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SOME COHOMOTOPY GROUPS OF SUSPENDED PROJECTIVE PLANES

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ABSTRACT. In this paper we compute some cohomotopy groups of the suspended complex and quaternionic projective plane by use of the exact sequence associated with the canonical cofiber sequence and a formula about a multiple of the identity class of the suspended projective plane.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this note all spaces, maps and homotopies are based. We denote by ΣX a suspension of a space X. For the normed fields $\mathbf{F} = \mathbf{R}$ (real), $\mathbf{C}$ (complex), $\mathbf{H}$ (quaternion) and $\mathbf{O}$ (octonion) with the usual norm, let $d = \dim_{\mathbf{R}} \mathbf{F}$.

The projective plane over $\mathbf{F}$ is denoted by $\mathbf{F}\mathrm{P}^2$. This is the space given by attaching a $2d$-cell to S^d by the Hopf map $h_d(\mathbf{F}) : S^{2d-1} \to S^d$. The inclusion map of S^d and the collapsing map to the top cell are denoted by

$$i_{\mathbf{F}} : S^d \to \mathbf{F}\mathrm{P}^2, \qquad p_{\mathbf{F}} : \mathbf{F}\mathrm{P}^2 \to S^{2d}$$

respectively. For a space X, let $\iota_X \in [X, X]$ be the identity class of X, $\iota_n = \iota_X$ for $X = S^n$ and $\iota_{\mathbf{F}} = \iota_X$ for $X = \mathbf{F}\mathrm{P}^2$. The n-th cohomotopy set of X is denoted by $\pi^n(X) = [X, S^n]$. We set $h_n(\mathbf{F}) = \Sigma^{n-d} h_d(\mathbf{F})$ for $n \geq d$.

The purpose of this note is to calculate cohomotopy groups of the suspended projective plane $\Sigma^k \mathbf{F}\mathrm{P}^2$ for the cases $\mathbf{F} = \mathbf{C}$ and $\mathbf{H}$. 2-primary versions of the calculations appeared in Master's theses of the third author [9] and the fourth author [12] in Shinshu University under the guidance of the other three authors together with Professor T. Matsuda.

The calculation will be done in the following way. Consider the exact sequence

$$\pi_{n+d+1}(S^k) \xrightarrow{h_{d+n+1}(\mathbf{F})^*} \pi_{n+2d}(S^k) \xrightarrow{\Sigma^n p_{\mathbf{F}}{}^*} [\Sigma^n \mathbf{F}\mathrm{P}^2, S^k]$$
$$\xrightarrow{\Sigma^n i_{\mathbf{F}}{}^*} \pi_{n+d}(S^k) \xrightarrow{h_{d+n}(\mathbf{F})^*} \pi_{n+2d-1}(S^k)$$

induced from the cofiber sequence

$$S^{2d-1} \xrightarrow{h_d(\mathbf{F})} S^d \xrightarrow{i_{\mathbf{F}}} \mathbf{F}\mathrm{P}^2 \xrightarrow{p_{\mathbf{F}}} S^{2d} \xrightarrow{h_{d+1}(\mathbf{F})} S^{d+1}.$$

From the above exact sequence we have the short exact sequence

$$0 \to \operatorname{Coker} h_{d+n+1}(\mathbf{F})^* \to [\Sigma^n \mathbf{F}\mathrm{P}^2, S^k] \to \operatorname{Ker} h_{n+d}(\mathbf{F})^* \to 0.$$

Then we determine the group extension by use of formulas of Toda brackets. For the 2-primary components, Coker $h_{d+n+1}(\mathbf{F})^*$ and Ker $h_{d+n}(\mathbf{F})^*$ are calculated in [9] and [12] for $\mathbf{F} = \mathbf{C}$ and $\mathbf{H}$, respectively.

The results are summarized in the following:

Theorem 1.1. *The cohomotopy groups* $[\Sigma^n \mathbf{C}\mathrm{P}^2, S^{n+k}]$ *in the range of* $-5 \leq k \leq 1$ *is isomorphic to the group given in the following table*:

$n \diagdown k$	1	0	-1	-2	-3	-4	-5
1	∞	0					
2	6	6	0				
3	∞+6	0	0	0			
4	12	0	6	6	0		
5		0	∞+6	3	3	0	
6		0	12	12+3	30	30	0
7			∞+12	2	30	6	6
8			24	2	60	6+24	30
9				2	∞+60	4	2+30
10				2	120	$(4)^2$+3	2+60
11					∞+120	2+4	60
12					240	2+4	120
13						2+4	∞+120
14						4	240
15							∞+240
16							240
17							

Theorem 1.2. *The cohomotopy group* $[\Sigma^n \mathbf{H}\mathrm{P}^2, S^{n+k}]$ *in the range of* $-3 \leq k \leq 3$ *is isomorphic to the group given in the following table*:

$n \diagdown k$	3	2	1	0	-1	-2	-3
1	$(2)^2$	2	2	0			
2	$(2)^2$	2	15+4	15+4	0		
3	∞+2	2	∞+15+4	$(2)^2$	$(2)^2$	0	
4	2	2	10+36	$(2)^3$	$(2)^3$	$(2)^3$	0
5		2	20+36	$(2)^2$	$(2)^4$	4+6	4+2+3
6		2	40+36	$(2)^2$	$(2)^3$	8+$(4)^2$+6+45	4+2+105
7			∞+40+36	$(2)^3$	∞+$(2)^2$	4+6	4+$(2)^4$+105
8			80+36	$(2)^4$	$(2)^3$	2+6	8+$(2)^2$+315
9				$(2)^3$	$(2)^4$	6	8+315
10				$(2)^2$	$(2)^3$	6	8+945
11					∞+$(2)^2$	6	∞+8+945
12					$(2)^2$	6	16+945
13						6	32+945
14						6	64+945
15							∞+128+945
16							128+945
17							

In the above tables, an integer n indicates a cyclic group $\mathbf{Z}_n$ of order n, the symbol "∞" an infinite cyclic group $\mathbf{Z}$, the symbol "+" the direct sum of the groups and $(n)^k$ indicates the direct sum of k-copies of $\mathbf{Z}_n$. Groups in the stable range (lower left area) and trivial groups (upper right area) are omitted.

In the stable range, Theorems 1.3 and 1.4 overlap with the results of [15], [10] and [7].

We use the notation and results of [13] freely.

2. Preliminaries

Consider an element $\alpha \in \pi_m(S^n)$ $(m > n \geq 2)$ such that $\Sigma\alpha$ and $\Sigma^2\alpha$ are of order t. Let $C_\alpha = S^n \cup_\alpha e^{m+1}$ be the mapping cone of α. The inclusion map of S^n and the collapsing map to the top cell e^{m+1} are denoted by $i : S^n \to C_\alpha$ and $p : C_\alpha \to S^{m+1}$, respectively. We shall use the identification $\Sigma^k C_\alpha = C_{\Sigma^k\alpha}$. Then we have the cofiber sequence

$$S^{m+k} \xrightarrow{\Sigma^k\alpha} S^{n+k} \xrightarrow{\Sigma^k i} \Sigma^k C_\alpha \xrightarrow{\Sigma^k p} S^{m+k+1} \xrightarrow{\Sigma^{k+1}\alpha} S^{n+k+1}.$$

Consider elements $\beta \in \pi_n(Z)$ and $\gamma \in [W, S^m]$ which satisfy $\beta \circ \alpha = 0$ and $\alpha \circ \gamma = 0$. We denote by $\overline{\beta} \in [C_\alpha, Z]$ an extension of β satisfying $i^*(\overline{\beta}) = \beta$ and by $\tilde{\gamma} \in [\Sigma W, C_\alpha]$ a coextension of γ satisfying $p_*(\tilde{\gamma}) = \Sigma\gamma$.

Making use of the homotopy exact sequence of the pair $(\Sigma C_\alpha, S^{n+1})$ and the theorem of Blakers-Massey [3], we easily obtain the following.

Lemma 2.1. (1) $\pi_{n+1}(\Sigma C_\alpha) \cong \mathbf{Z}\{\Sigma i\}$,

(2) $\pi_{m+2}(\Sigma C_\alpha) \cong \mathbf{Z}\{\widetilde{t\iota_{m+1}}\} \oplus \Sigma i \circ (\pi_{m+2}(S^{n+1})/\{\Sigma\alpha \circ \eta_{m+1}\})$.

By Theorem 10.3.10 of [16], we have the following.

Lemma 2.2. *Let Y be a 1-connected space. Then the commutator group of $[\Sigma C_\alpha, Y]$ and $\pi_{m+2}(Y) \circ \Sigma p$ is trivial.*

Hereafter, the commutativity of the homotopy group $[\Sigma C_\alpha, Y]$ is ensured by this lemma.

Consider the exact sequence

$$\pi_{n+2}(S^k) \xrightarrow{\Sigma^2\alpha^*} \pi_{m+2}(S^k) \xrightarrow{\Sigma p^*} [\Sigma C_\alpha, S^k] \xrightarrow{\Sigma i^*} \pi_{n+1}(S^k) \xrightarrow{\Sigma\alpha^*} \pi_{m+1}(S^k)$$

induced from the above cofiber sequence. Making use of this exact sequence and Lemma 2.2, we have the following.

Lemma 2.3. (1) $[\Sigma C_\alpha, S^{m+2}] \cong \mathbf{Z}\{\Sigma p\}$,

(2) $\Sigma p^* : \pi_{m+2}(S^k) \to [\Sigma C_\alpha, S^k]$ *is an isomorphism for* $k > n+2$,

(3) $[\Sigma C_\alpha, S^{n+2}] \cong \pi_{m+2}(S^{n+2})/\{\Sigma^2\alpha\}$,

(4) $[\Sigma C_\alpha, S^{n+1}] \cong \mathbf{Z}\{\overline{t\iota_{n+1}}\} \oplus (\pi_{m+2}(S^{n+1})/\{\eta_{n+1}\circ\Sigma^2\alpha\})\circ\Sigma p$.

From Theorem 1.3 of [11], we have

Proposition 2.4. (1) $[\Sigma C_\alpha, \Sigma C_\alpha] \cong \mathbf{Z}\{\Sigma\iota_{C_\alpha}\} \oplus \mathbf{Z}\{\widetilde{t\iota_{m+1}}\circ\Sigma p\} \oplus \Sigma i \circ (\pi_{m+2}(S^{n+1})/\{\eta_{n+1}\circ\Sigma^2\alpha, \Sigma\alpha\circ\eta_{m+1}\})\circ\Sigma p$,

(2) *If* $\Sigma : \pi_{m+2}(\Sigma C_\alpha)/\{\Sigma i\circ\eta_{n+1}\circ\Sigma^2\alpha\} \to \pi_{m+3}(\Sigma^2 C_\alpha)/\{\Sigma^2 i\circ\eta_{n+2}\circ\Sigma^3\alpha\}$ *is an isomorphism, then* $\Sigma : [\Sigma C_\alpha, \Sigma C_\alpha] \to [\Sigma^2 C_\alpha, \Sigma^2 C_\alpha]$ *is an isomorphism.*

Proof. Consider the exact sequence

$$\pi_{n+2}(\Sigma C_\alpha) \xrightarrow{\Sigma^2\alpha^*} \pi_{m+2}(\Sigma C_\alpha) \xrightarrow{\Sigma p^*} [\Sigma C_\alpha, \Sigma C_\alpha] \xrightarrow{\Sigma i^*} \pi_{n+1}(\Sigma C_\alpha) \xrightarrow{\Sigma\alpha^*} \pi_{m+1}(\Sigma C_\alpha).$$

By Lemmas 2.1 and 2.2, we have (1).

