok} f) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & \operatorname{Cok} f \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

2) Dualizing (3.2) with R, we get a map $\check{f} : A \xrightarrow{\binom{f}{j_f}} B \oplus Q$ with some projective module Q such that $\operatorname{Cok} \check{f} \overset{\mathrm{st}}{\cong} \underline{\operatorname{Cok}}\ \underline{f}$. And consider the commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & Q \arrow[d] & & \\
0 \arrow[r] & \operatorname{Im} \check{f} \arrow[r] \arrow[d] & B \oplus Q \arrow[r] \arrow[d] & \operatorname{Cok} \check{f} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Im} f \arrow[r] & B \arrow[r] \arrow[d] & \operatorname{Cok} f \arrow[r] & 0, \\
 & & 0 & &
\end{tikzcd}$$

where the middle column is a split exact sequence. If we put L' the kernel of epimorphism $\operatorname{Cok} \check{f} \to \operatorname{Cok} f$, then $\Omega^1_R(L')$ is the kernel of the natural map

$$A/\operatorname{Ker} \check{f} \cong \operatorname{Im} \check{f} \to \operatorname{Im} f \cong A/\operatorname{Ker} f.$$

Therefore $\Omega^1_R(L') \cong \operatorname{Ker} f / \operatorname{Ker} \check{f}$. $\square$

Corollary 3.7. *1)* $\underline{\mathrm{Ker}}\ \underline{f} \overset{\mathrm{st}}{\cong} \operatorname{Ker} f$ *if f is an epimorphism.*

2) $\underline{\mathrm{Cok}}\ \underline{f} \overset{\mathrm{st}}{\cong} \operatorname{Cok} f$ *if f is a split monomorphism.*

Notations. For a given homomorphism $f : A \to B$ in $\operatorname{mod} R$, put $A^\bullet = F_A{}^\bullet$, $B^\bullet = F_B{}^\bullet$ and $C^\bullet = C(f^\bullet)^\bullet$. Set $K^\bullet = F_{\underline{\mathrm{Ker}}\ \underline{f}}{}^\bullet$ and $L^\bullet = F_{\underline{\mathrm{Cok}}\ \underline{f}}{}^\bullet$. Chain maps $n_f{}^\bullet \in \operatorname{Hom}_{\mathsf{K}(\operatorname{mod} R)}(K^\bullet, A^\bullet)$ and $c_f{}^\bullet \in \operatorname{Hom}_{\mathsf{K}(\operatorname{mod} R)}(B^\bullet, L^\bullet)$ are induced from $\underline{n_f}$ and $\underline{c_f}$. Since $f^\bullet \circ n_f{}^\bullet = 0$ and $c_f{}^\bullet \circ f^\bullet = 0$, there exist $\varepsilon^\bullet \in \operatorname{Hom}_{\mathsf{K}(\operatorname{mod} R)}(K^\bullet, T^{-1}C^\bullet)$ and $\delta^\bullet \in \operatorname{Hom}_{\mathsf{K}(\operatorname{mod} R)}(C^\bullet, L^\bullet)$ such that

$$n_f{}^\bullet = n^\bullet \circ \varepsilon^\bullet, \text{ and } c_f{}^\bullet = \delta^\bullet \circ c^\bullet.$$

Notice that

$$C(\varepsilon^\bullet)^i = 0 \ (i \leq -1), \text{ and } C(\delta^\bullet)^j = 0 \ (j \geq -1)$$

because $\varepsilon^\bullet$ and $\delta^\bullet$ induce $1_{\underline{\mathrm{Ker}}\ \underline{f}}$ and $1_{\underline{\mathrm{Cok}}\ \underline{f}}$.

$$\begin{tikzcd}
K^\bullet \arrow[r, "n_f{}^\bullet"] \arrow[d, "\varepsilon^\bullet"] & A^\bullet \arrow[d, equal] & & \\
T^{-1}C^\bullet \arrow[r, "n^\bullet"] & A^\bullet \arrow[r, "f^\bullet"] & B^\bullet \arrow[r, "c^\bullet"] \arrow[d, equal] & C^\bullet \arrow[d, "\delta^\bullet"] \\
& & B^\bullet \arrow[r, "c_f{}^\bullet"] & L^\bullet.
\end{tikzcd}$$