Next we consider the commutative diagram

$$\begin{tikzcd}
\pi_{n+2}(\Sigma C_\alpha) \arrow[r, "\Sigma^2\alpha^*"] \arrow[d, "\Sigma"] & \pi_{m+2}(\Sigma C_\alpha) \arrow[r, "\Sigma p^*"] \arrow[d, "\Sigma"] & {[\Sigma C_\alpha, \Sigma C_\alpha]} \arrow[d, "\Sigma"] \\
\pi_{n+3}(\Sigma^2 C_\alpha) \arrow[r, "\Sigma^3\alpha^*"] & \pi_{m+3}(\Sigma^2 C_\alpha) \arrow[r, "\Sigma^2 p^*"] & {[\Sigma^2 C_\alpha, \Sigma^2 C_\alpha]}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\Sigma i^*"] & \pi_{n+1}(\Sigma C_\alpha) \arrow[r, "\Sigma\alpha^*"] \arrow[d, "\Sigma"] & \pi_{m+1}(\Sigma C_\alpha) \arrow[d, "\Sigma"] \\
{} \arrow[r, "\Sigma^2 i^*"] & \pi_{n+2}(\Sigma^2 C_\alpha) \arrow[r, "\Sigma^2\alpha^*"] & \pi_{m+2}(\Sigma^2 C_\alpha).
\end{tikzcd}$$

By Freudenthal's suspension theorem, $\Sigma : \pi_{n+i}(\Sigma C_\alpha) \to \pi_{n+i+1}(\Sigma^2 C_\alpha)$ is an isomorphism for $i < n+1$. Since $\pi_{n+2}(\Sigma C_\alpha) \cong \mathbf{Z}_2\{\Sigma i\circ\eta_{n+1}\}$, we have (2). This completes the proof. □

The following proposition is proved on p. 287 of [11] and is an unstable version of (2.2) of [4].

Proposition 2.5. $t\Sigma\iota_{C_\alpha} \equiv \Sigma i\circ\overline{t\iota_{n+1}} + \widetilde{t\iota_{m+1}}\circ\Sigma p \bmod \Sigma i\circ(\pi_{m+2}(S^{n+1})/\{\eta_{n+1}\circ\Sigma^2\alpha, \Sigma\alpha\circ\eta_{m+1}\})\circ\Sigma p$.

Proof. We consider the following commutative diagram

$$
\begin{tikzcd}
 & \pi_{m+2}(S^{m+1}) \arrow[r, "\Sigma p^*"] \arrow[d, "\Sigma\alpha_*"] & {[\Sigma C_\alpha, S^{m+1}]} \arrow[d, "\Sigma\alpha_*"] & & \\
\pi_{n+2}(S^{n+1}) \arrow[r, "\Sigma^2\alpha^*"] \arrow[d, "\Sigma i_*"] & \pi_{m+2}(S^{n+1}) \arrow[r, "\Sigma p^*"] \arrow[d, "\Sigma i_*"] & {[\Sigma C_\alpha, S^{n+1}]} \arrow[r, "\Sigma i^*"] \arrow[d, "\Sigma i_*"] & \pi_{n+1}(S^{n+1}) \arrow[r, "\Sigma\alpha^*"] \arrow[d, "\Sigma i_*"] & \pi_{m+1}(S^{n+1}) \arrow[d, "\Sigma i_*"] \\
\pi_{n+2}(\Sigma C_\alpha) \arrow[r, "\Sigma^2\alpha^*"] & \pi_{m+2}(\Sigma C_\alpha) \arrow[r, "\Sigma p^*"] \arrow[d, "\Sigma p_*"] & {[\Sigma C_\alpha, \Sigma C_\alpha]} \arrow[r, "\Sigma i^*"] \arrow[d, "\Sigma p_*"] & \pi_{n+1}(\Sigma C_\alpha) \arrow[r, "\Sigma\alpha^*"] & \pi_{m+1}(\Sigma C_\alpha) \\
 & \pi_{m+2}(S^{m+2}) \arrow[r, "\Sigma p^*"] \arrow[d, "\Sigma^2\alpha_*"] & {[\Sigma C_\alpha, S^{m+2}]} \arrow[d, "\Sigma^2\alpha_*"] & & \\
 & \pi_{m+2}(S^{n+2}) \arrow[r, "\Sigma p^*"] & {[\Sigma C_\alpha, S^{n+2}],} & &
\end{tikzcd}
$$

where the row and column sequences are exact. By chasing the diagram, we obtain the result. This completes the proof. $\square$

Consider the Hopf map $h_d(\mathbf{F}) : S^{2d-1} \to S^d$. By using the notation of [13], we have the following in the 2-primary components:

$$
\begin{aligned}
&h_n(\mathbf{R}) = 2\iota_n \ (n \geq 1), && h_n(\mathbf{C}) = \eta_n \ (n \geq 2), \\
&h_n(\mathbf{H}) = \nu_n \ (n \geq 4), && h_n(\mathbf{O}) = \sigma_n \ (n \geq 8).
\end{aligned}
$$

Let $o(\mathbf{F}) \in \mathbf{Z}$ be the order of the stable Hopf class $h(\mathbf{F}) = \Sigma^\infty h_d(\mathbf{F})$, i.e., $o(\mathbf{F}) = 2$, 24 or 240 for $\mathbf{F} = \mathbf{C}$, $\mathbf{H}$ or $\mathbf{O}$, respectively. We apply Proposition 2.5 for $\alpha = h_d(\mathbf{F})$. Then we have

Corollary 2.6. (1) $2\Sigma\iota_{\mathbf{C}} = \Sigma i_{\mathbf{C}} \circ \overline{2\iota_3} + \widetilde{2\iota_4} \circ \Sigma p_{\mathbf{C}}$ *on* $[\Sigma\mathbf{C}\mathrm{P}^2, \Sigma\mathbf{C}\mathrm{P}^2]$,
(2) $24\Sigma\iota_{\mathbf{H}} = \Sigma i_{\mathbf{H}} \circ \overline{24\iota_5} + \widetilde{24\iota_8} \circ \Sigma p_{\mathbf{H}}$ *on* $[\Sigma\mathbf{H}\mathrm{P}^2, \Sigma\mathbf{H}\mathrm{P}^2]$,
(3) $240\Sigma\iota_{\mathbf{O}} \equiv \Sigma i_{\mathbf{O}} \circ \overline{240\iota_9} + \widetilde{240\iota_{16}} \circ \Sigma p_{\mathbf{O}} \bmod \Sigma\iota_{\mathbf{O}} \circ \epsilon_9 \circ \Sigma p_{\mathbf{O}}$ *on* $[\Sigma\mathbf{O}\mathrm{P}^2, \Sigma\mathbf{O}\mathrm{P}^2]$, *where* ϵ_9 *is a generator of* $\pi_{17}(S^9)$.

Proof. By [13], $\pi_5(S^3) \cong \mathbf{Z}_2\{\eta_3^2\}$, $\pi_9(S^5) \cong \mathbf{Z}_2\{\nu_5 \circ \eta_8\}$, $\pi_{17}(S^9) \cong \mathbf{Z}_2\{\sigma_9 \circ \eta_{16}\} \oplus \mathbf{Z}_2\{\bar{\nu}_9\} \oplus \mathbf{Z}_2\{\epsilon_9\}$ and $\eta_9 \circ \sigma_{10} = \bar{\nu}_9 + \epsilon_9$. Apply Proposition 2.5 for $\alpha = h_d(\mathbf{F})$. Then we can see that the assertion has established. $\square$

Remark that Corollary 2.6 (1) is obtained from Theorem 8.1 of [1].

It is well known that

$$\Sigma \oplus h_d(\mathbf{F})_* : [\Sigma^{k-1} C_\alpha, S^{d-1}] \oplus [\Sigma^k C_\alpha, S^{2d-1}] \to [\Sigma^k C_\alpha, S^d]$$

is an isomorphism for all $k \geq 1$.

We recall some properties of Toda brackets [13].

Proposition 2.7 ([13]). *Consider elements $\alpha \in [Y, Z]$, $\beta \in [X, Y]$ and $\gamma \in [W, X]$ which satisfy $\alpha \circ \beta = 0$, $\beta \circ \gamma = 0$. Let $\{\alpha, \beta, \gamma\}$ be the Toda bracket, $i : Z \to Z \cup_\alpha CY$ and $p : X \cup_\gamma CW \to \Sigma W$ be the canonical maps. Then*

(1) $\overline{\alpha} \circ \widetilde{\gamma} \in \{\alpha, \beta, \gamma\}$,
(2) $\alpha \circ \overline{\beta} \in \{\alpha, \beta, \gamma\} \circ p$,
(3) $\widetilde{\beta} \circ \Sigma\gamma \in -i \circ \{\alpha, \beta, \gamma\}$.

3. Cohomotopy groups of $\Sigma^n \mathbf{C}\mathrm{P}^2$

Let $\mathbf{C}\mathrm{P}^2$ be the complex projective plane, i.e., $\mathbf{C}\mathrm{P}^2 = S^2 \cup_{\eta_2} e^4$.

In this section, we compute the cohomotopy groups of the suspended complex projective plane $\Sigma^n \mathbf{C}\mathrm{P}^2$. Our main tool is the following exact sequence

$$(\mathbf{C}; n, k) \qquad \pi_{n+3}(S^k) \xrightarrow{{\eta_{n+3}}^*} \pi_{n+4}(S^k) \xrightarrow{\Sigma^n p_{\mathbf{C}}{}^*} [\Sigma^n \mathbf{C}\mathrm{P}^2, S^k] \xrightarrow{\Sigma^n i_{\mathbf{C}}{}^*} \pi_{n+2}(S^k) \xrightarrow{{\eta_{n+2}}^*} \pi_{n+3}(S^k)$$

induced from the cofiber sequence

$$S^{n+3} \xrightarrow{\eta_{n+2}} S^{n+2} \xrightarrow{\Sigma^n i_{\mathbf{C}}} \Sigma^n \mathbf{C}\mathrm{P}^2 \xrightarrow{\Sigma^n p_{\mathbf{C}}} S^{n+4} \xrightarrow{\eta_{n+3}} S^{n+3}.$$

By Lemma 2.3, we have ([1])

$$\begin{aligned} [\Sigma^n \mathbf{C}\mathrm{P}^2, S^{n+4}] &\cong \mathbf{Z}\{\Sigma^n p_{\mathbf{C}}\}, \\ [\Sigma^n \mathbf{C}\mathrm{P}^2, S^{n+3}] &= 0, \\ [\Sigma^n \mathbf{C}\mathrm{P}^2, S^{n+2}] &\cong \mathbf{Z}\{\overline{2\iota_{n+2}}\} \end{aligned}$$

for $n \geq 1$.

Since $\eta_m \in \pi_{m+1}(S^m)$ is of order two for $m \geq 3$, we have in the p-primary components

$$[\Sigma^n \mathbf{C}\mathrm{P}^2, S^k]_{(p)} \cong \pi_{n+2}(S^k)_{(p)} \oplus \pi_{n+4}(S^k)_{(p)},$$

where p is an odd prime. We only compute the 2-primary components of the cohomotopy groups $[\Sigma^n \mathbf{C}\mathrm{P}^2, S^k]$. The odd primary components are easily obtained by [13].

We see ([13]) that

$${\eta_{n+2}}^* : \pi_{n+2}(S^{n+1}) \to \pi_{n+3}(S^{n+1})$$

is an isomorphism for $n \geq 2$. Hence we have

$$[\Sigma^n \mathbf{C}\mathrm{P}^2, S^{n+1}] \cong \mathrm{Coker}\, {\eta_{n+3}}^*,$$

where ${\eta_{n+3}}^* : \pi_{n+3}(S^{n+1}) \to \pi_{n+4}(S^{n+1})$, ${\eta_{n+3}}^*(\eta_{n+1}^2) = 4\nu_{n+1}$ for $n \geq 4$ by (5.5) of [13] and $\eta_3^3 = 2\nu'$ by (5.3) of [13]. From the exact sequence $(\mathbf{C}; n, n+1)$, we obtain

Proposition 3.1. (1) $[\Sigma\mathbf{C}\mathrm{P}^2, S^2] \cong \mathbf{Z}\{\eta_2 \circ \overline{2\iota_3}\}$,

(2) $[\Sigma^2\mathbf{C}\mathrm{P}^2, S^3] \cong \mathbf{Z}_2\{\nu' \circ \Sigma^2 p_{\mathbf{C}}\} \oplus \mathbf{Z}_3$,

(3) $[\Sigma^3\mathbf{C}\mathrm{P}^2, S^4] \cong \mathbf{Z}\{\nu_4 \circ \Sigma^3 p_{\mathbf{C}}\} \oplus \mathbf{Z}_2\{\Sigma\nu' \circ \Sigma^3 p_{\mathbf{C}}\} \oplus \mathbf{Z}_3$,

(4) $[\Sigma^n\mathbf{C}\mathrm{P}^2, S^{n+1}] \cong \mathbf{Z}_4\{\nu_{n+1} \circ \Sigma^n p_{\mathbf{C}}\} \oplus \mathbf{Z}_3$ *for* $n \geq 4$.

Consider the exact sequence $(\mathbf{C}; n, n)$. We obtain that $[\Sigma^n\mathbf{C}\mathrm{P}^2, S^n] \cong \operatorname{Coker}{\eta_{n+3}}^*$, where ${\eta_{n+3}}^* : \pi_{n+3}(S^n) \to \pi_{n+4}(S^n)$. Then we have

Proposition 3.2. (1) $[\Sigma^2\mathbf{C}\mathrm{P}^2, S^2] \cong \mathbf{Z}_2\{\eta_2 \circ \nu' \circ \Sigma^2 p_{\mathbf{C}}\} \oplus \mathbf{Z}_3$,

(2) $[\Sigma^n\mathbf{C}\mathrm{P}^2, S^n] = 0$ *for* $n \geq 3$.

Let $g_6(\mathbf{C}) : \Sigma^7\mathbf{C}\mathrm{P}^2 \to S^6$ be the S^1-transfer map ([8]). This is the adjoint of the composite

$$\Sigma\mathbf{C}\mathrm{P}^2 \hookrightarrow SU(3) \hookrightarrow SO(6) \hookrightarrow \Omega^6 S^6$$

of the canonical maps. We set $g_{n+6}(\mathbf{C}) = \Sigma^n g_6(\mathbf{C})$ for $n \geq 1$.

Proposition 3.3. (1) $[\Sigma^3\mathbf{C}\mathrm{P}^2, S^2] = 0$,

(2) $[\Sigma^4\mathbf{C}\mathrm{P}^2, S^3] \cong \mathbf{Z}_2\{\nu' \circ \overline{2\iota_6}\} \oplus \mathbf{Z}_3$,

(3) $[\Sigma^5\mathbf{C}\mathrm{P}^2, S^4] \cong \mathbf{Z}\{\nu_4 \circ \overline{2\iota_7}\} \oplus \mathbf{Z}_2\{\Sigma\nu' \circ \overline{2\iota_7}\} \oplus \mathbf{Z}_3$,

(4) $[\Sigma^6\mathbf{C}\mathrm{P}^2, S^5] \cong \mathbf{Z}_4\{\nu_5 \circ \overline{2\iota_8}\} \oplus \mathbf{Z}_3$,

(5) $[\Sigma^7\mathbf{C}\mathrm{P}^2, S^6] \cong \mathbf{Z}\{g_6(\mathbf{C})\} \oplus \mathbf{Z}_4\{\nu_6 \circ \overline{2\iota_9}\} \oplus \mathbf{Z}_3$ *and* $2g_6(\mathbf{C}) = [\iota_6, \iota_6] \circ \Sigma^7 p_{\mathbf{C}} + \nu_6 \circ \overline{2\iota_9}$,

(6) $[\Sigma^{n+1}\mathbf{C}\mathrm{P}^2, S^n] \cong \mathbf{Z}_8\{g_n(\mathbf{C})\} \oplus \mathbf{Z}_3$ *and* $2g_n(\mathbf{C}) = \nu_n \circ \overline{2\iota_{n+3}}$ *for* $n \geq 7$.

Proof. Making use of the exact sequence $(\mathbf{C}; n+1, n)$, we easily obtain that

$$[\Sigma^{n+1}\mathbf{C}\mathrm{P}^2, S^n] \cong \operatorname{Ker}{\eta_{n+3}}^*$$

except for $n = 6$, where ${\eta_{n+3}}^* : \pi_{n+3}(S^n) \to \pi_{n+4}(S^n)$. We shall only prove (5) and (6). Consider the EHP-exact sequence

$$[\Sigma^8\mathbf{C}\mathrm{P}^2, S^{11}] \xrightarrow{\Delta} [\Sigma^6\mathbf{C}\mathrm{P}^2, S^5] \xrightarrow{\Sigma} [\Sigma^7\mathbf{C}\mathrm{P}^2, S^6]$$
$$\xrightarrow{H} [\Sigma^7\mathbf{C}\mathrm{P}^2, S^{11}] \xrightarrow{\Delta} [\Sigma^5\mathbf{C}\mathrm{P}^2, S^5]$$

induced from the 2-local EHP fibration $S^5 \xrightarrow{\Sigma} \Omega S^6 \xrightarrow{H} \Omega S^{11}$.

Since $[\Sigma^8\mathbf{C}\mathrm{P}^2, S^{11}] = [\Sigma^5\mathbf{C}\mathrm{P}^2, S^5] = 0$ and $[\Sigma^7\mathbf{C}\mathrm{P}^2, S^{11}] \cong \mathbf{Z}\{\Sigma^7 p_{\mathbf{C}}\}$, we have the split short exact sequence

$$0 \to [\Sigma^6\mathbf{C}\mathrm{P}^2, S^5] \xrightarrow{\Sigma} [\Sigma^7\mathbf{C}\mathrm{P}^2, S^6] \xrightarrow{H} [\Sigma^7\mathbf{C}\mathrm{P}^2, S^{11}] \to 0.$$

By (10) of [8], we have $H(g_6(\mathbf{C})) = \pm\Sigma^7 p_{\mathbf{C}}$ and $2g_6(\mathbf{C}) = [\iota_6, \iota_6] \circ \Sigma^7 p_{\mathbf{C}} + \nu_6 \circ \overline{2\iota_9}$. It follows that $2g_n(\mathbf{C}) = \nu_n \circ \overline{2\iota_{n+3}}$ for $n \geq 7$. This completes the proof. $\square$

Remark that the order of $g_n(\mathbf{C})$ is 24 for $n \geq 7$ by Theorem 7.28 of [6].

Proposition 3.4. (1) $[\Sigma^4\mathbf{C}\mathrm{P}^2, S^2] \cong \mathbf{Z}_2\{\eta_2 \circ \nu' \circ \overline{2\iota_6}\} \oplus \mathbf{Z}_3$,
(2) $[\Sigma^5\mathbf{C}\mathrm{P}^2, S^3] \cong \mathbf{Z}_3$,
(3) $[\Sigma^6\mathbf{C}\mathrm{P}^2, S^4] \cong \mathbf{Z}_4\{\nu_4^2 \circ \Sigma^6 p_{\mathbf{C}}\} \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3$,
(4) $[\Sigma^n\mathbf{C}\mathrm{P}^2, S^n] \cong \mathbf{Z}_2\{\nu_n^2 \circ \Sigma^{n+2} p_{\mathbf{C}}\}$ *for* $n \geq 5$.

Proof. Making use of the exact sequence $(\mathbf{C}; n+2, n)$, we easily obtain that

$$[\Sigma^{n+2}\mathbf{C}\mathrm{P}^2, S^n] \cong \mathrm{Coker}\, \eta_{n+5}{}^*$$

for $n \geq 3$, where $\eta_{n+5}{}^* : \pi_{n+5}(S^n) \to \pi_{n+6}(S^n)$. For $n = 6$, we have $\Delta(\iota_{13}) \circ \eta_{11} = 0$ by (5.13) of [13]. And $\eta_{n+5}{}^* : \pi_{n+5}(S^n) \to \pi_{n+6}(S^n)$ is trivial for $n \geq 5$. □

Proposition 3.5. (1) $[\Sigma^5\mathbf{C}\mathrm{P}^2, S^2] \cong \mathbf{Z}_3$,
(2) $[\Sigma^6\mathbf{C}\mathrm{P}^2, S^3] \cong \mathbf{Z}_2\{\overline{\nu'\eta_6^2}\} \oplus \mathbf{Z}_{15}$,
(3) $[\Sigma^7\mathbf{C}\mathrm{P}^2, S^4] \cong \mathbf{Z}_2\{\Sigma\overline{\nu'\eta_6^2}\} \oplus \mathbf{Z}_{15}$,
(4) $[\Sigma^8\mathbf{C}\mathrm{P}^2, S^5] \cong \mathbf{Z}_4\{\overline{\nu_5\eta_8^2}\} \oplus \mathbf{Z}_{15}$,
(5) $[\Sigma^9\mathbf{C}\mathrm{P}^2, S^6] \cong \mathbf{Z}\{\overline{\Delta\iota_{13}}\} \oplus \mathbf{Z}_4\{\sigma'' \circ \Sigma^9 p_{\mathbf{C}}\} \oplus \mathbf{Z}_{15}$,
(6) $[\Sigma^{10}\mathbf{C}\mathrm{P}^2, S^7] \cong \mathbf{Z}_8\{\sigma' \circ \Sigma^{10} p_{\mathbf{C}}\} \oplus \mathbf{Z}_{15}$,
(7) $[\Sigma^{11}\mathbf{C}\mathrm{P}^2, S^8] \cong \mathbf{Z}\{\sigma_8 \circ \Sigma^{11} p_{\mathbf{C}}\} \oplus \mathbf{Z}_8\{\Sigma\sigma' \circ \Sigma^{11} p_{\mathbf{C}}\} \oplus \mathbf{Z}_{15}$,
(8) $[\Sigma^{n+3}\mathbf{C}\mathrm{P}^2, S^n] \cong \mathbf{Z}_{16}\{\sigma_n \circ \Sigma^{n+3} p_{\mathbf{C}}\} \oplus \mathbf{Z}_{15}$ *for* $n \geq 9$.
We have a relation: $2\overline{\nu_5\eta_8^2} = \sigma''' \circ \Sigma^8 p_{\mathbf{C}}$.