4. Monomorphisms, epimorphisms, and split morphisms

If $\underline{\mathrm{Ker}}\ \underline{f} = 0$, then $\underline{f}$ is injective. But the vanishing of $\underline{\mathrm{Ker}}\ \underline{f}$ is not a necessary condition for $\underline{f}$ to be injective; let A, B be two modules with $\operatorname{pd} B \geq 2$. Let f be a split monomorphism $A \to A \oplus B$. Obviously $\underline{n_f} = 0$ but $\underline{\mathrm{Ker}}\ \underline{f} \overset{\mathrm{st}}{\cong} \Omega^1_R(B)$ is not projective. We investigate what is an injective morphism in $\underline{\operatorname{mod}} R$.

Proposition 4.1. *With notations as in §3, the following are equivalent.*

1) *$\underline{f}$ is a monomorphism in $\underline{\operatorname{mod}} R$.*
2) *$\overline{\mathrm{Ext}}^1_R(f, -) : \operatorname{Ext}^1_R(B, -) \to \operatorname{Ext}^1_R(A, -)$ is surjective.*
3) *$\underline{n_f} = 0$.*
4) *There exists $\varphi^\bullet \in \operatorname{Hom}_{\mathsf{K}(\operatorname{mod} R)}(K^\bullet, T^{-1}B^\bullet)$ that makes the diagram commutative:*

$$\begin{tikzcd}
K^\bullet \arrow[r, "\varepsilon^\bullet"] \arrow[dr, "\varphi^\bullet"'] & T^{-1}C^\bullet \\
& T^{-1}B^\bullet \arrow[u, "T^{-1}c^\bullet"']
\end{tikzcd}$$

5) $\tau_{\leq -1}c^\bullet$ *is a split epimorphism in* $\mathsf{K}(\operatorname{mod} R)$.

6) $\underline{\Omega^1_R(f)}$ *is a split monomorphism and* $\underline{\mathrm{Ker}}\ \underline{f} \cong \underline{\mathrm{Cok}}\ \underline{\Omega^1_R(f)}$ *in* $\underline{\operatorname{mod}}\, R$.

Proof. 1) $\Leftrightarrow$ 3). A morphism $\underline{f}$ is called a monomorphism if and only if $\underline{f}\circ\underline{x} = 0$ always implies $\underline{x} = 0$, which is equivalent to $\underline{n_f} = 0$ from Lemma 3.3.

1) $\Leftrightarrow$ 2). An exact sequence

$$0 \to \underline{\mathrm{Ker}}\ \underline{f} \xrightarrow{\binom{n_f}{q_f}} A \oplus P \xrightarrow{(f\ \rho)} B \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{Ext}^1_R(B,\quad) \xrightarrow{\mathrm{Ext}^1_R(f,\ )} \mathrm{Ext}^1_R(A,\quad) \xrightarrow{\mathrm{Ext}^1_R(n_f,\ )} \mathrm{Ext}^1_R(\underline{\mathrm{Ker}}\ \underline{f},\quad) \to \cdots.$$

So $\mathrm{Ext}^1_R(f,\ )$ is surjective if and only if $\mathrm{Ext}^1_R(n_f,\ )$ is zero, which is equivalent to the condition $\underline{n_f} = 0$ from [1] (1.44).

3) $\Leftrightarrow$ 4). Lemma 2.1 shows that $\underline{n_f} = 0$ if and only if $n_f{}^\bullet = n^\bullet \circ \varepsilon^\bullet = 0$, that is, some $\varphi^\bullet : K^\bullet \to T^{-1}B^\bullet$ exists and $\varepsilon^\bullet = T^{-1}c^\bullet \circ \varphi^\bullet$ since $T^{-1}B^\bullet \xrightarrow{T^{-1}c^\bullet} T^{-1}C^\bullet \xrightarrow{n^\bullet} A^\bullet \to B^\bullet$ is a triangle.

4) $\Rightarrow$ 5). Applying $\tau_{\leq 0}$ to the diagram in 4), we get 5) since $\tau_{\leq 0}(T^{-1}c^\bullet) = T^{-1}(\tau_{\leq -1}c^\bullet)$ and $\tau_{\leq 0}\varepsilon^\bullet$ is the identity.