Proof. We only prove (4). The rest can be easily obtained by making use of the exact sequence $(\mathbf{C}; n+3, n)$ and the fact $\nu_n \circ \eta_{n+3} = 0$ for $n \geq 6$.

Consider the exact sequence $(\mathbf{C}; 8, 5)$:

$$\pi_{11}(S^5) \xrightarrow{\eta_{11}{}^*} \pi_{12}(S^5) \xrightarrow{\Sigma^8 p_{\mathbf{C}}{}^*} [\Sigma^8\mathbf{C}\mathrm{P}^2, S^5] \xrightarrow{\Sigma^8 i_{\mathbf{C}}{}^*} \pi_{10}(S^5) \xrightarrow{\eta_{10}{}^*} \pi_{11}(S^5),$$

where $\pi_{11}(S^5) \cong \mathbf{Z}_2\{\nu_5^2\}$, $\pi_{12}(S^5) \cong \mathbf{Z}_2\{\sigma'''\} \oplus \mathbf{Z}_{15}$, $\pi_{10}(S^5) \cong \mathbf{Z}_2\{\nu_5\eta_8^2\}$ and $\nu_5^2 \circ \eta_{11} = 0 = \nu_5 \circ \eta_8^3$ by [13]. From Corollary 2.6 (1), we see that

$$\begin{aligned} 2\overline{\nu_5\eta_8^2} &= \overline{\nu_5\eta_8^2} \circ 2\Sigma^8\iota_{\mathbf{C}} \\ &= \nu_5\eta_8^2 \circ \overline{2\iota_{10}} + \overline{\nu_5\eta_8^2} \circ \widetilde{2\iota_{11}} \circ \Sigma^8 p_{\mathbf{C}}. \end{aligned}$$

By Proposition 2.7 (2),

$$\nu_5\eta_8^2 \circ \overline{2\iota_{10}} \in \{\nu_5\eta_8^2, 2\iota_{10}, \eta_{10}\} \circ \Sigma^8 p_{\mathbf{C}} \subset \{\nu_5, 2\eta_8^2, \eta_{10}\} \circ \Sigma^8 p_{\mathbf{C}} = 0$$

and by Proposition 2.7 (1),

$$\begin{aligned}\overline{\nu_5\eta_8^2} \circ \widetilde{2\iota_{11}} &\in \{\nu_5\eta_8^2, \eta_{10}, 2\iota_{11}\}\\ &\subset \{\nu_5, \eta_8^3, 2\iota_{11}\}\\ &= \{\nu_5, 4\nu_8, 2\iota_{11}\}\\ &\supset \{\nu_5, 2\nu_8, 4\iota_{11}\} \ni \sigma'''.\end{aligned}$$

Thus we obtain that $2\overline{\nu_5\eta_8^2} = \sigma''' \circ \Sigma^8 p_{\mathbf{C}}$. From the above exact sequence, we have (4). This completes the proof. □

Proposition 3.6. (1) $[\Sigma^6\mathbf{C}\mathrm{P}^2, S^2] \cong \mathbf{Z}_2\{\eta_2 \circ \overline{\nu'\eta_6^2}\} \oplus \mathbf{Z}_{15}$,
(2) $[\Sigma^7\mathbf{C}\mathrm{P}^2, S^3] \cong \mathbf{Z}_2\{\epsilon_3 \circ \Sigma^7 p_{\mathbf{C}}\} \oplus \mathbf{Z}_3$,
(3) $[\Sigma^8\mathbf{C}\mathrm{P}^2, S^4] \cong \mathbf{Z}_8\{\nu_4 \circ g_7(\mathbf{C})\} \oplus \mathbf{Z}_2\{\epsilon_4 \circ \Sigma^8 p_{\mathbf{C}}\} \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3$,
(4) $[\Sigma^9\mathbf{C}\mathrm{P}^2, S^5] \cong \mathbf{Z}_4\{\nu_5 \circ g_8(\mathbf{C})\}$,
(5) $[\Sigma^{10}\mathbf{C}\mathrm{P}^2, S^6] \cong \mathbf{Z}_4\{\nu_6 \circ g_9(\mathbf{C})\} \oplus \mathbf{Z}_4\{\bar{\nu}_6 \circ \Sigma^{10} p_{\mathbf{C}}\} \oplus \mathbf{Z}_3$,
(6) $[\Sigma^{n+4}\mathbf{C}\mathrm{P}^2, S^n] \cong \mathbf{Z}_4\{\nu_n \circ g_{n+3}(\mathbf{C})\} \oplus \mathbf{Z}_2\{\bar{\nu}_n \circ \Sigma^{n+4} p_{\mathbf{C}}\}$ *for* $n = 7$, 8 *and* 9,
(7) $[\Sigma^{n+4}\mathbf{C}\mathrm{P}^2, S^n] \cong \mathbf{Z}_4\{\nu_n \circ g_{n+3}(\mathbf{C})\}$ *for* $n \geq 10$.
We have a relation: $2(\nu_n \circ g_{n+3}(\mathbf{C})) = \epsilon_n \circ \Sigma^{n+4} p_{\mathbf{C}}$ *for* $n \geq 5$.

Proof. We only show that $[\Sigma^{n+4}\mathbf{C}\mathrm{P}^2, S^n]$ for $n \geq 5$ contains a direct summand isomorphic to $\mathbf{Z}_4$. Consider the exact sequence $(\mathbf{C}; 9, 5)$:

$$\pi_{12}(S^5) \xrightarrow{\eta_{12}{}^*} \pi_{13}(S^5) \xrightarrow{\Sigma^9 p_{\mathbf{C}}{}^*} [\Sigma^9\mathbf{C}\mathrm{P}^2, S^5] \xrightarrow{\Sigma^9 i_{\mathbf{C}}{}^*} \pi_{11}(S^5) \xrightarrow{\eta_{11}{}^*} \pi_{12}(S^5),$$

where $\pi_{11}(S^5) \cong \mathbf{Z}_2\{\nu_5^2\}$, $\pi_{12}(S^5) \cong \mathbf{Z}_2\{\sigma'''\} \oplus \mathbf{Z}_{15}$, $\pi_{13}(S^5) \cong \mathbf{Z}_2\{\epsilon_5\}$ and $\nu_5^2 \circ \eta_{11} = 0 = \sigma''' \circ \eta_{12}$ by [13]. By (7.6) of [13] and Propositions 3.3 (6) and 2.7 (2),

$$\begin{aligned}2(\nu_5 \circ g_8(\mathbf{C})) &= \nu_5 \circ 2g_8(\mathbf{C})\\ &= \nu_5 \circ \nu_8 \circ \overline{2\iota_{11}}\\ &= \nu_5^2 \circ \overline{2\iota_{11}}\\ &\in \{\nu_5^2, 2\iota_{11}, \eta_{11}\} \circ \Sigma^9 p_{\mathbf{C}}\\ &\ni \epsilon_5 \circ \Sigma^9 p_{\mathbf{C}} \bmod 0.\end{aligned}$$

It follows that $2(\nu_5 \circ g_8(\mathbf{C})) = \epsilon_5 \circ \Sigma^9 p_{\mathbf{C}}$. For $n \geq 5$, we see that

$$2(\nu_n \circ g_{n+3}(\mathbf{C})) = \epsilon_n \circ \Sigma^{n+4} p_{\mathbf{C}}$$

and the kernel of $\eta_{n+6}{}^* : \pi_{n+6}(S^n) \to \pi_{n+7}(S^n)$ is generated by ν_n^2. This completes the proof. □

Proposition 3.7. (1) $[\Sigma^7\mathbf{C}\mathrm{P}^2, S^2] \cong \mathbf{Z}_2\{\eta_2 \circ \epsilon_3 \circ \Sigma^7 p_{\mathbf{C}}\} \oplus \mathbf{Z}_3$,
(2) $[\Sigma^8\mathbf{C}\mathrm{P}^2, S^3] \cong \mathbf{Z}_2\{\mu_3 \circ \Sigma^8 p_{\mathbf{C}}\} \oplus \mathbf{Z}_{15}$,

(3) $[\Sigma^9\mathbf{C}\mathrm{P}^2, S^4] \cong \mathbf{Z}_2\{\nu_4^3 \circ \Sigma^9 p_{\mathbf{C}}\} \oplus \mathbf{Z}_2\{\mu_4 \circ \Sigma^9 p_{\mathbf{C}}\} \oplus \mathbf{Z}_{15}$,
(4) $[\Sigma^{10}\mathbf{C}\mathrm{P}^2, S^5] \cong \mathbf{Z}_2\{\nu_5^3 \circ \Sigma^{10} p_{\mathbf{C}}\} \oplus \mathbf{Z}_4\{\overline{\sigma'''}\} \oplus \mathbf{Z}_{15}$,
(5) $[\Sigma^{11}\mathbf{C}\mathrm{P}^2, S^6] \cong \mathbf{Z}_4\{\sigma'' \circ \overline{2\iota_{13}}\} \oplus \mathbf{Z}_{15}$,
(6) $[\Sigma^{12}\mathbf{C}\mathrm{P}^2, S^7] \cong \mathbf{Z}_8\{\sigma' \circ \overline{2\iota_{14}}\} \oplus \mathbf{Z}_{15}$,
(7) $[\Sigma^{13}\mathbf{C}\mathrm{P}^2, S^8] \cong \mathbf{Z}\{\sigma_8 \circ \overline{2\iota_{15}}\} \oplus \mathbf{Z}_8\{\Sigma\sigma' \circ \overline{2\iota_{15}}\} \oplus \mathbf{Z}_{15}$,
(8) $[\Sigma^{14}\mathbf{C}\mathrm{P}^2, S^9] \cong \mathbf{Z}_{16}\{\sigma_9 \circ \overline{2\iota_{16}}\} \oplus \mathbf{Z}_{15}$,
(9) $[\Sigma^{15}\mathbf{C}\mathrm{P}^2, S^{10}] \cong \mathbf{Z}\{\Delta(\iota_{21}) \circ \Sigma^{15} p_{\mathbf{C}}\} \oplus \mathbf{Z}_{16}\{\sigma_{10} \circ \overline{2\iota_{17}}\} \oplus \mathbf{Z}_{15}$,
(10) $[\Sigma^{n+5}\mathbf{C}\mathrm{P}^2, S^n] \cong \mathbf{Z}_{16}\{\sigma_n \circ \overline{2\iota_{n+7}}\} \oplus \mathbf{Z}_{15}$ *for* $n \geq 11$.

We have relations: $2\overline{\sigma'''} = \mu_5 \circ \Sigma^{10} p_{\mathbf{C}}$, $2\sigma'' \circ \overline{2\iota_{13}} = \mu_6 \circ \Sigma^{11} p_{\mathbf{C}}$, $4\sigma' \circ \overline{2\iota_{14}} = \mu_7 \circ \Sigma^{12} p_{\mathbf{C}}$ *and* $8\sigma_n \circ \overline{2\iota_{n+7}} = \mu_n \circ \Sigma^{n+5} p_{\mathbf{C}}$ *for* $n \geq 9$.

Proof. We only prove (4). From the exact sequence $(\mathbf{C}; 10, 5)$ and from the fact that $\sigma''' \circ \eta_{12} = 0$ ([13]), we have the exact sequence

$$0 \to \mathbf{Z}_2\{\nu_5^3\} \oplus \mathbf{Z}_2\{\mu_5\} \xrightarrow{\Sigma^{10} p_{\mathbf{C}}{}^*} [\Sigma^{10}\mathbf{C}\mathrm{P}^2, S^5] \xrightarrow{\Sigma^{10} i_{\mathbf{C}}{}^*} \mathbf{Z}_2\{\sigma'''\} \oplus \mathbf{Z}_{15} \to 0.$$

By Corollary 2.6 (1),

$$\begin{aligned} 2\overline{\sigma'''} &= \overline{\sigma'''} \circ 2\Sigma^{10}\iota_{\mathbf{C}} \\ &= \sigma''' \circ \overline{2\iota_{12}} + \overline{\sigma'''} \circ \widetilde{2\iota_{13}} \circ \Sigma^{10} p_{\mathbf{C}}. \end{aligned}$$

By Lemma 6.5 of [13] and Proposition 2.7 (2),

$$\sigma''' \circ \overline{2\iota_{12}} \in \{\sigma''', 2\iota_{12}, \eta_{12}\} \circ \Sigma^{10} p_{\mathbf{C}} \ni \mu_5 \circ \Sigma^{10} p_{\mathbf{C}} \bmod \epsilon_5 \circ \eta_{13} \circ \Sigma^{10} p_{\mathbf{C}} = 0$$

and by (7.4) of [13] and Proposition 2.7 (1),

$$\begin{aligned} \Sigma(\overline{\sigma'''} \circ \widetilde{2\iota_{13}}) &\in \Sigma\{\sigma''', \eta_{12}, 2\iota_{13}\} \\ &\subset \{2\sigma'', \eta_{13}, 2\iota_{14}\} \\ &\supset \sigma'' \circ \{2\iota_{13}, \eta_{13}, 2\iota_{14}\} \\ &\ni \sigma'' \circ \eta_{13}^2 = 0 \bmod 2\pi_{15}(S^6) = 0. \end{aligned}$$

Since $\Sigma : \pi_{14}(S^5) \to \pi_{15}(S^6)$ is a monomorphism, we have $\overline{\sigma'''} \circ \widetilde{2\iota_{13}} = 0$. Thus we conclude that

$$[\Sigma^{10}\mathbf{C}\mathrm{P}^2, S^5] \cong \mathbf{Z}_2\{\nu_5^3 \circ \Sigma^{10} p_{\mathbf{C}}\} \oplus \mathbf{Z}_4\{\overline{\sigma'''}\} \oplus \mathbf{Z}_{15}$$

and $2\overline{\sigma'''} = \mu_5 \circ \Sigma^{10} p_{\mathbf{C}}$.

From the fact that $2\sigma'' = \Sigma\sigma'''$, $2\sigma' = \Sigma\sigma''$ and $2\sigma_9 = \Sigma^2\sigma'$, we have $\sigma'' \circ \overline{2\iota_{13}} = \Sigma\overline{\sigma'''}$, $2\sigma' \circ \overline{2\iota_{14}} = \Sigma\sigma'' \circ \overline{2\iota_{15}}$ and $2\sigma_9 \circ \overline{2\iota_{16}} = \Sigma^2(\sigma' \circ \overline{2\iota_{14}})$. This leads to the relations and completes the proof. □

4. Cohomotopy groups of $\Sigma^n\mathbf{H}\mathrm{P}^2$

Let $\mathbf{H}\mathrm{P}^2$ be the quaternionic projective plane, i.e., $\mathbf{H}\mathrm{P}^2 = S^4 \cup_{h_4(\mathbf{H})} e^8$. In this section, we compute the cohomotopy groups of the suspended quaternionic projective plane $\Sigma^n\mathbf{H}\mathrm{P}^2$. Consider the exact sequence

$$(\mathbf{H};n,k) \qquad \pi_{n+5}(S^k) \xrightarrow{h_{n+5}(\mathbf{H})^*} \pi_{n+8}(S^k) \xrightarrow{\Sigma^n p_{\mathbf{H}}{}^*} [\Sigma^n\mathbf{H}\mathrm{P}^2, S^k] \xrightarrow{\Sigma^n i_{\mathbf{H}}{}^*} \pi_{n+4}(S^k) \xrightarrow{h_{n+4}(\mathbf{H})^*} \pi_{n+7}(S^k)$$

induced from the cofiber sequence

$$S^{n+7} \xrightarrow{h_{n+4}(\mathbf{H})} S^{n+2} \xrightarrow{\Sigma^n i_{\mathbf{H}}} \Sigma^n\mathbf{H}\mathrm{P}^2 \xrightarrow{\Sigma^n p_{\mathbf{H}}} S^{n+8} \xrightarrow{h_{n+5}(\mathbf{H})} S^{n+5}.$$

For $p \geq 5$, we have in the p-primary components

$$[\Sigma^n\mathbf{H}\mathrm{P}^2, S^k]_{(p)} \cong \pi_{n+4}(S^k)_{(p)} \oplus \pi_{n+8}(S^k)_{(p)}$$

since $h_n(\mathbf{H})$ $(n \geq 5)$ is of order 24. By Lemma 2.3 and the fact that $\eta_n \circ \nu_{n+1} = 0$ for $n \geq 5$, we obtain that

(1) $[\Sigma\mathbf{H}\mathrm{P}^2, S^5] \cong \mathbf{Z}_2\{\nu_5 \circ \eta_8 \circ \Sigma p_{\mathbf{H}}\} \oplus \mathbf{Z}\{\overline{24\iota_5}\}$,
(2) $[\Sigma^{n-4}\mathbf{H}\mathrm{P}^2, S^n] \cong \mathbf{Z}\{\overline{24\iota_n}\}$ for $n \geq 6$.

Since $\nu' \circ \nu_6 = 0$, there exists an extension $\overline{\nu'} \in [\Sigma^2\mathbf{H}\mathrm{P}^2, S^3]$ of $\nu' \in \pi_6(S^3)$. By (5.3) of [13], we have the relation $H(\nu') = \eta_5$. We set $\bar{\eta}_5 = H(\overline{\nu'})$, where $H : [\Sigma^2\mathbf{H}\mathrm{P}^2, S^3] \to [\Sigma^2\mathbf{H}\mathrm{P}^2, S^5]$ is the generalized Hopf homomorphism and we also set $\bar{\eta}_n = \Sigma^{n-5}\bar{\eta}_5$ for $n \geq 5$.

Proposition 4.1. (1) $[\Sigma\mathbf{H}\mathrm{P}^2, S^4] \cong \mathbf{Z}_2\{\nu_4 \circ \eta_7^2 \circ \Sigma p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\Sigma\nu' \circ \eta_7^2 \circ \Sigma p_{\mathbf{H}}\}$,
(2) $[\Sigma^2\mathbf{H}\mathrm{P}^2, S^5] \cong \mathbf{Z}_2\{\nu_5 \circ \eta_8^2 \circ \Sigma^2 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\bar{\eta}_5\}$,
(3) $[\Sigma^3\mathbf{H}\mathrm{P}^2, S^6] \cong \mathbf{Z}\{\Delta(\iota_{13}) \circ \Sigma^3 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\bar{\eta}_6\}$,
(4) $[\Sigma^{n-3}\mathbf{H}\mathrm{P}^2, S^n] \cong \mathbf{Z}_2\{\bar{\eta}_n\}$ *for* $n \geq 7$.

Proof. We only prove (2). From the exact sequence $(\mathbf{H};2,5)$ and the fact that $\eta_5 \circ \nu_6 = 0$, we have the exact sequence

$$0 \to \pi_{10}(S^5) \xrightarrow{\Sigma^2 p_{\mathbf{H}}{}^*} [\Sigma^2\mathbf{H}\mathrm{P}^2, S^5] \xrightarrow{\Sigma^2 i_{\mathbf{H}}{}^*} \pi_6(S^5) \cong \mathbf{Z}_2\{\eta_5\} \to 0.$$

Assume that $2\bar{\eta}_5 = \nu_5 \circ \eta_8^2 \circ \Sigma^2 p_{\mathbf{H}}$. By Lemma 5.7 of [13] and Lemma 2 of [5], we have

$$0 = \Delta(2H(\overline{\nu'})) = \Delta(\nu_5 \circ \eta_8^2 \circ \Sigma^2 p_{\mathbf{H}}) = \eta_2 \circ \nu' \circ \eta_6^2 \circ p_{\mathbf{H}} \neq 0.$$

This is a contradiction. It follows that the above sequence splits. This completes the proof. ∎

Proposition 4.2. (1) $[\Sigma\mathbf{H}\mathrm{P}^2, S^3] \cong \mathbf{Z}_2\{\overline{\eta_3^2}\}$,
(2) $[\Sigma^{n-2}\mathbf{H}\mathrm{P}^2, S^n] \cong \mathbf{Z}_2\{\eta_n \circ \bar{\eta}_{n+1}\}$ *for* $n \geq 4$.