5) $\Rightarrow$ 6). Put $X^\bullet = C(\tau_{\leq -1}c^\bullet)$. Then a triangle

$$T^{-1}X^\bullet \to \tau_{\leq -1}B^\bullet \xrightarrow{\tau_{\leq -1}c^\bullet} \tau_{\leq -1}C^\bullet \to X^\bullet$$

induces a split exact sequence

$$0 \to H^{-2}(X^\bullet) \xrightarrow{\omega} H^{-1}(\tau_{\leq -1}B^\bullet) \to H^{-1}(\tau_{\leq -1}C^\bullet).$$

By definition, $H^{-1}(\tau_{\leq -1}B^\bullet) \overset{\mathrm{st}}{\cong} B$ and $H^{-1}(\tau_{\leq -1}C^\bullet) \overset{\mathrm{st}}{\cong} \underline{\mathrm{Ker}}\ \underline{f}$. We claim that $H^{-2}(X^\bullet) \overset{\mathrm{st}}{\cong} A$ and via this isomorphism, $\omega \overset{\mathrm{st}}{\cong} \Omega^1_R(f)$. Since $B^{-1} \to C^{-1} = X^{-1}$ is surjective, so is $d_X{}^{-2}$, which implies $H^{-2}(X^\bullet) \overset{\mathrm{st}}{\cong} \operatorname{Cok} d_X{}^{-3}$. Moreover, $\operatorname{Cok} d_X{}^{-3} = \operatorname{Cok} d_{C(c)}{}^{-3} \overset{\mathrm{st}}{\cong} \operatorname{Cok} d_A{}^{-2} = \Omega^1_R(A)$ as $\tau_{\leq -2}X^\bullet = \tau_{\leq -2}C(c)^\bullet$ and $C(c)^\bullet \cong A^\bullet$.

6) $\Rightarrow$ 4). With no assumption, we have the diagram

$$\begin{tikzcd}
F_{\Omega^1_R(A)}{}^\bullet \arrow[r, "\Omega^1_R(f)^\bullet"] \arrow[dd, "\alpha^\bullet"] & F_{\Omega^1_R(B)}{}^\bullet \arrow[r, "v^\bullet"] \arrow[dd, "\beta^\bullet"] & C(\Omega^1_R(f))^\bullet \arrow[dd, "\gamma^\bullet"] \arrow[dr, "t^\bullet"] & \\
 & & & K^\bullet \arrow[dl, "\varepsilon^\bullet"] \\
T^{-1}A^\bullet \arrow[r, "T^{-1}f^\bullet"] & T^{-1}B^\bullet \arrow[r, "T^{-1}c^\bullet"] & T^{-1}C^\bullet &
\end{tikzcd}$$

where $\alpha^\bullet$, $\beta^\bullet$ are canonical maps induced by $1_{\Omega^1_R(A)}$ and $1_{\Omega^1_R(B)}$, which induce $\gamma^\bullet$. The map $\underline{\mathrm{Cok}}\ \underline{\Omega^1_R(f)} \to \underline{\mathrm{Ker}}\ \underline{f}$ induces $t^\bullet$. Now if we assume the condition 6), there exists a chain map $s^\bullet : C(\Omega^1_R(f))^\bullet \to F_{\Omega^1_R(B)}{}^\bullet$ such that $v^\bullet \circ s^\bullet = 1_{C(\Omega^1_R(f))^\bullet}$ and $t^\bullet = 1_{K^\bullet}$. Hence $\varepsilon^\bullet = \gamma^\bullet = \gamma^\bullet \circ v^\bullet \circ s^\bullet = T^{-1} c^\bullet \circ \beta^\bullet \circ s^\bullet$ so we get the chain map $\varphi^\bullet = \beta^\bullet \circ s^\bullet$. □

If $\mathrm{Ext}^1_R(B,R) = 0$, then $H_{-1}(C^*_\bullet) = 0$, which implies $\varepsilon^\bullet = 1$; $K^\bullet = T^{-1}C^\bullet$. Thus we have the next lemma:

Lemma 4.2. *The following are equivalent for $B \in \underline{\mathrm{mod}}\, R$.*

1) *In $\underline{\mathrm{mod}}\, R$, every monomorphism to B splits.*
2) $\mathrm{Ext}^1_R(B,R) = 0$.