Proof. For $n \geq 3$, $h_{n+3}(\mathbf{H})^* : \pi_{n+3}(S^n) \to \pi_{n+6}(S^n)$ is an epimorphism. From the exact sequence $(\mathbf{H}; n-2, n)$ and the fact that $\eta_n^2 \circ \nu_{n+2} = 0$ for $n \geq 3$,

$$\Sigma^{n-2}{i_{\mathbf{H}}}^* : [\Sigma^{n-2}\mathbf{H}\mathrm{P}^2, S^n] \to \operatorname{Ker}{\nu_{n+2}}^* \cong \mathbf{Z}_2\{\eta_n^2\}$$

is an isomorphism. By the definition of $\bar{\eta}_n$, $\eta_n \circ \bar{\eta}_{n+1}$ is an extension of η_n^2 for $n \geq 4$. This completes the proof. $\square$

Proposition 4.3. (1) $[\Sigma\mathbf{H}\mathrm{P}^2, S^2] \cong \mathbf{Z}_2\{\eta_2 \circ \overline{\eta_3^2}\}$,
(2) $[\Sigma^2\mathbf{H}\mathrm{P}^2, S^3] \cong \mathbf{Z}_4\{\overline{\nu'}\} \oplus \mathbf{Z}_3\{\alpha_2(3) \circ \Sigma^2 p_{\mathbf{H}}\} \oplus \mathbf{Z}_5$,
(3) $[\Sigma^3\mathbf{H}\mathrm{P}^2, S^4] \cong \mathbf{Z}_4\{\Sigma\overline{\nu'}\} \oplus \mathbf{Z}\{\nu_4 \circ \overline{24\iota_7}\} \oplus \mathbf{Z}_3\{\alpha_2(4) \circ \Sigma^3 p_{\mathbf{H}}\} \oplus \mathbf{Z}_5$,
(4) $[\Sigma^4\mathbf{H}\mathrm{P}^2, S^5] \cong \mathbf{Z}_4\{\Sigma^2\overline{\nu'}\} \oplus \mathbf{Z}_2\{\sigma''' \circ \Sigma^4 p_{\mathbf{H}}\} \oplus \mathbf{Z}_9\{\overline{\alpha_1(5)}\} \oplus \mathbf{Z}_5$,
(5) $[\Sigma^5\mathbf{H}\mathrm{P}^2, S^6] \cong \mathbf{Z}_4\{\Sigma^3\overline{\nu'}\} \oplus \mathbf{Z}_4\{\sigma'' \circ \Sigma^5 p_{\mathbf{H}}\} \oplus \mathbf{Z}_9\{\overline{\alpha_1(6)}\} \oplus \mathbf{Z}_5$,
(6) $[\Sigma^6\mathbf{H}\mathrm{P}^2, S^7] \cong \mathbf{Z}_4\{\Sigma^4\overline{\nu'}\} \oplus \mathbf{Z}_8\{\sigma' \circ \Sigma^6 p_{\mathbf{H}}\} \oplus \mathbf{Z}_9\{\overline{\alpha_1(7)}\} \oplus \mathbf{Z}_5$,
(7) $[\Sigma^7\mathbf{H}\mathrm{P}^2, S^8] \cong \mathbf{Z}_4\{\Sigma^5\overline{\nu'}\} \oplus \mathbf{Z}_8\{\Sigma\sigma' \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}\{\sigma_8 \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_9\{\overline{\alpha_1(8)}\} \oplus \mathbf{Z}_5$,
(8) $[\Sigma^{n-1}\mathbf{H}\mathrm{P}^2, S^n] \cong \mathbf{Z}_4\{\Sigma^{n-3}\overline{\nu'}\} \oplus \mathbf{Z}_{16}\{\sigma_n \circ \Sigma^{n-1} p_{\mathbf{H}}\} \oplus \mathbf{Z}_9\{\overline{\alpha_1(n)}\} \oplus \mathbf{Z}_5$ *for* $n \geq 9$.

Proof. (1), (2) and (3) are easily obtained. Consider the exact sequence $(\mathbf{H}; n-1, n)$ for $n \geq 5$. Then the kernel of $h_{n+3}(\mathbf{H})^* : \pi_{n+3}(S^n) \to \pi_{n+6}(S^n)$ is isomorphic to $\mathbf{Z}_4\{2\nu_n\} \oplus \mathbf{Z}_3\{\alpha_1(n)\}$, where $2\nu_n = \Sigma^{n-3}\nu'$ for $n \geq 5$.

Consider the exact sequence $(\mathbf{H}; 4, 5)$:

$$\pi_9(S^5) \xrightarrow{h_9(\mathbf{H})^*} \pi_{12}(S^5) \xrightarrow{\Sigma^4 {p_{\mathbf{H}}}^*} [\Sigma^4\mathbf{H}\mathrm{P}^2, S^5] \xrightarrow{\Sigma^4 {i_{\mathbf{H}}}^*} \pi_8(S^5) \xrightarrow{h_8(\mathbf{H})^*} \pi_{11}(S^5),$$

where $\pi_9(S^5) \cong \mathbf{Z}_2\{\nu_5 \circ \eta_8\}$, $\nu_5 \circ \eta_8 \circ \nu_9 = 0$, $\pi_{12}(S^5) \cong \mathbf{Z}_2\{\sigma'''\} \oplus \mathbf{Z}_3\{\alpha_2(5)\} \oplus \mathbf{Z}_5$, $\pi_8(S^5) \cong \mathbf{Z}_8\{\nu_5\} \oplus \mathbf{Z}_3\{\alpha_1(5)\}$ and $\pi_{11}(S^5) \cong \mathbf{Z}_2\{\nu_5^2\}$ by [13].

Since $\overline{\nu'}$ is of order 4, we have the results for the 2-primary components.

Consider the 3-primary components. We have $\alpha_1(5)^2 = 0$ by (13.7) of [13]. By Corollary 2.6 (2),

$$\begin{aligned}
3\overline{\alpha_1(5)} &= \overline{\alpha_1(5)} \circ 24\Sigma^4\iota_{\mathbf{H}} \\
&= \overline{\alpha_1(5)} \circ \Sigma^4 i_{\mathbf{H}} \circ \overline{24\iota_8} + \overline{\alpha_1(5)} \circ \widetilde{24\iota_{11}} \circ \Sigma^4 p_{\mathbf{H}} \\
&= \alpha_1(5) \circ \overline{24\iota_8} + \overline{\alpha_1(5)} \circ \widetilde{24\iota_{11}} \circ \Sigma^4 p_{\mathbf{H}}.
\end{aligned}$$

By the definition of $\alpha_2(5)$ and Proposition 2.7 (2), we obtain

$$\alpha_1(5) \circ \overline{24\iota_8} \in \{\alpha_1(5), 24\iota_8, \alpha_1(8)\} \circ \Sigma^4 p_{\mathbf{H}} \ni \alpha_2(5) \circ \Sigma^4 p_{\mathbf{H}}$$

and by (13.8) of [13] and Proposition 2.7 (1),

$$\overline{\alpha_1(5)} \circ \widetilde{24\iota_{11}} \in \{\alpha_1(5), \alpha_1(8), 24\iota_{11}\} \ni 1/2\alpha_2(5).$$

It follows that $3\overline{\alpha_1(5)} = \alpha_2(5) \circ \Sigma^4 p_{\mathbf{H}}$ and $3\overline{\alpha_1(n)} = \alpha_2(n) \circ \Sigma^{n-1} p_{\mathbf{H}}$ for $n \geq 5$. This completes the proof. □

Proposition 4.4. (1) $[\Sigma^2\mathbf{H}\mathrm{P}^2, S^2] \cong \mathbf{Z}_4\{\eta_2 \circ \overline{\nu'}\} \oplus \mathbf{Z}_{15}$,

(2) $[\Sigma^3\mathbf{H}\mathrm{P}^2, S^3] \cong \mathbf{Z}_2\{\nu' \circ \bar{\eta}_6\} \oplus \mathbf{Z}_2\{\epsilon_3 \circ \Sigma^3 p_{\mathbf{H}}\}$,

(3) $[\Sigma^4\mathbf{H}\mathrm{P}^2, S^4] \cong \mathbf{Z}_2\{\nu_4 \circ \bar{\eta}_7\} \oplus \mathbf{Z}_2\{\Sigma\nu' \circ \bar{\eta}_7\} \oplus \mathbf{Z}_2\{\epsilon_4 \circ \Sigma^4 p_{\mathbf{H}}\}$,

(4) $[\Sigma^5\mathbf{H}\mathrm{P}^2, S^5] \cong \mathbf{Z}_2\{\nu_5 \circ \bar{\eta}_8\} \oplus \mathbf{Z}_2\{\epsilon_5 \circ \Sigma^5 p_{\mathbf{H}}\}$,

(5) $[\Sigma^6\mathbf{H}\mathrm{P}^2, S^6] \cong \mathbf{Z}_2\{\bar{\nu}_6 \circ \Sigma^6 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\epsilon_6 \circ \Sigma^6 p_{\mathbf{H}}\}$,

(6) $[\Sigma^n\mathbf{H}\mathrm{P}^2, S^n] \cong (\Sigma^n p_{\mathbf{H}})^*\pi_{n+8}(S^n)$ *for* $n \geq 7$.

Proof. Since $\eta_{2*} : [\Sigma^2\mathbf{H}\mathrm{P}^2, S^3] \to [\Sigma^2\mathbf{H}\mathrm{P}^2, S^2]$ is an isomorphism, we have (1).

From the exact sequence $(\mathbf{H}; n, n)$ and the fact which $\bar{\eta}_n$ is of order two for $n \geq 5$, we obtain (2), (3) and (4).

Consider the homomorphism $h_{11}(\mathbf{H})^* : \pi_{11}(S^6) \to \pi_{14}(S^6)$, where $\pi_{11}(S^6) \cong \mathbf{Z}\{\Delta(\iota_{13})\}$ and $\pi_{14}(S^6) \cong \mathbf{Z}_8\{\bar{\nu}_6\} \oplus \mathbf{Z}_2\{\epsilon_6\} \oplus Z_3\{[\iota_6, \iota_6] \circ \alpha_1(11)\}$. By Lemma 6.2 of [13],

$$h_{11}(\mathbf{H})^*(\Delta(\iota_{13})) = \Delta(\iota_{13}) \circ h_{11}(\mathbf{H}) = 2\bar{\nu}_6 + [\iota_6, \iota_6] \circ \alpha_1(11).$$

From the fact that $\pi_{n+4}(S^n) = 0$ for $n \geq 6$, we have (5). Also, from the fact $\pi_{n+5}(S^n) = 0$ for $n \geq 7$, we have (6). □

The following proposition is easily obtain by making use of the exact sequence $(\mathbf{H}; n+1, n)$.

Proposition 4.5. (1) $[\Sigma^3\mathbf{H}\mathrm{P}^2, S^2] \cong \mathbf{Z}_2\{\eta_2 \circ \nu' \circ \bar{\eta}_6\} \oplus \mathbf{Z}_2\{\eta_2 \circ \epsilon_3 \circ \Sigma^3 p_{\mathbf{H}}\}$,

(2) $[\Sigma^4\mathbf{H}\mathrm{P}^2, S^3] \cong \mathbf{Z}_2\{\nu' \circ \eta_6 \circ \bar{\eta}_7\} \oplus \mathbf{Z}_2\{\mu_3 \circ \Sigma^4 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_3 \circ \epsilon_4 \circ \Sigma^4 p_{\mathbf{H}}\}$,

(3) $[\Sigma^5\mathbf{H}\mathrm{P}^2, S^4] \cong \mathbf{Z}_2\{\nu_4 \circ \eta_7 \circ \bar{\eta}_8\} \oplus \mathbf{Z}_2\{\Sigma\nu' \circ \eta_7 \circ \bar{\eta}_8\} \oplus \mathbf{Z}_2\{\mu_4 \circ \Sigma^5 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_4 \circ \epsilon_5 \circ \Sigma^5 p_{\mathbf{H}}\}$,

(4) $[\Sigma^6\mathbf{H}\mathrm{P}^2, S^5] \cong \mathbf{Z}_2\{\nu_5 \circ \eta_8 \circ \bar{\eta}_9\} \oplus \mathbf{Z}_2\{\mu_5 \circ \Sigma^6 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_5 \circ \epsilon_6 \circ \Sigma^6 p_{\mathbf{H}}\}$,

(5) $[\Sigma^7\mathbf{H}\mathrm{P}^2, S^6] \cong \mathbf{Z}\{\overline{12\Delta(\iota_{13})}\} \oplus \mathbf{Z}_2\{\mu_6 \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_6 \circ \epsilon_7 \circ \Sigma^7 p_{\mathbf{H}}\}$,

(6) $[\Sigma^{n+1}\mathbf{H}\mathrm{P}^2, S^n] \cong \mathrm{Coker}\, \nu_{n+6}{}^*$, *where* $\nu_{n+6}{}^* : \pi_{n+6}(S^n) \to \pi_{n+9}(S^n)$ *for* $n \geq 7$.

We show

Proposition 4.6. (1) $[\Sigma^4\mathbf{H}\mathrm{P}^2, S^2] \cong \mathbf{Z}_2\{\eta_2 \circ \nu' \circ \eta_6 \circ \bar{\eta}_7\} \oplus \mathbf{Z}_2\{\eta_2^2 \circ \epsilon_4 \circ \Sigma^4 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_2 \circ \mu_4 \circ \Sigma^4 p_{\mathbf{H}}\}$,

(2) $[\Sigma^5\mathbf{H}\mathrm{P}^2, S^3] \cong \mathbf{Z}_4\{\epsilon' \circ \Sigma^5 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_3 \circ \mu_4 \circ \Sigma^5 p_{\mathbf{H}}\} \oplus \mathbf{Z}_3\{\alpha_1(3) \circ \overline{\alpha_1(6)}\}$,

(3) $[\Sigma^6\mathbf{H}\mathrm{P}^2, S^4] \cong \mathbf{Z}_4\{\nu_4 \circ \Sigma^4\overline{\nu'}\} \oplus \mathbf{Z}_8\{\underline{\nu_4 \circ \sigma' \circ \Sigma^5 p_{\mathbf{H}}}\} \oplus \mathbf{Z}_4\{\underline{\Sigma\epsilon' \circ \Sigma^6 p_{\mathbf{H}}}\} \oplus \mathbf{Z}_2\{\eta_4 \circ \mu_5 \circ \Sigma^6 p_{\mathbf{H}}\} \oplus \mathbf{Z}_3\{\alpha_1(4) \circ \overline{\alpha_1(7)}\} \oplus \mathbf{Z}_9\{[\iota_4, \iota_4] \circ \overline{\alpha_1(7)}\} \oplus \mathbf{Z}_5$,

(4) $[\Sigma^7\mathbf{H}\mathrm{P}^2, S^5] \cong \mathbf{Z}_4\{\nu_5 \circ \sigma_8 \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_5 \circ \mu_6 \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_3\{\beta_1(5) \circ \Sigma^7 p_{\mathbf{H}}\}$,

(5) $[\Sigma^8\mathbf{H}\mathrm{P}^2, S^6] \cong \mathbf{Z}_2\{\nu_6 \circ \sigma_9 \circ \Sigma^8 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_6 \circ \mu_7 \circ \Sigma^8 p_{\mathbf{H}}\} \oplus \mathbf{Z}_3\{\beta_1(6) \circ \Sigma^8 p_{\mathbf{H}}\}$,

(6) $[\Sigma^{n+2}\mathbf{H}\mathrm{P}^2, S^n] \cong \mathbf{Z}_2\{\eta_n \circ \mu_{n+1} \circ \Sigma^{n+2}p_{\mathbf{H}}\} \oplus \mathbf{Z}_3\{\beta_1(n) \circ \Sigma^{n+2}p_{\mathbf{H}}\}$ *for* $n \geq 7$.

Proof. For $n \geq 5$, $h_{n+6}(\mathbf{H})^* : \pi_{n+6}(S^n) \to \pi_{n+9}(S^n)$ is monomorphic by [13]. It follows that

$$[\Sigma^{n+2}\mathbf{H}\mathrm{P}^2, S^n] \cong \mathrm{Coker}\, {\nu_{n+7}}^* : \pi_{n+7}(S^n) \to \pi_{n+10}(S^n).$$

By (7.19) of [13], we have $\sigma''' \circ \nu_{12} = 4x\nu_5 \circ \sigma_8$, $\sigma'' \circ \nu_{13} = 2x\nu_6 \circ \sigma_9$ and $\sigma' \circ \nu_{14} = x\nu_7 \circ \sigma_{10}$ for x odd. This completes the proof. $\square$

We show

Proposition 4.7. (1) $[\Sigma^5\mathbf{H}\mathrm{P}^2, S^2] \cong \mathbf{Z}_4\{\eta_2 \circ \epsilon' \circ \Sigma^5 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\eta_2^2 \circ \mu_4 \circ \Sigma^5 p_{\mathbf{H}}\} \oplus \mathbf{Z}_3$,

(2) $[\Sigma^6\mathbf{H}\mathrm{P}^2, S^3] \cong \mathbf{Z}_4\{\mu' \circ \Sigma^6 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\nu' \circ \epsilon_6 \circ \Sigma^6 p_{\mathbf{H}}\} \oplus \mathbf{Z}_3\{\alpha_3(3) \circ \Sigma^6 p_{\mathbf{H}}\} \oplus \mathbf{Z}_{35}$,

(3) $[\Sigma^7\mathbf{H}\mathrm{P}^2, S^4] \cong \mathbf{Z}_4\{\Sigma\mu' \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\Sigma\nu' \circ \epsilon_7 \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\nu_4 \circ \sigma' \circ \eta_{14} \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\nu_4 \circ \bar{\nu}_7 \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\nu_4 \circ \epsilon_7 \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_3\{\alpha_3(3) \circ \Sigma^7 p_{\mathbf{H}}\} \oplus \mathbf{Z}_{35}$,

(4) $[\Sigma^8\mathbf{H}\mathrm{P}^2, S^5] \cong \mathbf{Z}_8\{\zeta_5 \circ \Sigma^8 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\nu_5 \circ \bar{\nu}_8 \circ \Sigma^8 p_{\mathbf{H}}\} \oplus \mathbf{Z}_2\{\nu_5 \circ \epsilon_8 \circ \Sigma^8 p_{\mathbf{H}}\} \oplus \mathbf{Z}_9\{\alpha_3'(5) \circ \Sigma^8 p_{\mathbf{H}}\} \oplus \mathbf{Z}_{35}$,

(5) $[\Sigma^9\mathbf{H}\mathrm{P}^2, S^6] \cong \mathbf{Z}_8\{\zeta_6 \circ \Sigma^9 p_{\mathbf{H}}\} \oplus \mathbf{Z}_9\{\alpha_3'(6) \circ \Sigma^9 8 p_{\mathbf{H}}\} \oplus \mathbf{Z}_{35}$,

(6) $[\Sigma^{10}\mathbf{H}\mathrm{P}^2, S^7] \cong \mathbf{Z}_8\{\zeta_7 \circ \Sigma^{10} p_{\mathbf{H}}\} \oplus \mathbf{Z}_{27}\{\overline{\alpha_2(7)}\} \oplus \mathbf{Z}_{35}$,

(7) $[\Sigma^{11}\mathbf{H}\mathrm{P}^2, S^8] \cong \mathbf{Z}\{\sigma_8 \circ \overline{24\iota_{15}}\} \oplus \mathbf{Z}_8\{\zeta_8 \circ \Sigma^{11} p_{\mathbf{H}}\} \oplus \mathbf{Z}_{27}\{\overline{\alpha_2(8)}\} \oplus \mathbf{Z}_{35}$,

(8) $[\Sigma^{12}\mathbf{H}\mathrm{P}^2, S^9] \cong \mathbf{Z}_{16}\{\sigma_9 \circ \overline{24\iota_{16}}\} \oplus \mathbf{Z}_{27}\{\overline{\alpha_2(9)}\} \oplus \mathbf{Z}_{35}$,

(9) $[\Sigma^{13}\mathbf{H}\mathrm{P}^2, S^{10}] \cong \mathbf{Z}_{32}\{\overline{4\sigma_{10}}\} \oplus \mathbf{Z}_{27}\{\overline{\alpha_2(10)}\} \oplus \mathbf{Z}_{35}$,

(10) $[\Sigma^{14}\mathbf{H}\mathrm{P}^2, S^{11}] \cong \mathbf{Z}_{64}\{\overline{2\sigma_{11}}\} \oplus \mathbf{Z}_{27}\{\overline{\alpha_2(11)}\} \oplus \mathbf{Z}_{35}$,

(11) $[\Sigma^{15}\mathbf{H}\mathrm{P}^2, S^{12}] \cong \mathbf{Z}\{\Delta(\iota_{25}) \circ \Sigma^{15} p_{\mathbf{H}}\} \oplus \mathbf{Z}_{128}\{\overline{\sigma_{12}}\} \oplus \mathbf{Z}_{27}\{\overline{\alpha_2(12)}\} \oplus \mathbf{Z}_{35}$,

(12) $[\Sigma^{n+3}\mathbf{H}\mathrm{P}^2, S^n] \cong \mathbf{Z}_{128}\{\overline{\sigma_n}\} \oplus \mathbf{Z}_{27}\{\overline{\alpha_2(n)}\} \oplus \mathbf{Z}_{35}$ *for* $n \geq 13$.