Proof. 2) $\Rightarrow$ 1). Let $\underline{f} : A \to B$ be a monomorphism and let us use the same notations as in Proposition 4.1. If $\mathrm{Ext}^1_R(B,R) = 0$, then $H_{-1}(C^*_\bullet) = 0$, which implies $T^{-1}C^\bullet = K^\bullet$, that is, $\varepsilon^\bullet$ is an isomorphism. Since $\underline{n_f} = 0$, $n^\bullet = n_f{}^\bullet = 0$ hence $f^\bullet$ is a split monomorphism.

1) $\Rightarrow$ 2). If $\mathrm{Ext}^1_R(B,R) \neq 0$, then there exists a non-split short exact sequence

$$0 \to R \to A \xrightarrow{f} B \to 0.$$

We see $\underline{f}$ is a monomorphism because $\underline{\mathrm{Ker}}\ \underline{f} \overset{\mathrm{st}}{\cong} R$. But $\underline{f}$ does not split. □

Dually, we get

Lemma 4.3. *The following are equivalent for $A \in \underline{\mathrm{mod}}\, R$.*

1) *In $\underline{\mathrm{mod}}\, R$, every epimorphism from A splits.*
2) $\mathrm{Ext}^1_R(\mathrm{Tr}\, A, R) = 0$.

Remark. The condition that $\underline{\Omega^1_R(f)}$ is a split monomorphism does not automatically induce $\underline{\mathrm{Ker}}\ \underline{f} \overset{\mathrm{st}}{\cong} \underline{\mathrm{Cok}}\ \underline{\Omega^1_R(f)}$. For instance, let $z \in R$ be an non-zero-divisor of R. Let f be an endomorphism of $R/(z^2)$ as $f = z$. Then $\Omega^1_R(f)$ is an endomorphism of R, so we have $\underline{\mathrm{Cok}}\ \underline{\Omega^1_R(f)} = 0$. But $\underline{\mathrm{Ker}}\ \underline{f} \overset{\mathrm{st}}{\cong} R/(z)$ is not projective.

Theorem 4.4. *The following are equivalent for a ring R.*

1) *Every monomorphism in $\underline{\mathrm{mod}}\, R$ splits.*
2) *Every epimorphism in $\underline{\mathrm{mod}}\, R$ splits.*
3) *R is self-injective.*
4) *Every short exact sequence $0 \to A \to B \to C \to 0$ induces an exact sequence of standard resolutions $0 \to F_A{}^\bullet \to F_B{}^\bullet \to F_C{}^\bullet \to 0$.*

5) *Every short exact sequence* $0 \to A \to B \to C \to 0$ *remains exact when dualized by* R; $0 \to C^* \to B^* \to A^* \to 0$ *is exact.*

Proof. The equivalence between 3) and 1) (or 2)) follows from Lemma 4.2 (Lemma 4.3) respectively. We have already shown in Lemma 2.7 that 4) and 5) are equivalent. Obviously 3) implies 5), so it suffices to prove that 5) implies 3). Let M be an arbitrary object of $\operatorname{mod} R$. Consider a projective cover of M:

$$0 \to \Omega_R^1(M) \to P \to M \to 0.$$

If the dualized sequence remains exact, that means $\operatorname{Ext}_R^1(M, R) = 0$. □

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References

[1] M. Auslander and M. Bridger, *The stable module theory*, Mem. Amer. Math. Soc. **94**, 1969.

[2] M. Amasaki, *Homogeneous prime ideals and graded modules fitting into long Bourbaki sequences*, preprint, 1999.

[3] T. Kawasaki, *Local cohomology modules of indecomposable surjective-Buchsbaum modules over Gorenstein local rings*, J. Math. Soc. Japan **48**(1996), 551–566.

[4] J. Miyachi, *Derived categories with applications to representations of algebras*, seminar note, 2000.

[5] Y. Yoshino, *Maximal Buchsbaum modules of finite projective dimension*, J. Algebra **159**(1993), 240–264.

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