Proof. We have the following table of the kernel of the homomorphism $h_{n+7}(\mathbf{H})^* : \pi_{n+7}(S^n) \to \pi_{n+10}(S^n)$ by [13],

n	2	$3 \leq n \leq 6$	7	8
Kernel $\cong$	$\mathbf{Z}_3$	$\mathbf{Z}_5$	$\mathbf{Z}_3+\mathbf{Z}_5$	$\mathbf{Z}+\mathbf{Z}_3+\mathbf{Z}_5$
generator			$\alpha_2(7)$	$8\sigma_8, \alpha_2(8)$

n	9	10	11	$n \geq 12$
Kernel $\cong$	$\mathbf{Z}_2+\mathbf{Z}_3+\mathbf{Z}_5$	$\mathbf{Z}_4+\mathbf{Z}_3+\mathbf{Z}_5$	$\mathbf{Z}_8+\mathbf{Z}_3+\mathbf{Z}_5$	$\mathbf{Z}_{16}+\mathbf{Z}_3+\mathbf{Z}_5$
generator	$8\sigma_9, \alpha_2(9)$	$4\sigma_{10}, \alpha_2(10)$	$2\sigma_{11}, \alpha_2(11)$	$\sigma_n, \alpha_2(n)$

Consider an extension $\sigma_9 \circ \overline{24\iota_{16}} \in [\Sigma^{12}\mathbf{H}\mathrm{P}^2, S^9]$ of $8\sigma_9$. By (9.2) of [13] and Proposition 2.7 (2), we have

$$\begin{aligned} 2(\sigma_9 \circ \overline{24\iota_{16}}) &= 2\sigma_9 \circ \overline{24\iota_{16}} \\ &\in \{2\sigma_9, 8\iota_{16}, \nu_{16}\} \circ \Sigma^{12}p_{\mathbf{H}} \\ &\ni \zeta_9 \circ \Sigma^{12}p_{\mathbf{H}} \bmod 0. \end{aligned}$$

It follows that $2(\sigma_9 \circ \overline{24\iota_{16}}) = \zeta_9 \circ \Sigma^{12}p_{\mathbf{H}}$ and $[\Sigma^{12}\mathbf{H}\mathrm{P}^2, S^9]_{(2)} \cong \mathbf{Z}_{16}\{\sigma_9 \circ \overline{24\iota_{16}}\}$.

Since $8\sigma_9 \circ \nu_{16} = 4\sigma_{10} \circ \nu_{17} = 2\sigma_{11} \circ \nu_{18} = \sigma_{12} \circ \nu_{19} = 0$ by [13], there exist extensions

$$\begin{aligned} &\overline{8\sigma_9} \in [\Sigma^{12}\mathbf{H}\mathrm{P}^2, S^9], && \overline{4\sigma_{10}} \in [\Sigma^{13}\mathbf{H}\mathrm{P}^2, S^{10}], \\ &\overline{2\sigma_{11}} \in [\Sigma^{14}\mathbf{H}\mathrm{P}^2, S^{11}], && \overline{\sigma_{12}} \in [\Sigma^{15}\mathbf{H}\mathrm{P}^2, S^{12}]. \end{aligned}$$

We set $\overline{\sigma_n} = \Sigma^{n-12}\overline{\sigma_{12}}$ for $n \geq 12$.

Since $\pi_{20}(S^9) \cong \mathbf{Z}_8\{\zeta_9\} \oplus \mathbf{Z}_2\{\overline{\nu_9} \circ \nu_{17}\} \oplus \mathbf{Z}_9 \oplus \mathbf{Z}_7$ by [13] and $\nu_{17} \circ \Sigma^{12}p_{\mathbf{H}} = 0$,

$$3\overline{8\sigma_9} \equiv \sigma_9 \circ \overline{24\iota_{16}} \bmod \zeta_9 \circ p_{\mathbf{H}}.$$

So we obtain $2\overline{8\sigma_9} = x\zeta_9 \circ \Sigma^{12}p_{\mathbf{H}}$ for x odd. By the similar argument, we obtain

$$4\overline{4\sigma_{10}} = z\zeta_{10} \circ \Sigma^{13}p_{\mathbf{H}}, \quad 8\overline{2\sigma_{11}} = y\zeta_{11} \circ \Sigma^{14}p_{\mathbf{H}}$$

and

$$16\overline{\sigma_{12}} = w\zeta_{12} \circ \Sigma^{15}p_{\mathbf{H}},$$

where z, y and w are odd. This leads to (9), (10), (11) and (12) in the 2-primary components.

Consider the 3-primary components of $[\Sigma^{n+3}\mathbf{H}\mathrm{P}^2, S^n]$ for $n \geq 7$. From the exact sequence $(\mathbf{H}; n+3, n)$ and $\alpha_2(n) \circ \alpha_1(n+7) = 0$ ([13]), we have the exact sequence

$$0 \to \mathbf{Z}_9\{\alpha_3'(n)\} \to [\Sigma^{n+3}\mathbf{H}\mathrm{P}^2, S^n]_{(3)} \to \mathbf{Z}_3\{\alpha_2(n)\} \to 0.$$

Since $\alpha_2(n) \circ \alpha_1(n+7) = 0$, there exists an extension $\overline{\alpha_2(n)} \in [\Sigma^{n+3}\mathbf{H}\mathrm{P}^2, S^n]$ of $\alpha_2(n)$. By Corollary 2.5 (2) and Proposition 2.7,

$$\begin{aligned} 3\overline{\alpha_2(n)} &= \alpha_2(n) \circ \overline{24\iota_{n+7}} + \overline{\alpha_2(n)} \circ \widetilde{24\iota_{n+11}} \circ \Sigma^{n+3}p_{\mathbf{H}} \\ &\in \{\alpha_2(n), 3\iota_{n+7}, \alpha_1(n+7)\} \circ \Sigma^{n+3}p_{\mathbf{H}} \\ &\quad + \{\alpha_2(n), \alpha_1(n+7), 3\iota_{n+10}\} \circ \Sigma^{n+3}p_{\mathbf{H}}. \end{aligned}$$

Here, we recall $\alpha_3(n) \in \{\alpha_2(n), 3\iota_{n+7}, \alpha_1(n+7)\}$, $\alpha'_3(n) \in \{\alpha_2(n), \alpha_1(n+7), 3\iota_{n+10}\}$ and $3\alpha'_3(n) = \alpha_3(n)$ by [13]. Thus we have

$$\begin{aligned} 3\overline{\alpha_2(n)} &= \alpha'_3(n) \circ \Sigma^{n+3} p_{\mathbf{H}} + \alpha_3(n) \circ \Sigma^{n+3} p_{\mathbf{H}} \\ &= 4\alpha'_3(n) \circ \Sigma^{n+3} p_{\mathbf{H}}. \end{aligned}$$

This completes the proof. □

Let $\mathrm{ext}(\nu_{11}) \in [\Sigma^6 \mathbf{O}P^2, S^{11}]$ be an extension of ν_{11}. We set $\mathrm{ext}(\nu) = \Sigma^\infty \mathrm{ext}(\nu_{11}) \in \{\mathbf{O}\mathrm{P}^2, S^5\}$.

Example. $120\, \mathrm{ext}(\nu_{11}) = x\zeta_{11}\Sigma^6 p_{\mathbf{O}}$ for x odd.

Proof. By Corollary 2.6 (3),

$$\begin{aligned} o(\mathbf{O})\, \mathrm{ext}(\nu_{11}) &= \mathrm{ext}(\nu_{11}) \circ o(\mathbf{O})\Sigma^6 \iota_{\mathbf{O}} \\ &\equiv \nu_{11} \circ \overline{o(\mathbf{O})\iota_{14}} + \mathrm{ext}(\nu_{11}) \circ \widetilde{o(\mathbf{O})\iota_{21}} \circ p_{\mathbf{O}} \\ &\quad \mathrm{mod}\ \nu_{11} \circ \bar{\nu}_{14} \circ \Sigma^6 p_{\mathbf{O}} = 0. \end{aligned}$$

By Proposition 2.7 (2), we obtain

$$\begin{aligned} \nu_{11} \circ \overline{o(\mathbf{O})\iota_{14}} &\in \{\nu_{11}, 16\iota_{14}, \sigma_{14}\} \circ \Sigma^6 p_{\mathbf{O}} \\ &\supset \{\nu_{11}, 8\iota_{14}, 2\sigma_{14}\} \circ \Sigma^6 p_{\mathbf{O}} \\ &\ni \zeta_{11} \circ \Sigma^6 p_{\mathbf{O}} \ \mathrm{mod}\ 0 \end{aligned}$$

and

$$\mathrm{ext}(\nu_{11}) \circ \widetilde{o(\mathbf{O})\iota_{21}} \in \{\nu_{11}, \sigma_{14}, 16\iota_{21}\} \ni \pm\zeta_{11} \ \mathrm{mod}\ 0.$$

So we have $o(\mathbf{O})\, \mathrm{ext}(\nu_{11}) = 0$ or $2\zeta_{11} \circ \Sigma^6 p_{\mathbf{O}}$. By Theorem 7.4 of [13], the order of ζ_n is 8 for $n \geq 5$ and $\pi_{n+11}(S^n)$ is generated by ζ_n and $\bar{\nu}_n \circ \nu_{n+8}$ if $n \geq 6$ and $n \neq 12$. Therefore ${\Sigma^{n-5} p_{\mathbf{O}}}^* : \pi_{n+11}(S^n) \to [\Sigma^{n-5}\mathbf{O}\mathrm{P}^2, S^n]$ is a monomorphism if $n \geq 6$ and $n \neq 12$.

In the stable range, we have

$$o(\mathbf{O})\, \mathrm{ext}(\nu) = o(\mathbf{O})\iota \circ \mathrm{ext}(\nu) \in 2\langle 8\iota, \nu, \sigma\rangle \circ p_{\mathbf{O}} = 2\zeta \circ p_{\mathbf{O}}.$$

This implies the relation $o(\mathbf{O})\, \mathrm{ext}(\nu_{11}) = 2\zeta_{11} \circ \Sigma^6 p_{\mathbf{O}}$. This leads to the assertion. □

Additional remark, added in proof. In the proof of Proposition 4.3, we obtained th fact that

$$3\overline{\alpha_1(5)} = \alpha_2(5) \circ \Sigma^4 p_{\mathbf{H}}.$$

We shall give another proof of the relation. By using the EHP-sequence and by the fact that $[\Sigma^6\mathbf{H}\mathrm{P}^2, S^{11}] = 0$, $\Sigma : [\Sigma^4\mathbf{H}\mathrm{P}^2, S^5] \to [\Sigma^5\mathbf{H}\mathrm{P}^2, S^6]$ is a monomorphism. So we have $3\overline{\alpha_1(5)} = 3\iota_5 \circ \overline{\alpha_1(5)}$. And we see that

$$3\iota_5 \circ \overline{\alpha_1(5)} \in \{3\iota_5, \alpha_1(5), \alpha_1(8)\} \circ \Sigma^4 p_{\mathbf{H}} \ \mathrm{mod}\ 0.$$

We know $\langle 3\iota, \alpha_1, \alpha_1 \rangle = \alpha_2$ in the stable range. This leads to the relation.

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